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New Proofs of Ramanujan's Partition Identities for Moduli 5 and 7

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The purpose of this paper is to present new proofs of modular identities (1.1) and (1.2). These identities are closely related to Ramanujan's partition identities of moduli 5 and 7. The common feature in these proofs is that they both begin with the trigonometric identity (1.3).

1. INTRODUCTION

Let $|q|<1$ and $(a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1})$. In [18] (p. 139, (4.5), p. 145, (8.1)), Ramanujan stated without proofs the two identities

$$\sum_{n=0}^{\infty}\left(\frac{q^{5n+1}}{(1-q^{5n+1})^2}+\frac{q^{5n+4}}{(1-q^{5n+4})^2}-\frac{q^{5n+2}}{(1-q^{5n+2})^2}-\frac{q^{5n+3}}{(1-q^{5n+3})^2}\right)$$
$$=q\,\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty} \tag{1.1}$$

and

$$\sum_{n=0}^{\infty}\left(q^{7n+1}\frac{(1+q^{7n+1})}{(1-q^{7n+1})^3}+q^{7n+2}\frac{(1+q^{7n+2})}{(1-q^{7n+2})^3}\right.$$
$$+q^{7n+4}\frac{(1+q^{7n+4})}{(1-q^{7n+4})^3}-q^{7n+3}\frac{(1+q^{7n+3})}{(1-q^{7n+3})^3}$$
$$\left.-q^{7n+5}\frac{(1+q^{7n+5})}{(1-q^{7n+5})^3}-q^{7n+6}\frac{(1+q^{7n+6})}{(1-q^{7n+6})^3}\right)$$
$$=q(q;q)_\infty^3\,(q^7;q^7)_\infty^3+8q^2\,\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}. \tag{1.2}$$

The first two proofs of (1.1) were given by W. N. Bailey [2], [3]. He used his own ${}_6\psi_6$ formula for one proof and Weierstrass elliptic functions for the other. A proof of (1.2) was first given by N. J. Fine [8]. In 1984, using Hecke's theory of modular forms, S. Raghavan [15] gave further proofs of both (1.1) and (1.2).

In this paper, new proofs of (1.1) and (1.2) will be given. For the proof of (1.1), Ramanujan's ${}_1\psi_1$ summation is used. For the proof of (1.2), Bailey's ${}_6\psi_6$ summation and a modular identity are used. The common feature of these two proofs is that they both begin with the trigonometric identity,

$$\left(\csc\theta + 4\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\sin(2n+1)\theta\right)^2 = \csc^2\theta + C_0 - 8\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos 2n\theta, \tag{1.3}$$

where $|q|<1$ and $C_0 = 8\sum_{n=1}^{\infty} nq^n/1-q^{2n}$.

To prove (1.3), we first observe that for $n \geqslant 1$

$$\csc\theta\sin(2n+1)\theta = 1 + 2\sum_{k=1}^{n}\cos 2k\theta.$$

It follows that

$$\left(\csc\theta + 4\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\sin(2n+1)\theta\right)^2 = \csc^2\theta + \sum_{n=0}^{\infty} C_{2n}\cos 2n\theta,$$

where the constants C_{2n} are independent of θ. Upon expansion, the sum on the left side equals

$$\csc^2\theta + 8\frac{q}{1-q} + 8\sum_{n=1}^{\infty}\left(\frac{q^{2n+1}}{1-q^{2n+1}}\left(1 + 2\sum_{m=1}^{n}\cos 2m\theta\right)\right) + 8\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\frac{q^{2m+1}}{1-q^{2m+1}}$$
$$\times(\cos(2(n-m)\theta) - \cos(2(n+m+1)\theta)).$$

Comparing coefficients of $\cos 2n\theta$ $(n \geqslant 1)$, we find that

$$C_{2n} = 16\sum_{m=1}^{\infty}\frac{q^{2n+2m-1}}{1-q^{2n+2m-1}} + 16\sum_{m=1}^{\infty}\frac{q^{2m-1}}{1-q^{2m-1}}\frac{q^{2n+2m-1}}{1-q^{2n+2m-1}} - 8\sum_{m=1}^{n}\frac{q^{2m-1}}{1-q^{2m-1}}\frac{q^{2n-2m+1}}{1-q^{2n-2m+1}}.$$

Multiplying both sides by $(1-q^{2n})/q^{2n}$, we deduce that

$$\frac{C_{2n}(1-q^{2n})}{q^{2n}} = 16 \sum_{m=1}^{\infty} \left(\frac{q^{2m-1}}{1-q^{2n+2m-1}} - \frac{q^{2n+2m-1}}{1-q^{2n+2m-1}} \right)$$
$$+16 \sum_{m=1}^{\infty} \left(\frac{q^{2m-1}}{1-q^{2m-1}} - \frac{q^{2m-1}}{1-q^{2n+2m-1}} \right)$$
$$-8 \sum_{m=1}^{n} \left(1 + \frac{q^{2m-1}}{1-q^{2m-1}} + \frac{q^{2n-2m+1}}{1-q^{2n-2m+1}} \right).$$

Hence,

$$C_{2n} = -\frac{8nq^{2n}}{1-q^{2n}}, \qquad n \geqslant 1.$$

The identity (1.3) is similar to the identity given in Ramanujan's paper [16, Section 4]. The proof of (1.3) given here follows the idea of Ramanujan. A less elementary proof of (1.3) can be found in [4, Chapter 18, Entry 16(second part)(iii) and 17(ii)].

