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# Ramanujan's Singular Moduli

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**Abstract.** In his first notebook, in scattered places, Ramanujan recorded without proofs the values of over 100 class invariants and over 30 singular moduli. This paper is devoted to establishing all of Ramanujan's representations for singular moduli. For those of odd index, new algorithms needed to be developed.

**Key words:** Ramanujan-Weber class invariants, singular moduli, modular equations, theta-functions

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## 1. Introduction

If, as usual in the theory of elliptic functions, $k = k(q)$ denotes the modulus, then the singular modulus $k_n$ is defined by $k_n = k(e^{-\pi\sqrt{n}})$, where $n$ denotes a positive integer. Following Ramanujan, set $\alpha = k^2$ and $\alpha_n = k_n^2$.

Closely associated with the singular modulus are the class invariants $G_n$ and $g_n$. Let

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q| < 1,$$

and set

$$\chi(q) = (-q;q^2)_\infty.$$

For $q = \exp(-\pi\sqrt{n})$, the *class invariants* $G_n$ and $g_n$ are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \qquad \text{and} \qquad g_n := 2^{-1/4}q^{-1/24}\chi(-q).$$

In the notation of Weber [12], $G_n =: 2^{-1/4} f(\sqrt{-n})$ and $g_n =: 2^{-1/4} f_1(\sqrt{-n})$. Now [1, p. 124]

$$\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24} \qquad \text{and} \qquad \chi(-q) = 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}.$$

Thus,

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24} \qquad \text{and} \qquad g_n = 2^{-1/12}(1-\alpha_n)^{1/12}\alpha_n^{-1/24}. \tag{1.1}$$

Watson [11] remarked, "For reasons which had commended themselves to Weber and Ramanujan independently, it is customary to determine $G_n$ for odd values of $n$, and $g_n$ for even values of $n$." It is well known that $G_n$ and $g_n$ are algebraic [4, p. 214, Theorem 10.23; p. 257, Theorem 12.17]. It follows trivially from (1.1) that $\alpha_n$ is also algebraic. However, much more can be proved. Using results of Deuring [5], Chan and Huang [3] have proved the following theorem.

**Theorem 1.1.**
(a) *If* $n \equiv 1 \pmod 4$, *then* $G_n$ *and* $2\alpha_n$ *are units.*
(b) *If* $n \equiv 3 \pmod 8$, *then* $2^{-1/12}G_n$ *and* $2^2\alpha_n$ *are units.*
(c) *If* $n \equiv 7 \pmod 8$, *then* $2^{-1/4}G_n$ *and* $2^4\alpha_n$ *are units.*
(d) *If* $n \equiv 2 \pmod 4$, *then* $g_n$ *and* $\alpha_n$ *are units.*

It is clear from (1.1) that if the value of $G_n$ (or $g_n$) can be determined, then $\alpha_n$ can be computed by solving a quadratic equation. However, the expression that one obtains generally is unattractive and does not evince the fact that $\alpha_n$ can be expressed in terms of units in certain algebraic number fields. Thus, formulas for $\alpha_n$ that facilitate their representations via units are desirable.

In his second letter to Hardy [9, p. xxix], Ramanujan asserted that

$$\begin{aligned} k_{210} = {} & \left(\sqrt{2}-1\right)^4\left(2-\sqrt{3}\right)^2\left(\sqrt{7}-\sqrt{6}\right)^4\left(8-3\sqrt{7}\right)^2 \\ & \times \left(\sqrt{10}-3\right)^4\left(4-\sqrt{15}\right)^4\left(\sqrt{15}-\sqrt{14}\right)^2\left(6-\sqrt{35}\right)^2. \end{aligned}$$

This was first proved by Watson [10], who used the following remarkable formula that he found in Ramanujan's first notebook [8, vol. 1, p. 320] and which enables one to calculate $\alpha_n$ for even $n$.

**Theorem 1.2.** *Set*

$$\begin{gathered} g_n^6 = uv, \\ u^2 + 1/u^2 = 2U, \qquad v^2 + 1/v^2 = 2V, \\ W = \sqrt{U^2+V^2-1}, \end{gathered}$$

*and*

$$2S = U + V + W + 1.$$

*Then*

$$\begin{aligned} \alpha_n = {} & \left\{\sqrt{S}-\sqrt{S-1}\right\}^2\left\{\sqrt{S-U}-\sqrt{S-U-1}\right\}^2 \\ & \times \left\{\sqrt{S-V}-\sqrt{S-V-1}\right\}^2\left\{\sqrt{S-W}-\sqrt{S-W-1}\right\}^2. \end{aligned}$$

Watson's proof of Theorem 1.2 is a *verification*; it does not shed any light on how Ramanujan might have discovered the formula. Ramanathan [7] stated Ramanujan's Theorem 1.2 but did not find another proof. Regrettably, we also have not found a more transparent proof. Watson's proof crucially depends upon the symmetry in the quadratic equation for $\sqrt{\alpha_n}$ in terms of $g_n$ arising from (1.1). This symmetry does not appear in the quadratic equation for $\sqrt{\alpha_n}$ in terms of $G_n$. However, we show that the algorithm implicit in Theorem 1.2 can be adopted to determine $\alpha_n$ for odd $n$ as well.

Furthermore, Watson [10] inexplicably claimed, "...this is the sole instance in which Ramanujan has calculated the value of $k$ for an even integer $n$." In fact, twenty additional values of $k_n$ for even $n$ are found in the first notebook. At the beginning of Section 2, Theorem 2.1 gives thirteen of these values.

On page 82 of his first notebook, Ramanujan offers three additional theorems for calculating $\alpha_n$ when $n$ is even. The first (Theorem 2.2) expresses $\alpha_{4p}$ as a product of units involving $G_p$. The second (Theorem 2.4) expresses $\alpha_{16p}$ as a product of units involving $G_p$. The third (Theorem 2.5) enables one to determine $\alpha_{8p}$ as a product of two fourth powers of units, provided that $\alpha_{2p}$ can be expressed as a product of units of a certain form. In Section 2 we prove these results and calculate eight examples of Ramanujan as illustrations.

The calculation of $\alpha_n$ when $n$ is odd is slightly more difficult. On page 80 in his first notebook, Ramanujan recorded the values of $\alpha_{21}$, $\alpha_{33}$, and $\alpha_{45}$ in terms of units. This list is repeated, with the addition of $\alpha_{15}$, at the bottom of page 262 in his second notebook. On pages 345 and 346 in his first notebook, Ramanujan recorded units that appear in representations of $\alpha_n$ when $n = 3, 5, 7, 9, 13, 15, 17, 25, 55$. (Inexplicably, the units for $\alpha_7$ and $\alpha_{15}$ are recorded twice.) Ramanujan also indicated that he had intended to calculate $\alpha_{39}$, but no factors are given. Of course, the result for $n = 15$ is superseded by the complete formula given on page 262 in the second notebook. It is unclear to us why Ramanujan only listed portions of $\alpha_n$ and not complete formulas. Initially in our investigations we employed computational "trial and error" to "guess" the complete formulas for $\alpha_n$, $n = 5, 9, 13, 17, 25, 55$. We remark that the values for $\alpha_3$ and $\alpha_7$ are easily determined from the formula

$$\alpha_n = \frac{1}{2} G_n^{-12}\left(G_n^{12} - \sqrt{G_n^{24} - 1}\right), \tag{1.2}$$

which is readily obtained from (1.1). For further values of $n$, however, (1.2) becomes unwieldy, and so better algorithms were sought.