We complete our introduction by defining certain theta functions that will be used in the sequel. Following Ramanujan, for $|ab| < 1$, define the general theta function $f(a, b)$ by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty,$$

where the last equality is obtained by the Jacobi triple product identity. In particular, set

$$f(-q) := f(-q, -q^2) = (q; q)_\infty,$$
$$\varphi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_\infty}{(q; -q)_\infty},$$

and

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}.$$

We will also denote $p(n)$ to be the number of unrestricted partitions of n.

2. The Modulus 5

In this section, we will prove (1.1) and deduce Ramanujan's partition identity of modulus 5. To prove (1.1), we need two lemmas.

LEMMA 2.1. *If $|a|<1$ and $|b|<1$, then*

$$a\varphi(ab)\,\psi(a^2b^2)\frac{f(-b/a,\,-a^3b)}{f(-a^2,\,-b^2)} = \sum_{k=1}^{\infty}\left(\frac{a^kb^{k-1}}{1-a^{2k}b^{2k-2}}-\frac{a^{k-1}b^k}{1-a^{2k-2}b^{2k}}\right). \tag{2.1}$$

LEMMA 2.2. *If $ab=cd$, then*

(i) $$f(a,b)\,f(c,d)+f(-a,-b)\,f(-c,-d)=2f(ac,bd)\,f(ad,bc) \tag{2.2}$$

and

(ii) $$f(a,b)\,f(c,d)-f(-a,-b)\,f(-c,-d) = 2af\left(\frac{b}{c},\frac{c}{b}\,abcd\right)f\left(\frac{b}{d},\frac{d}{b}\,abcd\right). \tag{2.3}$$

For a proof of Lemma 2.1, see [4, Chapter 17, (85)]. For a proof of Lemma 2.2, see [4, Chapter 16, Entry 29].

Using (1.3), Lemma 2.1 and 2.2, we will show

THEOREM 2.3. *For $|q|<1$, the identity* (1.1) *holds.*

Proof. From (1.3), we know that the difference between

$$\left(\csc\theta+4\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\sin(2n+1)\,\theta\right)^2$$

and

$$\csc^2\theta-8\sum_{n=1}^{\infty}n\frac{q^{2n}}{1-q^{2n}}\cos 2n\theta$$

is independent of θ.

So for arbitrary θ and θ_1, we deduce that

$$\begin{aligned}
&\csc^2\theta-\csc^2\theta_1-8\sum_{n=1}^{\infty}n\frac{q^{2n}}{1-q^{2n}}(\cos 2n\theta-\cos 2n\theta_1)\\
&=\left(\csc\theta+\csc\theta_1+4\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}(\sin(2n+1)\,\theta+\sin(2n+1)\,\theta_1\right)\\
&\quad\times\left(\csc\theta-\csc\theta_1+4\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}(\sin(2n+1)\,\theta-\sin(2n+1)\,\theta_1)\right).
\end{aligned} \tag{2.4}$$

Now, replace q by q^5 and let $\theta = u/2$ and $\theta_1 = u$, with u given by $e^{iu} = q^2$. By writing the trigonometric functions in terms of q, we can rewrite (2.4) as

$$\frac{q^2}{(1-q^2)^2} - \frac{q^4}{(1-q^4)^2} + \sum_{n=1}^{\infty} \left(n \frac{q^{10n}(q^{2n}+q^{-2n})}{1-q^{10n}} - n \frac{q^{10n}(q^{4n}+q^{-4n})}{1-q^{10n}} \right)$$
$$= \left(\frac{q}{1-q^2} + \frac{q^2}{1-q^4} + \sum_{n=0}^{\infty} \left(\frac{q^{12n+6} - q^{8n+4} + q^{14n+7} - q^{6n+3}}{1-q^{10n+5}} \right) \right)$$
$$\times \left(\frac{q}{1-q^2} - \frac{q^2}{1-q^4} + \sum_{n=0}^{\infty} \left(\frac{q^{12n+6} - q^{8n+4} - q^{14n+7} + q^{6n+3}}{1-q^{10n+5}} \right) \right). \quad (2.5)$$

Let S_1 and S_2 be the first and second series on the right side of (2.5), respectively. Then by interchanging summation, we have

$$S_1 = \frac{q}{1-q^2} + \frac{q^2}{1-q^4} + \sum_{n=0}^{\infty} (q^{12n+6} - q^{8n+4} + q^{14n+7} - q^{6n+3}) \sum_{m=0}^{\infty} q^{(10n+5)m}$$
$$= \frac{q}{1-q^2} + \frac{q^2}{1-q^4} + \sum_{n=0}^{\infty} \left(\frac{q^{5n+6}}{1-q^{10n+12}} - \frac{q^{5n+4}}{1-q^{10n+8}} + \frac{q^{5n+7}}{1-q^{10n+14}} - \frac{q^{5n+3}}{1-q^{10n+6}} \right).$$

Therefore,

$$S_1 = \sum_{n=0}^{\infty} \left(\frac{q^{5n+1}}{1-q^{10n+2}} - \frac{q^{5n+4}}{1-q^{10n+8}} \right) + \sum_{n=0}^{\infty} \left(\frac{q^{5n+2}}{1-q^{10n+4}} - \frac{q^{5n+3}}{1-q^{10n+6}} \right). \quad (2.6)$$

By letting $a = q$ and $b = q^4$ in Lemma 2.1, we find that

$$\sum_{n=0}^{\infty} \left(\frac{q^{5n+1}}{1-q^{10n+2}} - \frac{q^{5n+4}}{1-q^{10n+8}} \right) = q \frac{f(-q^3, -q^7)}{f(-q^2, -q^8)} \varphi(q^5)\, \psi(q^{10}).$$

Similarly, by letting $a = q^2$ and $b = q^3$ in Lemma 2.1, we find that

$$\sum_{n=0}^{\infty} \left(\frac{q^{5n+2}}{1-q^{10n+4}} - \frac{q^{5n+3}}{1-q^{10n+6}} \right) = q^2 \frac{f(-q, -q^9)}{f(-q^4, -q^6)} \varphi(q^5)\, \psi(q^{10}).$$

Therefore, by (2.6),

$$S_1 = q\varphi(q^5)\,\psi(q^{10})\left(\frac{f(-q^3, -q^7)}{f(-q^2, -q^8)} + q\,\frac{f(-q, -q^9)}{f(-q^4, -q^6)}\right). \tag{2.7}$$

A similar argument gives

$$S_2 = q\varphi(q^5)\,\psi(q^{10})\left(\frac{f(-q^3, -q^7)}{f(-q^2 - q^8)} - q\,\frac{f(-q, -q^9)}{f(-q^4, -q^6)}\right). \tag{2.8}$$

Multiplying (2.7) by (2.8), we deduce that

$$\begin{aligned} S_1 S_2 = q^2 & \frac{\varphi^2(q^5)\,\psi^2(q^{10})}{f^2(-q^2, -q^8)\,f^2(-q^4, -q^6)} \\ & \times (f(-q^3, -q^7)\,f(-q^4, -q^6) + qf(-q, -q^9)\,f(-q^2, -q^8)) \\ & \times (f(-q^3, -q^7)\,f(-q^4, -q^6) - qf(-q, -q^9)\,f(-q^2, -q^8)). \end{aligned} \tag{2.9}$$

Setting $a = -q$, $b = -q^4$, $c = q^2$ and $d = q^3$ in Lemma 2.2, we deduce that

$$\begin{aligned} & f(-q, -q^4)\,f(q^2, q^3) + f(q, q^4)\,f(-q^2, -q^3) \\ & \quad = 2f(-q^3, -q^7)\,f(-q^4, -q^6). \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} & f(-q, -q^4)\,f(q^2, q^3) - f(q, q^4)\,f(-q^2, -q^3) \\ & \quad = -2qf(-q^2, -q^8)\,f(-q, -q^9). \end{aligned} \tag{2.11}$$

Subtracting (2.11) from (2.10) and simplifying, we arrive at

$$\begin{aligned} & f(-q^3, -q^7)\,f(-q^4, -q^6) + qf(-q, -q^9)\,f(-q^2, -q^8) \\ & \quad = f(q, q^4)\,f(-q^2, -q^3). \end{aligned}$$

Adding (2.10) and (2.11) and simplifying, we find that

$$\begin{aligned} & f(-q^3, -q^7)\,f(-q^4, -q^6) - qf(-q, -q^9)\,f(-q^2, -q^8) \\ & \quad = f(-q, -q^4)\,f(q^2, q^3). \end{aligned}$$