In Section 3 we adopt the algorithm of Theorem 1.2 and reformulate it in Theorem 3.1 in terms of $G_n$ to calculate some values of $\alpha_n$ when $n$ is odd. Theorem 3.2 provides a list of all of Ramanujan's values. Although Theorem 3.1 yields a systematic procedure for calculating $\alpha_n$ when $n$ is odd, the calculations are often cumbersome and the representations that we obtain, although expressed in terms of units, are frequently more complicated than we would like. Thus, we establish three simple lemmas that provide an alternative procedure for calculating all of Ramanujan's singular moduli for odd $n$.

Ramanathan [7] and J.M. and P.B. Borwein [2] previously determined some of Ramanujan's values for $\alpha_n$.

Ramanujan calculated over 100 class invariants including 48 that had not been heretofore determined. However, the values of all the invariants used in this paper can be found in Weber's treatise [12] as well as in Ramanujan's notebooks [8].

In all cases, Ramanujan simply recorded in his notebooks the values or factors of singular moduli without explaining their meanings. It took us several years to discover that these radical expressions were singular moduli.

## 2. Singular moduli for even *n*

We begin with a list of thirteen values for $\alpha_n$ found on scattered pages in the first notebook.

**Theorem 2.1 ([8, vol. 1, pp. 214, 288, 289, 310, 312, 313]).** *We have*

$$\begin{aligned}
\alpha_2 &= \left(\sqrt{2}-1\right)^2,\\
\alpha_6 &= \left(2-\sqrt{3}\right)^2\left(\sqrt{3}-\sqrt{2}\right)^2 = \frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{6}+\sqrt{2}+1} = \left(\frac{\sqrt{6}-\sqrt{2}-1}{\sqrt{2}-1}\right)^2,\\
\alpha_{10} &= \left(\sqrt{10}-3\right)^2\left(3-2\sqrt{2}\right)^2 = \frac{3\sqrt{2}-\sqrt{5}-2}{3\sqrt{2}+\sqrt{5}+2},\\
\alpha_{18} &= \left(5\sqrt{2}-7\right)^2\left(7-4\sqrt{3}\right)^2 = \left(2-\sqrt{3}\right)^4\left(\sqrt{2}-1\right)^6 = \frac{7\sqrt{2}-2\sqrt{6}-5}{7\sqrt{2}+2\sqrt{6}+5},\\
\alpha_{22} &= \left(10-3\sqrt{11}\right)^2\left(3\sqrt{11}-7\sqrt{2}\right)^2,\\
\alpha_{30} &= \left(5-2\sqrt{6}\right)^2\left(4-\sqrt{15}\right)^2\left(\sqrt{6}-\sqrt{5}\right)^2\left(2-\sqrt{3}\right)^2,\\
\alpha_{42} &= \left(8-3\sqrt{7}\right)^2\left(7-4\sqrt{3}\right)^2\left(3-2\sqrt{2}\right)^2\left(\sqrt{7}-\sqrt{6}\right)^2,\\
\alpha_{58} &= \left(13\sqrt{58}-99\right)^2\left(99-70\sqrt{2}\right)^2,\\
\alpha_{70} &= \left(15-4\sqrt{14}\right)^2\left(8-3\sqrt{7}\right)^2\left(3\sqrt{14}-5\sqrt{5}\right)^2\left(6-\sqrt{35}\right)^2,\\
\alpha_{78} &= \left(2-\sqrt{3}\right)^6\left(3\sqrt{3}-\sqrt{26}\right)^2\left(\sqrt{13}-2\sqrt{3}\right)^4\left(5-2\sqrt{6}\right)^2,\\
\alpha_{102} &= \left(\frac{\sqrt{51}-7}{\sqrt{2}}\right)^4\left(5-2\sqrt{6}\right)^4\left(\sqrt{51}-5\sqrt{2}\right)^2\left(2-\sqrt{3}\right)^4,\\
\alpha_{130} &= \left(5\sqrt{130}-57\right)^2\left(\sqrt{10}-3\right)^4\left(\sqrt{26}-5\right)^4\left(3-2\sqrt{2}\right)^4,
\end{aligned}$$

*and*

$$\alpha_{190} = \left(\frac{3\sqrt{19}-13}{\sqrt{2}}\right)^4\left(37\sqrt{19}-51\sqrt{10}\right)^2\left(2\sqrt{5}-\sqrt{19}\right)^4\left(\sqrt{19}-3\sqrt{2}\right)^4.$$

**Proof:** The value of $\alpha_2$ was, in fact, established in Example 1, Section 2 of Chapter 17 in the second notebook [1, p. 97]. Ramanathan [7] and the Borweins [2, p. 139] also determined $\alpha_2$.

All of the remaining values for $\alpha_n$ are easily determined from Theorem 1.2. The required values for $g_n$ can be obtained from the tables of Weber [12, pp. 721–725]. In each instance, we list the values for $u$, $v$, $U$, $V$, $W$, and $S$ in the table below. The reader can easily verify the calculations.

| $n$ | $u$ | $v$ | $U$ | $V$ | $W$ | $S$ |
|---|---|---|---|---|---|---|
| 6 | 1 | $1+\sqrt{2}$ | 1 | 3 | 3 | 4 |
| 10 | 1 | $2+\sqrt{5}$ | 1 | 9 | 9 | 10 |
| 18 | 1 | $5+2\sqrt{6}$ | 1 | 49 | 49 | 50 |
| 22 | 1 | $7+5\sqrt{2}$ | 1 | 99 | 99 | 100 |
| 30 | $2+\sqrt{5}$ | $3+\sqrt{10}$ | 9 | 19 | 21 | 25 |
| 42 | $2\sqrt{2}+\sqrt{7}$ | $3\sqrt{3}+2\sqrt{7}$ | 15 | 55 | 57 | 64 |
| 58 | 1 | $70+13\sqrt{29}$ | 1 | 9801 | 9801 | 9802 |
| 70 | $9+4\sqrt{5}$ | $7+5\sqrt{2}$ | 161 | 99 | 189 | 225 |
| 78 | $18+5\sqrt{13}$ | $5+\sqrt{26}$ | 649 | 51 | 651 | 676 |
| 102 | $7+5\sqrt{2}$ | $35+6\sqrt{34}$ | 99 | 2449 | 2451 | 2500 |
| 130 | $38+17\sqrt{5}$ | $18+5\sqrt{13}$ | 2889 | 649 | 2961 | 3250 |
| 190 | $38+17\sqrt{5}$ | $117+37\sqrt{10}$ | 2889 | 27,379 | 27,531 | 28,900 |

The second and third formulas for $\alpha_6$ and $\alpha_{18}$, and the second formula for $\alpha_{10}$ can be easily verified by direct calculations.

The Borweins [2, p. 139] calculated $\alpha_n$ for $1 \le n \le 9$. Ramanathan [7] also established $\alpha_{130}$ by using Theorem 1.2. □

Recall the definition of $F(x)$ given in [1, pp. 91, 102]. If ${}_2F_1(a, b; c; x)$ denotes the ordinary hypergeometric function, then

$$F(x) = \exp\left( -\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)} \right), \quad 0 < x < 1.$$

**Theorem 2.2 ([8, vol. 1, p. 82]).** *If* $p > 0$, $n \ge 1$, *and*

$$e^{-\pi\sqrt{p}} = F\left( \frac{1 - \sqrt{1 - 1/n^2}}{2} \right), \tag{2.1}$$

*then*

$$e^{-2\pi\sqrt{p}} = F\left( \left(\sqrt{n+1} - \sqrt{n}\right)^4 \left(\sqrt{n} - \sqrt{n-1}\right)^4 \right). \tag{2.2}$$

From (1.2) and (2.1), $n = G_p^{12}$. Hence, in Theorem 2.2 Ramanujan provides an algorithm for determining $\alpha_{4p}$ from the value of $\alpha_p$, or from $G_p$, namely,

$$\alpha_{4p} = \left( \sqrt{G_p^{12} + 1} - \sqrt{G_p^{12}} \right)^4 \left( \sqrt{G_p^{12}} - \sqrt{G_p^{12} - 1} \right)^4. \tag{2.3}$$

Before proving Theorem 2.2, we verify four examples recorded by Ramanujan.

*Examples 2.3 ([8, vol. 1, p. 82]).* We have

$$\alpha_4 = \left(\sqrt{2}-1\right)^4,$$
$$\alpha_{12} = \left(\sqrt{3}-\sqrt{2}\right)^4\left(\sqrt{2}-1\right)^4,$$
$$\alpha_{28} = \left(\sqrt{2}-1\right)^8\left(2\sqrt{2}-\sqrt{7}\right)^4,$$

and

$$\alpha_{60} = \left(\sqrt{10}-3\right)^4\left(\sqrt{2}-1\right)^4\left(\sqrt{6}-\sqrt{5}\right)^4\left(\sqrt{3}-\sqrt{2}\right)^4.$$

The value of $\alpha_4$ was also recorded in the second notebook [1, p. 97]. Both $\alpha_4$ and $\alpha_{28}$ were also determined by Ramanathan [7], and the Borwein brothers have determined $\alpha_4$ and $\alpha_{12}$ [2, pp. 139, 151].

**Proof:** Let $p=1$, so that trivially $G_1=1$. Then from (2.3),

$$\alpha_4 = \left(\sqrt{2}-1\right)^4(1-0)^4 = \left(\sqrt{2}-1\right)^4.$$

Let $p=3$, so that, from Weber's tables [12, p. 721], $G_3 = 2^{1/12}$ and $n=2$. Thus, from (2.3),

$$\alpha_{12} = \left(\sqrt{3}-\sqrt{2}\right)^4\left(\sqrt{2}-1\right)^4.$$

Let $p=7$, so that, from Weber's tables [12, p. 721], $G_7 = 2^{1/4}$ and $n=8$. Thus, from (2.3),

$$\alpha_{28} = \left(3-2\sqrt{2}\right)^4\left(2\sqrt{2}-\sqrt{7}\right)^4 = \left(\sqrt{2}-1\right)^8\left(2\sqrt{2}-\sqrt{7}\right)^4.$$

Let $p=15$. From Weber's treatise [12, p. 721], $G_{15} = 2^{-1/12}(1+\sqrt{5})^{1/3}$. Thus, $n=(1+\sqrt{5})^4/2 = 4(7+3\sqrt{5})$. Hence, from (2.3),

$$\alpha_{60} = \left(\sqrt{29+12\sqrt{5}} - 2\sqrt{7+3\sqrt{5}}\right)^4\left(2\sqrt{7+3\sqrt{5}} - \sqrt{27+12\sqrt{5}}\right)^4.$$

To denest these radicals, we employ the following denesting theorem [6]. If $a^2-qb^2=d^2$, a perfect square, then

$$\sqrt{a+b\sqrt{q}} = \sqrt{\frac{a+d}{2}} + (\operatorname{sgn} b)\sqrt{\frac{a-d}{2}}, \tag{2.4}$$

where we have corrected a misprint. To that end, from (2.4),

$$\begin{aligned}\alpha_{60} &= \left(\sqrt{\frac{29+11}{2}} + \sqrt{\frac{29-11}{2}} - 2\sqrt{\frac{7+2}{2}} - 2\sqrt{\frac{7-2}{2}}\right)^4 \\ &\quad\times\left(2\sqrt{\frac{7+2}{2}} + 2\sqrt{\frac{7-2}{2}} - \sqrt{\frac{27+3}{2}} - \sqrt{\frac{27-3}{2}}\right)^4\end{aligned}$$

$$= \left(\sqrt{20}+3-3\sqrt{2}-\sqrt{10}\right)^4\left(3\sqrt{2}+\sqrt{10}-\sqrt{15}-\sqrt{12}\right)^4$$
$$= \left\{\left(\sqrt{10}-3\right)\left(\sqrt{2}-1\right)\right\}^4\left\{\left(\sqrt{6}-\sqrt{5}\right)\left(\sqrt{3}-\sqrt{2}\right)\right\}^4. \qquad \square$$

**Proof of Theorem 2.2:** From [1, p. 215, Eq. (24.21)], we find that

$$\beta = \frac{1}{\alpha^2}\left(1-\sqrt{1-\alpha}\right)^4, \tag{2.5}$$

where $\beta$ has degree 2 over $\alpha$. Note that, by (2.2), we are required to show that

$$\beta = \left(\sqrt{n-1}-\sqrt{n}\right)^4\left(\sqrt{n+1}-\sqrt{n}\right)^4. \tag{2.6}$$

By (2.1), with $n := G_p^{12}$ and $\alpha := \alpha_p$,

$$\alpha = \frac{n-\sqrt{n^2-1}}{2n},$$

which implies that

$$\alpha^{-1} = \left(\sqrt{n(n+1)}+\sqrt{n(n-1)}\right)^2. \tag{2.7}$$

Since $4\alpha(1-\alpha) = n^{-2}$, we deduce that

$$\frac{1}{1-\alpha} = \left(\sqrt{n(n+1)}-\sqrt{n(n-1)}\right)^2. \tag{2.8}$$

Hence, from (2.5), (2.7), and (2.8),

$$\beta = \left(\sqrt{n(n+1)}+\sqrt{n(n-1)}\right)^4\left(1-\frac{1}{\sqrt{n(n+1)}-\sqrt{n(n-1)}}\right)^4$$
$$= \left(\sqrt{n+1}+\sqrt{n-1}\right)^4\left(\sqrt{n}-\frac{\sqrt{n+1}+\sqrt{n-1}}{2}\right)^4$$
$$= \left(\sqrt{n^2+n}+\sqrt{n^2-n}-n-\sqrt{n^2-1}\right)^4$$
$$= \left(\sqrt{n-1}-\sqrt{n}\right)^4\left(\sqrt{n+1}-\sqrt{n}\right)^4,$$

and so (2.6) has been shown. $\square$

**Theorem 2.4 ([8, vol. 1, p. 82]).** *Under the same hypotheses as Theorem* 2.2,

$$e^{-4\pi\sqrt{p}} = F\Bigg(\left(\sqrt{n+1}+\sqrt{n}\right)^8\left\{\sqrt{2n}+1-\sqrt{2\sqrt{n}\left(\sqrt{n+1}+\sqrt{2}\right)}\right\}^4$$
$$\times\left\{\sqrt{2n}-1-\sqrt{2\sqrt{n}\left(\sqrt{n+1}-\sqrt{2}\right)}\right\}^4\Bigg). \tag{2.9}$$