Therefore, by (2.9),

$$\begin{aligned} S_1 S_2 &= q^2\,\frac{\varphi^2(q^5)\,\psi^2(q^{10})\,f(q, q^4)\,f(-q^2, -q^3)\,f(-q, -q^4)\,f(q^2, q^3)}{f^2(-q^2, -q^8)\,f^2(-q^4, -q^6)} \\ &= q^2\,\frac{f^5(-q^{10})}{f(-q^2)}. \end{aligned} \tag{2.12}$$

The left side of (2.5) can be rewritten as

$$\frac{q^2}{(1-q^2)^2}-\frac{q^4}{(1-q^4)^2}+\left(\sum_{n=1}^{\infty} n(q^{12n}+q^{8n}) \sum_{m=0}^{\infty} q^{10nm}\right.$$

$$\left.-\sum_{n=1}^{\infty} n(q^{14n}+q^{6n}) \sum_{m=0}^{\infty} q^{10nm}\right)$$

$$=\sum_{n=0}^{\infty}\left(\frac{q^{10n+2}}{(1-q^{10n+2})^2}+\frac{q^{10n+8}}{(1-q^{10n+8})^2}\right.$$

$$\left.-\frac{q^{10n+4}}{(1-q^{10n+4})^2}-\frac{q^{10n+6}}{(1-q^{10n+6})^2}\right), \tag{2.13}$$

where we have inverted the orders of summation. Hence, by (2.12) and (2.13),

$$\sum_{n=0}^{\infty}\left(\frac{q^{10n+2}}{(1-q^{10n+2})^2}+\frac{q^{10n+8}}{(1-q^{10n+8})^2}-\frac{q^{10n+4}}{(1-q^{10n+4})^2}-\frac{q^{10n+6}}{(1-q^{10n+6})^2}\right)$$

$$=q^2\frac{f^5(-q^{10})}{f(-q^2)}. \tag{2.14}$$

Replacing q^2 by q in (2.14), we deduce Theorem 2.3.

The proof of Theorem 2.3 given here is elementary. It uses simple properties of theta function all known to Ramanujan. The deepest theorem used is his ${}_1\psi_1$ summation, which is used to prove Lemma 2.1.

From Theorem 2.3, we deduce

COROLLARY 2.4. *For* $|q|<1$,

$$\sum_{n=0}^{\infty} p(5n+4)\, q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{2.15}$$

Proof. We rewrite the left hand side of (1.1) as

$$\sum_{n=0}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2}=\sum_{n=0}^{\infty}\left(\frac{n}{5}\right)\sum_{k=1}^{\infty} kq^{nk},$$

where $(\frac{n}{5})$ is the Legendre symbol. By Euler's famous identity

$$(q;q)_\infty^{-1}=\sum_{n=0}^{\infty} p(n)\, q^n, \tag{2.16}$$

we write the right hand side of (1.1) as

$$q(q^5; q^5)_\infty^5 \sum_{n=0}^{\infty} p(n) q^n.$$

Equating terms in the expansion of both sides which involve powers of q which are multiples of 5, we obtain

$$\begin{aligned}(q^5; q^5)_\infty^5 \sum_{n=0}^{\infty} p(5n+4) q^{5n+5} &= 5 \sum_{n=0}^{\infty} \left(\frac{n}{5}\right) \sum_{k=1}^{\infty} kq^{5nk} \\ &= 5q^5 \frac{(q^{25}; q^{25})_\infty^5}{(q^5; q^5)_\infty},\end{aligned}$$

where the last equality is obtained by using (1.1) with q replaced by q^5. Replacing q^5 by q, we have (2.15).

Remark. The proof of (2.15) given here can be found in Bailey's paper [2]. There are many proofs of (2.15) in the literature. These proofs were given by H. B. C. Darling [6], L. J. Mordell [13], H. Rademacher and H. S. Zuckerman [14], D. Kruswijk [12].

An immediate consequence of Corollary 2.4 is

COROLLARY 2.5. *For each nonnegative integer n, $p(5n+4) \equiv 0$ $(mod\ 5)$.*

A more careful analysis of Corollary 2.4 gives

COROLLARY 2.6. *For each nonnegative integer n, $p(25n+24) \equiv 0$ $(mod\ 25)$.*

Proof. See [17].

3. THE MODULUS 7

We will prove (1.2) in this section and deduce Ramanujan's partition identity for modulus 7. We need three lemmas.

LEMMA 3.1. *For $|q| < 1$,*

$$\begin{aligned}&-2\frac{\cos\theta}{\sin^3\theta} + 16 \sum_{n=1}^{\infty} n^2 \frac{q^{2n}}{1-q^{2n}} \sin 2n\theta \\ &\quad = 2\left(\csc\theta + 4 \sum_{n=0}^{\infty} \frac{q^{2n+1}}{1-q^{2n+1}} \sin(2n+1)\theta\right) \\ &\qquad \times \left(-\csc\theta\cot\theta + 4 \sum_{n=0}^{\infty} (2n+1) \frac{q^{2n+1}}{1-q^{2n+1}} \cos(2n+1)\theta\right). \qquad (3.1)\end{aligned}$$

Proof. Differentiate (1.3) with respect to θ.