Thus, together Theorems 2.2 and 2.4 yield the formula

$$\alpha_{16p} = \left(\sqrt{G_p^{12}+1}+\sqrt{G_p^{12}}\right)^8 \left\{\sqrt{2G_p^{12}+1-\sqrt{2\sqrt{G_p^{12}}\left(\sqrt{G_p^{12}+1}+\sqrt{2}\right)}}\right\}^4 \times \left\{\sqrt{2G_p^{12}-1-\sqrt{2\sqrt{G_p^{12}}\left(\sqrt{G_p^{12}+1}-\sqrt{2}\right)}}\right\}^4.$$

For example, if $p = 1$, then $n = G_1 = 1$, and a simple calculation shows that

$$\alpha_{16} = \left(\sqrt{2}+1\right)^4 (2^{1/4}-1)^8.$$

**Proof:** From Theorem 2.2,

$$\begin{aligned} e^{-4\pi\sqrt{p}} &= F^2\left(\left(\sqrt{n+1}-\sqrt{n}\right)^4\left(\sqrt{n}-\sqrt{n-1}\right)^4\right) \\ &=: F^2\left(\frac{4x}{(1+x)^2}\right). \end{aligned} \tag{2.10}$$

Also, from Entry 2(v) of Chapter 17 in the second notebook [1, p. 93], for $0 < x < 1$,

$$F(x^2) = F^2\left(\frac{4x}{(1+x)^2}\right). \tag{2.11}$$

Thus, from (2.11), (2.9), and (2.10), it suffices to show that

$$x = \left(\sqrt{n+1}+\sqrt{n}\right)^4 \left\{\sqrt{2n+1-\sqrt{2\sqrt{n}\left(\sqrt{n+1}+\sqrt{2}\right)}}\right\}^2 \times \left\{\sqrt{2n-1-\sqrt{2\sqrt{n}\left(\sqrt{n+1}-\sqrt{2}\right)}}\right\}^2. \tag{2.12}$$

From (2.10), it follows that

$$\frac{2}{\left(\sqrt{n+1}-\sqrt{n}\right)^2\left(\sqrt{n}-\sqrt{n-1}\right)^2} = \frac{1}{\sqrt{x}}+\sqrt{x}. \tag{2.13}$$

Let $u = \sqrt{x}$. Since $u$ tends to 0 as $n$ tends to $\infty$, the solution of (2.13) that we seek is

$$\begin{aligned} u &= \left(\sqrt{\frac{1}{2\left(\sqrt{n+1}-\sqrt{n}\right)^2\left(\sqrt{n}-\sqrt{n-1}\right)^2}+\frac{1}{2}} - \sqrt{\frac{1}{2\left(\sqrt{n+1}-\sqrt{n}\right)^2\left(\sqrt{n}-\sqrt{n-1}\right)^2}-\frac{1}{2}}\right)^2 \\ &= \frac{1}{2}\left(\sqrt{n+1}+\sqrt{n}\right)^2\left(\sqrt{\left(\sqrt{n}+\sqrt{n-1}\right)^2+\left(\sqrt{n+1}-\sqrt{n}\right)^2} - \sqrt{\left(\sqrt{n}+\sqrt{n-1}\right)^2-\left(\sqrt{n+1}-\sqrt{n}\right)^2}\right)^2 \end{aligned}$$

$$= \left(\sqrt{n+1}+\sqrt{n}\right)^2\Big(\sqrt{2n+\sqrt{n(n-1)}-\sqrt{n(n+1)}} \\ -\sqrt{\sqrt{n(n-1)}+\sqrt{n(n+1)}-1}\Big)^2$$

$$= \left(\sqrt{n+1}+\sqrt{n}\right)^2\Big(2n-1+2\sqrt{n(n-1)} \\ -2\sqrt{(2n+1)\sqrt{n(n+1)}+(2n-1)\sqrt{n(n-1)}-4n}\Big). \tag{2.14}$$

Comparing the proposed value of $u$ from (2.12) with that of (2.14) above, we see that it suffices to show that

$$2\sqrt{(2n+1)\sqrt{n(n+1)}+(2n-1)\sqrt{n(n-1)}-4n} \\ = \left(\sqrt{2n}+1\right)\sqrt{2\sqrt{n}\left(\sqrt{n+1}-\sqrt{2}\right)}+\left(\sqrt{2n}-1\right)\sqrt{2\sqrt{n}\left(\sqrt{n+1}+\sqrt{2}\right)}. \tag{2.15}$$

If we square both sides of (2.15), it is a routine matter to show that (2.15) indeed is a correct equality. This therefore completes the proof. □

The next theorem enables one to determine $\alpha_{8p}$ from the value of $\alpha_{2p}$.

**Theorem 2.5 ([8, vol. 1, p. 82]).** *If* $n \geq 1,\ p > 0$, *and*

$$e^{-\pi\sqrt{2p}} = F\left(\left(\sqrt{n+1}-\sqrt{n}\right)^2\left(\sqrt{n}-\sqrt{n-1}\right)^2\right), \tag{2.16}$$

*then*

$$e^{-2\pi\sqrt{2p}} = F\left(\left\{\frac{\sqrt{n}+1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{\left(\sqrt{n}+1\right)\left(\sqrt{n}+\sqrt{n+1}\right)}\right\}^4 \\ \times\left\{\frac{\sqrt{n}-1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{\left(\sqrt{n}-1\right)\left(\sqrt{n}+\sqrt{n+1}\right)}\right\}^4\right). \tag{2.17}$$

Observe that

$$\left\{\frac{\sqrt{n}\pm1+\sqrt{n+1}}{\sqrt{2}}-\sqrt{\left(\sqrt{n}\pm1\right)\left(\sqrt{n}+\sqrt{n+1}\right)}\right\} \\ \times\left\{\frac{\sqrt{n}\pm1+\sqrt{n+1}}{\sqrt{2}}+\sqrt{\left(\sqrt{n}\pm1\right)\left(\sqrt{n}+\sqrt{n+1}\right)}\right\} = 1. \tag{2.18}$$

Thus, if $\alpha_{2p}$ can be expressed as a product of units of the form $(\sqrt{n+1}-\sqrt{n})^2(\sqrt{n}-\sqrt{n-1})^2$, then $\alpha_{8p}$ can be expressed as a product of two fourth powers of units. Before proving Theorem 2.5, we present three examples recorded by Ramanujan.

*Examples 2.6 ([8, vol. 1, p. 82]).* We have

$$\alpha_8 = \left(\sqrt{3+2\sqrt{2}} - \sqrt{2+2\sqrt{2}}\right)^4,$$
$$\alpha_{24} = \left(\sqrt{6+3\sqrt{3}} - \sqrt{5+3\sqrt{3}}\right)^4 \left(\sqrt{2+\sqrt{3}} - \sqrt{1+\sqrt{3}}\right)^4,$$

and

$$\alpha_{40} = \left(2\sqrt{2}+\sqrt{5} - 2\sqrt{3+\sqrt{10}}\right)^4 \left(\sqrt{2}+\sqrt{5} - \sqrt{6+2\sqrt{10}}\right)^4.$$

**Proof:** Let $p = 1$. Then from Theorem 2.1, $\alpha_2 = (\sqrt{2}-1)^2$. Thus, $n = 1$, and from Theorem 2.5,

$$\alpha_8 = \left(\sqrt{2}+1-\sqrt{2\left(1+\sqrt{2}\right)}\right)^4. \tag{2.19}$$

But from (2.4),

$$\sqrt{3+2\sqrt{2}} = \sqrt{\frac{3+1}{2}} + \sqrt{\frac{3-1}{2}} = \sqrt{2}+1.$$

Using this in (2.19), we achieve the desired representation of $\alpha_8$.

Let $p = 3$. From Theorem 2.1, $\alpha_6 = (2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2$. Thus, $n = 3$, and from Theorem 2.5,

$$\alpha_{24} = \left(\frac{3+\sqrt{3}}{\sqrt{2}} - \sqrt{5+3\sqrt{3}}\right)^4 \left(\frac{1+\sqrt{3}}{\sqrt{2}} - \sqrt{1+\sqrt{3}}\right)^4. \tag{2.20}$$

But from (2.4),

$$\sqrt{6+3\sqrt{3}} = \sqrt{\frac{6+3}{2}} + \sqrt{\frac{6-3}{2}} = \frac{3+\sqrt{3}}{\sqrt{2}}$$

and

$$\sqrt{2+\sqrt{3}} = \sqrt{\frac{2+1}{2}} + \sqrt{\frac{2-1}{2}} = \frac{\sqrt{3}+1}{\sqrt{2}}.$$

Using these calculations in (2.20), we complete the verification of Ramanujan's representation for $\alpha_{24}$.

Let $p = 5$. From Theorem 2.1, $\alpha_{10} = (\sqrt{10}-3)^2(3-2\sqrt{2})^2$. Thus, $n = 9$, and from Theorem 2.5,

$$\alpha_{40} = \left(2\sqrt{2}+\sqrt{5}-\sqrt{4\left(3+\sqrt{10}\right)}\right)^4 \left(\sqrt{2}+\sqrt{5}-\sqrt{2\left(3+\sqrt{10}\right)}\right)^4,$$

which is what is claimed. □

**Proof of Theorem 2.5:** From (2.16),

$$\begin{aligned} e^{-2\pi\sqrt{2p}} &= F^2\left(\left(\sqrt{n+1}-\sqrt{n}\right)^2\left(\sqrt{n}-\sqrt{n-1}\right)^2\right) \\ &=: F^2\left(\frac{4x}{(1+x)^2}\right). \end{aligned}$$

Hence, as in (2.13),

$$\frac{2}{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n}-\sqrt{n-1}\right)} = \frac{1}{\sqrt{x}}+\sqrt{x}, \tag{2.21}$$

and, by (2.17) and (2.5), it suffices to prove that

$$\begin{aligned} x = &\left\{\frac{\sqrt{n}+1+\sqrt{n+1}}{\sqrt{2}} - \sqrt{\left(\sqrt{n}+1\right)\left(\sqrt{n}+\sqrt{n+1}\right)}\right\}^2 \\ &\times\left\{\frac{\sqrt{n}-1+\sqrt{n+1}}{\sqrt{2}} - \sqrt{\left(\sqrt{n}-1\right)\left(\sqrt{n}+\sqrt{n+1}\right)}\right\}^2. \end{aligned} \tag{2.22}$$

Let $u=\sqrt{x}$. In view of the form (2.21), it is natural to assume that

$$u = (a_1-b_1)(a_2-b_2),$$

where

$$a_1^2-b_1^2 = 1 = a_2^2-b_2^2. \tag{2.23}$$

Then, by (2.21),

$$\left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n}+\sqrt{n-1}\right) = \frac{1}{2}\left(u+\frac{1}{u}\right) = a_1a_2+b_1b_2. \tag{2.24}$$

If $s:=\sqrt{n}+\sqrt{n+1}$, the values of $a_1, b_1, a_2,$ and $b_2$ that satisfy (2.23) and (2.24) are

$$a_1 = \frac{s+1}{\sqrt{2}}, \qquad a_2 = \frac{s-1}{\sqrt{2}}, \qquad b_1 = \sqrt{s}\sqrt{\sqrt{n}+1}, \qquad \text{and} \qquad b_2 = \sqrt{s}\sqrt{\sqrt{n}-1}.$$

Then, as already observed in (2.18), (2.23) is satisfied. Furthermore,

$$\begin{aligned} a_1a_2+b_1b_2 &= n+\sqrt{n(n+1)}+\left(\sqrt{n}+\sqrt{n+1}\right)\sqrt{n-1} \\ &= \left(\sqrt{n+1}+\sqrt{n}\right)\left(\sqrt{n}+\sqrt{n-1}\right), \end{aligned}$$

and so (2.24) is satisfied. Hence, (2.22) has been shown, and the proof of Theorem 2.5 is complete. □

## 3. Singular Moduli for odd *n*

From (1.1), we find that, in the notation of Theorem 1.2,

$$2g_n^{12} = 2u^2v^2 = \frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n}. \tag{3.1}$$

By elementary manipulation, we find from the other equality of (1.1) that

$$2iG_n^{12} = \frac{i}{2\alpha_n G_n^{12}} - \frac{2\alpha_n G_n^{12}}{i}. \tag{3.2}$$

If we set

$$(g_n^*)^{12} := iG_n^{12} \qquad \text{and} \qquad \sqrt{\alpha_n^*} := \frac{2\alpha_n G_n^{12}}{i},$$

then (3.2) takes the form

$$2(g_n^*)^{12} = \frac{1}{\sqrt{\alpha_n^*}} - \sqrt{\alpha_n^*}. \tag{3.3}$$

Comparing (3.1) and (3.3), we deduce the following theorem from Theorem 1.2.

**Theorem 3.1.** *Set*

$$(g_n^*)^6 = uv,$$
$$u^2 + 1/u^2 = 2U, \qquad v^2 + 1/v^2 = 2V,$$
$$W = \sqrt{U^2 + V^2 - 1},$$

*and*

$$2S = U + V + W + 1.$$

*Then*

$$\alpha_n^* = \left\{\sqrt{S} - \sqrt{S-1}\right\}^2 \left\{\sqrt{S-U} - \sqrt{S-U-1}\right\}^2 \times \left\{\sqrt{S-V} - \sqrt{S-V-1}\right\}^2 \left\{\sqrt{S-W} - \sqrt{S-W-1}\right\}^2.$$

The next theorem gives the twelve values of $\alpha_n$, when $n$ is odd, that are found in Ramanujan's notebooks. In those instances when two representations are given, the former one is that which is in the notebooks, or that which contains the units provided by Ramanujan in his notebooks. The Borweins [2, pp. 139, 151] calculated $\alpha_n$, for $n = 3, 5, 7, 9,$ and 15, and Ramanathan [7] determined $\alpha_n$, for $n = 3, 7, 9,$ and 15.