LEMMA 3.2 (Bailey's ${}_6\psi_6$ Formula). *If* $|qa^2/bcde| < 1$, *then*

$$
{}_6\psi_6\left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e \end{matrix} ; q, \frac{qa^2}{bcde}\right]
$$
$$
= \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_\infty}. \tag{3.2}
$$

Proof. See [1] or [10, pp. 128].

We also need the following identity.

LEMMA 3.3. *Suppose* $q^2 = e^{2\pi i z}$. *Let*

$$
G_1(z) := -q^{-4/7} \frac{f(-q^4, -q^{10})}{f(-q^2, -q^{12})}, \qquad G_2(z) := q^{-2/7} \frac{f(-q^6, -q^8)}{f(-q^4, -q^{10})}
$$

and

$$
G_3(z) := -q^{6/7} \frac{f(-q^2, -q^{12})}{f(-q^6, -q^8)},
$$

then

$$
\frac{G_1^2(z)}{G_3(z)} + \frac{G_2^2(z)}{G_1(z)} + \frac{G_3^2(z)}{G_2(z)} = -8 - \frac{1}{q^2}\left(\frac{(q^2; q^2)_\infty}{(q^{14}; q^{14})_\infty}\right)^4. \tag{3.3}
$$

Proof. See [9, Lemma (3.9)] or [5, Chapter 25, Entry 31]. For a proof given by S. Ramanujan, refer to [18, pp. 243].

We are now ready to give a proof of

THEOREM 3.4. *For* $|q| < 1$, *the identity* (1.2) *holds.*

Proof. If we replace q by q^7 and let $q = e^{i\theta}$ in Lemma 3.1, we find that

$$
\sum_{n=1}^{\infty} n^2 \frac{q^{14n}(q^{2n} - q^{-2n})}{1 - q^{14n}} + \frac{q^2(1+q^2)}{(1-q^2)^3}
$$
$$
= \left(\frac{-1}{(q - q^{-1})} + \sum_{n=0}^{\infty} \frac{q^{14n+7}(q^{2n+1} - q^{-2n-1})}{1 - q^{14n+7}}\right)
$$
$$
\times \left(\frac{q + q^{-1}}{(q - q^{-1})^2} + \sum_{n=0}^{\infty} (2n+1) \frac{q^{14n+7}(q^{2n+1} + q^{-2n-1})}{1 - q^{14n+7}}\right). \tag{3.4}
$$

Similarly, if we replace q by q^7 and let $q^2=e^{i\theta}$ in Lemma 3.1, we find that

$$\sum_{n=1}^{\infty} n^2 \frac{q^{14n}(q^{4n}-q^{-4n})}{1-q^{14n}}+\frac{q^4(1+q^4)}{(1-q^4)^3}$$
$$=\left(\frac{-1}{(q^2-q^{-2})}+\sum_{n=0}^{\infty}\frac{q^{14n+7}(q^{4n+2}-q^{-4n-2})}{1-q^{14n+7}}\right)$$
$$\times\left(\frac{q^2+q^{-2}}{(q^2-q^{-2})^2}+\sum_{n=0}^{\infty}(2n+1)\frac{q^{14n+7}(q^{4n+2}+q^{-4n-2})}{1-q^{14n+7}}\right). \quad (3.5)$$

With q replaced by q^7 and $q^3=e^{i\theta}$ in Lemma 3.1, we deduce that

$$-\sum_{n=1}^{\infty} n^2 \frac{q^{14n}(q^{6n}-q^{-6n})}{1-q^{14n}}-\frac{q^6(1+q^6)}{(1-q^6)^3}$$
$$=\left(\frac{1}{(q^3-q^{-3})}-\sum_{n=0}^{\infty}\frac{q^{14n+7}(q^{6n+3}-q^{-6n-3})}{1-q^{14n+7}}\right)$$
$$\times\left(\frac{q^3+q^{-3}}{(q^3-q^{-3})^2}+\sum_{n=0}^{\infty}(2n+1)\frac{q^{14n+7}(q^{6n+3}+q^{-6n-3})}{1-q^{14n+7}}\right). \quad (3.6)$$