**Theorem 3.2.** *We have*

$$\alpha_3 = \frac{2-\sqrt{3}}{4},$$

$$\alpha_5 = \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^3 \left(\sqrt{\frac{3+\sqrt{5}}{4}} - \sqrt{\frac{\sqrt{5}-1}{4}}\right)^4$$

$$= \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^3 \left(\frac{\sqrt{5}+1}{2} - \sqrt{\frac{\sqrt{5}+1}{2}}\right)^2,$$

$$\alpha_7 = \frac{8-3\sqrt{7}}{16},$$

$$\alpha_9 = \frac{1}{2}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4 \left(\sqrt{\frac{3+\sqrt{3}}{4}} - \sqrt{\frac{\sqrt{3}-1}{4}}\right)^8$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)^4 \left(\sqrt{4+2\sqrt{3}} - \sqrt{3+2\sqrt{3}}\right)^2,$$

$$\alpha_{13} = \frac{1}{2}\left(\frac{\sqrt{13}-3}{2}\right)^3 \left(\sqrt{\frac{7+\sqrt{13}}{4}} - \sqrt{\frac{3+\sqrt{13}}{4}}\right)^4$$

$$= \frac{1}{2}\left(\frac{\sqrt{13}-3}{2}\right)^3 \left(\sqrt{\frac{19+5\sqrt{13}}{2}} - \sqrt{\frac{17+5\sqrt{13}}{2}}\right)^2,$$

$$\alpha_{15} = \frac{1}{16}\left(\frac{\sqrt{5}-1}{2}\right)^4 \left(2-\sqrt{3}\right)^2 \left(4-\sqrt{15}\right),$$

$$\alpha_{17} = \frac{1}{2}\left(\sqrt{\frac{7+\sqrt{17}}{4}} - \sqrt{\frac{3+\sqrt{17}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{4+\sqrt{17}}}{4}} - \sqrt{\frac{\sqrt{4+\sqrt{17}}-1}{4}}\right)^8,$$

$$\alpha_{21} = \frac{1}{2}\left(\frac{3-\sqrt{7}}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{7}-\sqrt{3}}{2}\right)^3 \left(\sqrt{\frac{5+\sqrt{7}}{4}} - \sqrt{\frac{1+\sqrt{7}}{4}}\right)^4$$

$$\times \left(\sqrt{\frac{3+\sqrt{7}}{4}} - \sqrt{\frac{\sqrt{7}-1}{4}}\right)^4,$$

$$\alpha_{25} = \frac{1}{2}\left(161-72\sqrt{5}\right)\left(\sqrt{\frac{5+\sqrt{5}}{4}} - \sqrt{\frac{1+\sqrt{5}}{4}}\right)^8,$$

$$\alpha_{33} = \frac{1}{2}(2-\sqrt{3})^3\left(\frac{\sqrt{11}-3}{\sqrt{2}}\right)^2\left(\sqrt{\frac{7+3\sqrt{3}}{4}}-\sqrt{\frac{3+3\sqrt{3}}{4}}\right)^4$$
$$\times\left(\sqrt{\frac{5+\sqrt{3}}{4}}-\sqrt{\frac{1+\sqrt{3}}{4}}\right)^4,$$
$$\alpha_{45} = \frac{1}{2}(\sqrt{5}-2)^3\left(\frac{\sqrt{5}-\sqrt{3}}{\sqrt{2}}\right)^4\left(\sqrt{\frac{7+3\sqrt{5}}{4}}-\sqrt{\frac{3+3\sqrt{5}}{4}}\right)^4$$
$$\times\left(\sqrt{\frac{3+\sqrt{5}}{2}}-\sqrt{\frac{1+\sqrt{5}}{2}}\right)^4,$$

*and*

$$\alpha_{55} = 4(\sqrt{5}-2)^2(10-3\sqrt{11})(3\sqrt{5}-2\sqrt{11})$$
$$\times\left(\sqrt{\frac{7+\sqrt{5}}{8}}-\sqrt{\frac{\sqrt{5}-1}{8}}\right)^{12}\left(\sqrt{\frac{4+\sqrt{5}}{2}}-\sqrt{\frac{2+\sqrt{5}}{2}}\right)^4.$$

**Proof of Theorem 3.2 for $n=3, 5, 7, 9, 13$:** These five values are easily computed by using Theorem 3.1. The required values for $G_n$ may be found in Weber's tables [12, p. 721]. In each instance, we list the values for $u$, $v$, $U$, $V$, $W$, and $S$ in the table below. The reader should easily be able to verify the calculations.

| $n$ | $u$ | $v$ | $U$ | $V$ | $W$ | $S$ |
|---|---|---|---|---|---|---|
| 3 | $\exp(\pi i/4)$ | $\sqrt{2}$ | 0 | $\frac{5}{4}$ | $\frac{3}{4}$ | $\frac{3}{2}$ |
| 5 | $\exp(\pi i/4)$ | $\sqrt{2+\sqrt{5}}$ | 0 | $\sqrt{5}$ | 2 | $\frac{1}{2}(3+\sqrt{5})$ |
| 7 | $\exp(\pi i/4)$ | $2^{3/2}$ | 0 | $\frac{65}{16}$ | $\frac{63}{16}$ | $\frac{9}{2}$ |
| 9 | $\exp(\pi i/4)$ | $2+\sqrt{3}$ | 0 | 7 | $4\sqrt{3}$ | $4+2\sqrt{3}$ |
| 13 | $\exp(\pi i/4)$ | $\sqrt{18+5\sqrt{13}}$ | 0 | $5\sqrt{13}$ | 18 | $\frac{1}{2}(19+5\sqrt{13})$ |

□

Except for $n=55$, we have also used Theorem 3.1 to calculate the remaining values in Theorem 3.2. However, the following lemmas lead to simpler calculations.

**Lemma 3.3.** *If $r$ is any positive real number and $t=\sqrt{(r+1)/8}$, then*

$$r-\sqrt{r^2-1} = \left(\sqrt{t+\frac{1}{2}}-\sqrt{t-\frac{1}{2}}\right)^4. \tag{3.4}$$

**Proof:** The equality (3.4) can be readily verified by elementary algebra. □

**Lemma 3.4.** *If r and t are as given in Lemma* 3.3, *then*

$$r-\sqrt{r^2-1}=\left(\sqrt{\frac{\sqrt{t+\frac{1}{2}}+1}{2}}-\sqrt{\frac{\sqrt{t+\frac{1}{2}}-1}{2}}\right)^8. \tag{3.5}$$

**Proof:** It is readily verified that

$$\left(\sqrt{\frac{\sqrt{t+\frac{1}{2}}+1}{2}}-\sqrt{\frac{\sqrt{t+\frac{1}{2}}-1}{2}}\right)^2=\sqrt{t+\frac{1}{2}}-\sqrt{t-\frac{1}{2}}. \tag{3.6}$$

Using Lemma 3.3 in (3.6), we deduce (3.5).

We frequently set $G = G_n$ below, when the value of $n$ is understood. □

**Proof of Theorem 3.2 for *n*= 5, 9, 13, 15, 17, 25:** Let $n = 5$. From Weber's tables [12, p. 721],

$$G_5^{12}=\left(\frac{\sqrt{5}+1}{2}\right)^3=\sqrt{5}+2. \tag{3.7}$$

If $r = G_5^{12}$ in Lemma 3.3, then

$$t=\sqrt{\frac{3+\sqrt{5}}{8}}=\frac{\sqrt{5}+1}{4}$$

and

$$G^{12}-\sqrt{G^{24}-1}=\left(\sqrt{\frac{\sqrt{5}+3}{4}}-\sqrt{\frac{\sqrt{5}-1}{4}}\right)^4. \tag{3.8}$$

Thus, the given value for $\alpha_5$ follows immediately from (1.2), (3.7), and (3.8).

Let $n = 9$. From Weber's tables [12, p. 721],

$$G_9^{12}=\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)^4=7+4\sqrt{3}. \tag{3.9}$$

Applying Lemma 3.4 with $r = G_9^{12}$, we find that

$$t=\sqrt{\frac{8+4\sqrt{3}}{8}}=\frac{\sqrt{3}+1}{2},$$

$$\sqrt{t+\frac{1}{2}}=\sqrt{\frac{2+\sqrt{3}}{2}}=\frac{1+\sqrt{3}}{2},$$

and

$$G^{12}-\sqrt{G^{24}-1}=\left(\sqrt{\frac{3+\sqrt{3}}{4}}-\sqrt{\frac{\sqrt{3}-1}{4}}\right)^{8}. \tag{3.10}$$

Thus, by (1.2), (3.9), and (3.10), we deduce Ramanujan's value for $\alpha_9$.

Let $n=13$. From Weber's tables [12, p. 721],

$$G_{13}^{12}=\left(\frac{\sqrt{13}+3}{2}\right)^{3}=18+5\sqrt{13}. \tag{3.11}$$

Then in Lemma 3.3, set $r=G_{13}^{12}$ to deduce that

$$t=\sqrt{\frac{19+5\sqrt{13}}{8}}=\frac{5+\sqrt{13}}{4}$$

and

$$G^{12}-\sqrt{G^{24}-1}=\left(\sqrt{\frac{7+\sqrt{13}}{4}}-\sqrt{\frac{3+\sqrt{13}}{4}}\right)^{4}. \tag{3.12}$$

Hence, the given value for $\alpha_{13}$ follows from (1.2), (3.11), and (3.12).

Let $n=15$. From Weber's tables [12, p. 721],

$$G_{15}^{12}=\frac{1}{2}\left(\sqrt{5}+1\right)^{4}=28+12\sqrt{5}. \tag{3.13}$$

Apply Lemma 3.3 with $r=G_{15}^{12}$. Then

$$t=\sqrt{\frac{29+12\sqrt{5}}{8}},$$

and so

$$\begin{aligned}
G^{12}-\sqrt{G^{24}-1}&=\left(\sqrt{\sqrt{\frac{29+12\sqrt{5}}{8}}+\frac{1}{2}}-\sqrt{\sqrt{\frac{29+12\sqrt{5}}{8}}-\frac{1}{2}}\right)^{4}\\
&=\left(\sqrt{\frac{29+12\sqrt{5}}{2}}-\sqrt{\frac{27+12\sqrt{5}}{2}}\right)^{2}\\
&=28+12\sqrt{5}-\sqrt{\left(29+12\sqrt{5}\right)\left(27+12\sqrt{5}\right)}\\
&=28+12\sqrt{5}-16\sqrt{3}-7\sqrt{15}\\
&=\left(2-\sqrt{3}\right)^{2}\left(4-\sqrt{15}\right).
\end{aligned} \tag{3.14}$$

Hence, by (1.2), (3.13), and (3.14), the desired result follows.

Let $n = 17$. From Weber's book [12, p. 721],

$$G_{17}^{12} = \left(\sqrt{\frac{5+\sqrt{17}}{8}} + \sqrt{\frac{\sqrt{17}-3}{8}}\right)^{12} = 20 + 5\sqrt{17} + \sqrt{\left(20+5\sqrt{17}\right)^2 - 1},$$

after a lengthy calculation. We now apply Lemma 3.3 with $r = 20 + 5\sqrt{17}$. Then

$$t = \sqrt{\frac{21+5\sqrt{17}}{8}} = \frac{5+\sqrt{17}}{4},$$

and so

$$G_{17}^{-12} = \left(\sqrt{\frac{7+\sqrt{17}}{4}} - \sqrt{\frac{3+\sqrt{17}}{4}}\right)^{4}. \tag{3.15}$$

Next, set $r = 20 + 5\sqrt{17} + \sqrt{(20+5\sqrt{17})^2 - 1}$ in Lemma 3.4. Then

$$\begin{aligned} t &= \sqrt{\frac{21+5\sqrt{17}+\sqrt{\left(20+5\sqrt{17}\right)^2-1}}{8}} \\ &= \sqrt{\frac{42+10\sqrt{17}+4\sqrt{206+50\sqrt{17}}}{16}} \\ &= \frac{3+\sqrt{17}+2\sqrt{4+\sqrt{17}}}{4} \end{aligned}$$

and

$$\sqrt{t+\frac{1}{2}} = \sqrt{\frac{5+\sqrt{17}+2\sqrt{4+\sqrt{17}}}{4}} = \frac{1+\sqrt{4+\sqrt{17}}}{2}.$$

Thus,

$$G^{12} - \sqrt{G^{24}-1} = \left(\sqrt{\frac{3+\sqrt{4+\sqrt{17}}}{4}} - \sqrt{\frac{\sqrt{4+\sqrt{17}}-1}{4}}\right)^{8}. \tag{3.16}$$

Using (3.15) and (3.16) in (1.2), we complete the proof.

Let $n = 25$. From Weber's treatise [12, p. 722],

$$G_{25}^{12} = \left(\frac{\sqrt{5}+1}{2}\right)^{12} = 161 + 72\sqrt{5}. \tag{3.17}$$

With $r = G_{25}^{12}$ in Lemma 3.4,

$$t = \sqrt{\frac{162 + 72\sqrt{5}}{8}} = \frac{6 + 3\sqrt{5}}{2}$$

and

$$\sqrt{t + \frac{1}{2}} = \sqrt{\frac{7 + 3\sqrt{5}}{2}} = \frac{3 + \sqrt{5}}{2}.$$

Hence,

$$G^{12} - \sqrt{G^{24} - 1} = \left(\sqrt{\frac{5 + \sqrt{5}}{4}} - \sqrt{\frac{1 + \sqrt{5}}{4}}\right)^8 . \tag{3.18}$$

Putting (3.17) and (3.18) in (1.2), we complete the proof. □

The next lemma will enable us to calculate $\alpha_{21}, \alpha_{33}, \alpha_{45}$, and $\alpha_{55}$.