If we add (3.4), (3.5), and (3.6), we find that the left side is

$$\sum_{n=1}^{\infty} n^2 \frac{q^{14n}(q^{2n}-q^{-2n})}{1-q^{14n}}+\frac{q^2(1+q^2)}{(1-q^2)^3}+\sum_{n=1}^{\infty} n^2 \frac{q^{14n}(q^{4n}-q^{-4n})}{1-q^{14n}}$$
$$+\frac{q^4(1+q^4)}{(1-q^4)^3}-\sum_{n=1}^{\infty} n^2 \frac{q^{14n}(q^{6n}-q^{-6n})}{1-q^{14n}}-\frac{q^6(1+q^6)}{(1-q^6)^3}$$
$$=\sum_{n=0}^{\infty}\left(q^{14n+2}\frac{(1+q^{14n+2})}{(1-q^{14n+2})^3}+q^{14n+4}\frac{(1+q^{14n+4})}{(1-q^{14n+4})^3}\right.$$
$$+q^{14n+8}\frac{(1+q^{14n+8})}{(1-q^{14n+8})^3}-q^{14n+6}\frac{(1+q^{14n+6})}{(1-q^{14n+6})^3}$$
$$\left.-q^{14n+10}\frac{(1+q^{14n+10})}{(1-q^{14n+10})^3}-q^{14n+12}\frac{(1+q^{14n+12})}{(1-q^{14n+12})^3}\right),$$

where we have inverted the orders of summation. We note that this expression is the same as the left side of (1.2) when q is replaced by q^2. So, to prove Theorem 3.4, it suffices to show that the right side of the sum of (3.4), (3.5) and (3.6) is

$$q^2(q^2;q^2)_\infty^3\,(q^{14};q^{14})_\infty^3+8q^4\frac{(q^{14};q^{14})_\infty^7}{(q^2;q^2)_\infty}.$$

By interchanging summation, the right side of the sum of (3.4), (3.5) and (3.6) can be written as

$$\left[\sum_{n=1}^{\infty}\left(\frac{q^{7n-6}}{1-q^{14n-12}}-\frac{q^{7n-1}}{1-q^{14n-2}}\right)\right]\left(\sum_{n=-\infty}^{\infty}\frac{q^{7n+1}+q^{(7n+1)3}}{(1-q^{(7n+1)2})^2}\right)$$
$$+\left[\sum_{n=1}^{\infty}\left(\frac{q^{7n-5}}{1-q^{14n-10}}-\frac{q^{7n-2}}{1-q^{14n-4}}\right)\right]\left(\sum_{n=-\infty}^{\infty}\frac{q^{7n+2}+q^{(7n+2)3}}{(1-q^{(7n+2)2})^2}\right)$$
$$-\left[\sum_{n=1}^{\infty}\left(\frac{q^{7n-4}}{1-q^{14n-8}}-\frac{q^{7n-3}}{1-q^{14n-6}}\right)\right]\left(\sum_{n=-\infty}^{\infty}\frac{q^{7n+3}+q^{(7n+3)3}}{(1-q^{(7n+3)2})^2}\right).$$

By Lemma 2.1, we can rewrite the expression above as

$$q\varphi(q^7)\,\psi(q^{14})\frac{f(-q^5,-q^9)}{f(-q^2,-q^{12})}\left(\sum_{n=-\infty}^{\infty}\frac{q^{7n+1}+q^{(7n+1)3}}{(1-q^{(7n+1)2})^2}\right)$$
$$+q^2\varphi(q^7)\,\psi(q^{14})\frac{f(-q^3,-q^{11})}{f(-q^4,-q^{10})}\left(\sum_{n=-\infty}^{\infty}\frac{q^{7n+2}+q^{(7n+2)3}}{(1-q^{(7n+2)2})^2}\right)$$
$$-q^3\varphi(q^7)\,\psi(q^{14})\frac{f(-q,-q^{13})}{f(-q^6,-q^8)}\left(\sum_{n=-\infty}^{\infty}\frac{q^{7n+3}+q^{(7n+3)3}}{(1-q^{(7n+3)2})^2}\right). \tag{3.7}$$

By replacing q by q^7 in Lemma 3.2, we deduce that

$${}_6\psi_6\left[\begin{matrix} q^7\sqrt{a}, & -q^7\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & aq^7/b, & aq^7/c, & aq^7/d, & aq^7/e \end{matrix}; q^7, \frac{q^7a^2}{bcde}\right]$$
$$=\frac{(aq^7, aq^7/bc, aq^7/bd, aq^7/be, aq^7/cd, aq^7/ce, aq^7/be, q^7, q^7/a; q^7)_\infty}{(aq^7/b, aq^7/c, aq^7/d, aq^7/e, q^7/b, q^7/c, q^7/d, q^7/e, q^7a^2/bcde; q^7)_\infty}. \tag{3.8}$$

Setting $a=-q^2$, $b=c=q$ and $d=e=-q$ in (3.8), we find that

$$\sum_{n=-\infty}^{\infty}\frac{q^{7n+1}+q^{(7n+1)3}}{(1-q^{(7n+1)2})^2}=q\frac{(-q^2,-q^5,-q^7,-q^7,q^7,q^7,q^7,q^7;q^7)_\infty}{(-q,-q,q,q,-q^6,-q^6,q^6,q^6;q^7)_\infty}. \tag{3.9}$$

Similarly, setting $a=-q^4$, $b=c=q^2$ and $d=e=-q^2$ in (3.8), we find that

$$\sum_{n=-\infty}^{\infty}\frac{q^{7n+2}+q^{(7n+2)3}}{(1-q^{(7n+2)2})^2}$$
$$=q^2\frac{(-q^4,-q^3,-q^7,-q^7,q^7,q^7,q^7,q^7;q^7)_\infty}{(-q^2,-q^2,q^2,q^2,-q^5,-q^5,q^5,q^5;q^7)_\infty}. \tag{3.10}$$

Finally, with $a=-q^6$, $b=c=q^3$ and $d=e=-q^3$ in (3.8), we find that

$$\sum_{n=-\infty}^{\infty} \frac{q^{7n+3}+q^{(7n+3)3}}{(1-q^{(7n+3)2})^2}$$
$$=q^3 \frac{(-q, -q^6, -q^7, -q^7, q^7, q^7, q^7, q^7; q^7)_\infty}{(-q^3, -q^3, q^3, q^3, -q^4, -q^4, q^4, q^4; q^7)_\infty}. \tag{3.11}$$

Using (3.9), (3.10), (3.11), we can simplify (3.7) as

$$q^4 \frac{\varphi(q^7)\, \psi(q^{14})(q^7, q^7; q^7)_\infty\, (q^{14}; q^{14})_\infty^5}{f(-q^2, -q^{12})\, f(-q^4, -q^{10})\, f(-q^6, -q^8)} \left[q^{-2} \left(\frac{f(-q^4, -q^{10})}{f(-q^2, -q^{12})} \right)^2 \right.$$
$$\times \frac{f(-q^6, -q^8)}{f(-q^2, -q^{12})} + \left(\frac{f(-q^6, -q^8)}{f(-q^4, -q^{10})} \right)^2 \frac{f(-q^2, -q^{12})}{f(-q^4, -q^{10})}$$
$$\left. -q^2 \left(\frac{f(-q^2, -q^{12})}{f(-q^6, -q^8)} \right)^2 \frac{f(-q^4, -q^{10})}{f(-q^6, -q^8)} \right]$$
$$=q^4 \frac{(q^{14}; q^{14})_\infty^7}{(q^2; q^2)_\infty} \left(-\frac{G_1^2(z)}{G_3(z)} - \frac{G_2^2(z)}{G_1(z)} - \frac{G_3^2(z)}{G_2(z)} \right)$$
$$=q^4 \frac{(q^{14}; q^{14})_\infty^7}{(q^2; q^2)_\infty} \left(8 + q^{-2} \frac{(q^2; q^2)_\infty^4}{(q^{14}; q^{14})_\infty^4} \right)$$
$$=q^2 (q^2; q^2)_\infty^3\, (q^{14}; q^{14})_\infty^3 + 8q^4 \frac{(q^{14}; q^{14})_\infty^7}{(q^2; q^2)_\infty},$$

where we have used Lemma 3.3. This is what we wanted to show, and, replacing q^2 by q, we complete the proof of Theorem 3.4.

COROLLARY 3.5. *For* $|q|<1$,

$$\sum_{n=0}^{\infty} p(7n+5)\, q^n = 7 \frac{(q^7; q^7)_\infty^3}{(q; q)_\infty^4} + 49q \frac{(q^7; q^7)_\infty^7}{(q; q)_\infty^8}. \tag{3.12}$$

Proof. We rewrite the left hand side of (1.2) as

$$\sum_{n=0}^{\infty} \left(\frac{n}{7} \right) \frac{q^n(1+q^n)}{(1-q^n)^3} = \sum_{n=0}^{\infty} \left(\frac{n}{7} \right) q^n(1+q^n) \sum_{k=1}^{\infty} \frac{k(k+1)}{2} (q^n)^{k-1}$$
$$= \sum_{n=0}^{\infty} \left(\frac{n}{7} \right) \sum_{k=1}^{\infty} k^2 q^{nk}.$$

By Jacobi's triple product identity

$$\sum_{n=0}^{\infty} (-1)^n\, (2n+1)\, q^{n(n+1)/2} = (q; q)_\infty^3$$

and Euler's identity (2.16), the right hand side of (1.2) can be expressed as

$$q \sum_{n=0}^{\infty} (-1)^n (2n+1)\, q^{n(n+1)/2} (q^7; q^7)_\infty^3 + 8q^2 (q^7; q^7)_\infty^7 \sum_{n=0}^{\infty} p(n)\, q^n.$$

Equating terms in the expansion of both sides which involve powers of q which are multiples of 7, we obtain

$$49 \sum_{n=0}^{\infty} \left(\frac{n}{7}\right) \sum_{k=1}^{\infty} k^2 q^{7nk} = -7q^7 \left(\sum_{n=0}^{\infty} (-1)^n (2n+1)(q^{49})^{n(n+1)/2} \right) (q^7; q^7)_\infty^3 + 8(q^7; q^7)_\infty^7 \sum_{n=0}^{\infty} p(7n+5)\, q^{7n+7}.$$

By Theorem 3.4, with q replaced by q^7, we can rewrite the left hand side of the identity above and deduce that

$$49q^7 (q^7; q^7)_\infty^3 (q^{49}; q^{49})_\infty^3 + 392 q^{14} \frac{(q^{49}; q^{49})_\infty^7}{(q^7; q^7)_\infty} = -7q^7 (q^{49}; q^{49})_\infty^3 (q^7; q^7)_\infty^3 + 8(q^7; q^7)_\infty^7 \sum_{n=0}^{\infty} p(7n+5)\, q^{7n+7}.$$

Simplifying, we find that

$$\sum_{n=0}^{\infty} p(7n+5)\, q^{7n} = 7 \frac{(q^{49}; q^{49})_\infty^3}{(q^7; q^7)_\infty^4} + 49q^7 \frac{(q^{49}; q^{49})_\infty^7}{(q^7; q^7)_\infty^8}.$$

Replacing q^7 by q, we obtain (3.12).

Remark. The proof given here can be found in [18, pp. 145]. Other proofs of Corollary 3.5 may be found in papers by O. Kolberg [11], H. Rademacher and H. S. Zuckerman [14], L. J. Mordell [13] and F. G. Garvan [9].

An immediate consequence of Corollary 3.5 is

COROLLARY 3.6. *For each nonnegative integer n, $p(7n+5) \equiv 0 \ (mod\ 7)$.*

A more careful analysis of Corollary 3.5 gives

COROLLARY 3.7. *For each nonnegative integer n, $p(49n+47) \equiv 0$ $(mod\ 49)$.*

Proof. See [17].

4. A Shorter Proof of Theorem 3.4.

In this section, we will include another proof of Theorem 3.4. This short and elegant proof is given by F. Garvan. In Lemma 3.2, we let $a=t^2$, $b=c=d=e=t$ to obtain

$$\frac{(q;q)_\infty^6 (t^2;q)_\infty (t^{-2}q;q)_\infty}{(t;q)_\infty^4 (t^{-1}q;q)_\infty^4} = \sum_{n=0}^{\infty} \frac{q^n(1+tq^n)}{(1-tq^n)^3} - \sum_{n=1}^{\infty} \frac{t^{-2}q^n(1+t^{-1}q^n)}{(1-t^{-1}q^n)^3}. \tag{4.1}$$

If we replace q^2 by q in (3.3) and multiply both sides by $q^2(q^7;q^7)_\infty^7/(q;q)_\infty$, we obtain

$$\begin{aligned} &q(q^7;q^7)_\infty^3 (q;q)_\infty^3 + 8q^2 \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} \\ &= (q^7;q^7)_\infty^6 \left(q \frac{(q^2;q^7)_\infty (q^5;q^7)_\infty}{(q;q^7)_\infty^4 (q^6;q^7)_\infty^4} + q^2 \frac{(q^3;q^7)_\infty (q^4;q^7)_\infty}{(q^2;q^7)_\infty^4 (q^5;q^7)_\infty^4} \right. \\ &\quad \left. - q^3 \frac{(q;q^7)_\infty (q^6;q^7)_\infty}{(q^3;q^7)_\infty^4 (q^4;q^7)_\infty^4} \right). \end{aligned} \tag{4.2}$$

If we multiply both sides of (4.1) by t, substitute q^7 for q and use the substitutions $t=q, q^2, q^3$ we find that (4.2) can be written as

$$\begin{aligned} &q(q;q)_\infty^3 (q^7;q^7)_\infty^3 + 8q^2 \frac{(q^7;q^7)_\infty^7}{(q;q)_\infty} \\ &= \sum_{n=0}^{\infty} \left(q^{7n+1} \frac{(1+q^{7n+1})}{(1-q^{7n+1})^3} + q^{7n+2} \frac{(1+q^{7n+2})}{(1-q^{7n+2})^3} + q^{7n+4} \frac{(1+q^{7n+4})}{(1-q^{7n+4})^3} \right. \\ &\quad \left. - q^{7n+3} \frac{(1+q^{7n+3})}{(1-q^{7n+3})^3} - q^{7n+5} \frac{(1+q^{7n+5})}{(1-q^{7n+5})^3} - q^{7n+6} \frac{(1+q^{7n+6})}{(1-q^{7n+6})^3} \right) \end{aligned}$$

which is (1.2).

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