**Lemma 3.5.** *Let $r = uv$, where $v > u$ and $u$ is a unit. Set $u = u_1 + u_2$, where $u_1, u_2, > 0$ and $u_1^2 - u_2^2 = 1$. Furthermore, let*

$$a^2 = 1 + 2vu_1 + v^2 \qquad \textit{and} \qquad b^2 = 1 - 2vu_1 + v^2,$$

*where $a, b > 0$. Then*

$$r - \sqrt{r^2 - 1} = \left(\sqrt{\sqrt{\frac{a+b+2}{16}} + \frac{1}{2}} - \sqrt{\sqrt{\frac{a+b+2}{16}} - \frac{1}{2}}\right)^4 \times \left(\sqrt{\sqrt{\frac{a-b+2}{16}} + \frac{1}{2}} - \sqrt{\sqrt{\frac{a-b+2}{16}} - \frac{1}{2}}\right)^4 . \tag{3.19}$$

**Proof:** The right side of (3.19) equals

$$\begin{aligned}
&\left(\sqrt{\frac{a+b+2}{4}} - \sqrt{\frac{a+b-2}{4}}\right)^2 \left(\sqrt{\frac{a-b+2}{4}} - \sqrt{\frac{a-b-2}{4}}\right)^2 \\
&= \left(\frac{a+b}{2} - \sqrt{\left(\frac{a+b}{2}\right)^2 - 1}\right) \left(\frac{a-b}{2} - \sqrt{\left(\frac{a-b}{2}\right)^2 - 1}\right) \\
&= \left(\frac{a^2-b^2}{4} + \sqrt{\left(\frac{a^2-b^2}{4}\right)^2 - \frac{a^2+b^2}{2} + 1}\right. \\
&\quad \left. - \left(\frac{a+b}{2}\sqrt{\left(\frac{a-b}{2}\right)^2 - 1} + \frac{a-b}{2}\sqrt{\left(\frac{a+b}{2}\right)^2 - 1}\right)\right).
\end{aligned} \tag{3.20}$$

Set

$$r' = \frac{a^2-b^2}{4} + \sqrt{\left(\frac{a^2-b^2}{4}\right)^2 - \frac{a^2+b^2}{2} + 1}.$$

Then, by an elementary calculation,

$$\sqrt{r'^2-1} = \frac{a+b}{2}\sqrt{\left(\frac{a-b}{2}\right)^2 - 1} + \frac{a-b}{2}\sqrt{\left(\frac{a+b}{2}\right)^2 - 1}.$$

Hence, we see that the right side of (3.20) equals $r' - \sqrt{r'^2-1}$, and it therefore remains to show that $r = r'$.

In fact, from the definitions of $a^2$ and $b^2$,

$$\begin{aligned} r' &= vu_1 + \sqrt{v^2u_1^2 + 1 - (1+v^2)} \\ &= vu_1 + v\sqrt{u_1^2 - 1} \\ &= vu_1 + vu_2 = vu = r, \end{aligned}$$

which completes the proof. □

**Proof of Theorem 3.2 for $n = 21, 33, 45, 55$:** Let $n = 21$. From Weber's treatise [12, p. 722],

$$r := G_{21}^{12} = \left(\frac{3+\sqrt{7}}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^3 = \left(8+3\sqrt{7}\right)\left(2\sqrt{7}+3\sqrt{3}\right). \tag{3.21}$$

Set $u_1 = 2\sqrt{7}$ and $v = 8+3\sqrt{7}$ in Lemma 3.5. Then

$$a^2 = 1 + 2\left(8+3\sqrt{7}\right)2\sqrt{7} + \left(8+3\sqrt{7}\right)^2 = 212 + 80\sqrt{7} = \left(10+4\sqrt{7}\right)^2$$

and

$$b^2 = 1 - 2\left(8+3\sqrt{7}\right)2\sqrt{7} + \left(8+3\sqrt{7}\right)^2 = 44 + 16\sqrt{7} = \left(4+2\sqrt{7}\right)^2.$$

Hence, $a = 10+4\sqrt{7}$ and $b = 4+2\sqrt{7}$. Moreover,

$$\sqrt{\frac{a+b+2}{16}} = \sqrt{\frac{16+6\sqrt{7}}{16}} = \frac{3+\sqrt{7}}{4}$$

and

$$\sqrt{\frac{a-b+2}{16}} = \sqrt{\frac{8+2\sqrt{7}}{16}} = \frac{1+\sqrt{7}}{4}.$$

Thus, by Lemma 3.5,

$$G^{12}-\sqrt{G^{24}-1}=\left(\sqrt{\frac{5+\sqrt{7}}{4}}-\sqrt{\frac{1+\sqrt{7}}{4}}\right)^4\left(\sqrt{\frac{3+\sqrt{7}}{4}}-\sqrt{\frac{\sqrt{7}-1}{4}}\right)^4. \quad (3.22)$$

On using (3.21) and (3.22) in (1.2), we complete the proof.

Let $n=33$. From Weber's treatise [12, p. 722],

$$G_{33}^{12}=\left(2+\sqrt{3}\right)^3\left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)^2=\left(26+15\sqrt{3}\right)\left(10+3\sqrt{11}\right). \quad (3.23)$$

Apply Lemma 3.5 with $u_1=10$ and $v=26+15\sqrt{3}$. Then

$$a^2=1+2\left(26+15\sqrt{3}\right)10+\left(26+15\sqrt{3}\right)^2=1872+1080\sqrt{3}=\left(30+18\sqrt{3}\right)^2$$

and

$$b^2=1-2\left(26+15\sqrt{3}\right)10+\left(26+15\sqrt{3}\right)^2=832+480\sqrt{3}=\left(20+12\sqrt{3}\right)^2.$$

Thus, $a=30+18\sqrt{3}$ and $b=20+12\sqrt{3}$, so that

$$\sqrt{\frac{a+b+2}{16}}=\sqrt{\frac{52+30\sqrt{3}}{16}}=\frac{5+3\sqrt{3}}{4}$$

and

$$\sqrt{\frac{a-b+2}{16}}=\sqrt{\frac{12+6\sqrt{3}}{16}}=\frac{3+\sqrt{3}}{4}.$$

Thus, by Lemma 3.5,

$$G^{12}-\sqrt{G^{24}-1}=\left(\sqrt{\frac{7+3\sqrt{3}}{4}}-\sqrt{\frac{3+3\sqrt{3}}{4}}\right)^4\left(\sqrt{\frac{5+\sqrt{3}}{4}}-\sqrt{\frac{1+\sqrt{3}}{4}}\right)^4. \quad (3.24)$$

Upon substituting (3.23) and (3.24) in (1.2), we complete the proof.

Let $n=45$. From Weber's book [12, p. 723],

$$G_{45}^{12}=\left(\sqrt{5}+2\right)^3\left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}\right)^4=\left(38+17\sqrt{5}\right)\left(31+8\sqrt{15}\right). \quad (3.25)$$

We apply Lemma 3.5 with $u_1=31$ and $v=38+17\sqrt{5}$. Thus,

$$a^2=1+2\left(38+17\sqrt{5}\right)31+\left(38+17\sqrt{5}\right)^2=5246+2346\sqrt{5}=\left(51+23\sqrt{5}\right)^2$$

and

$$b^2 = 1 - 2\big(38 + 17\sqrt{5}\big)31 + \big(38 + 17\sqrt{5}\big)^2 = 534 + 238\sqrt{5} = \big(17 + 7\sqrt{5}\big)^2.$$

Hence, $a = 51 + 23\sqrt{5}$ and $b = 17 + 7\sqrt{5}$, so that

$$\sqrt{\frac{a+b+2}{16}} = \sqrt{\frac{70+30\sqrt{5}}{16}} = \frac{5+3\sqrt{5}}{4}$$

and

$$\sqrt{\frac{a-b+2}{16}} = \sqrt{\frac{36+16\sqrt{5}}{16}} = \frac{2+\sqrt{5}}{2}.$$

So, by Lemma 3.5,

$$G^{12} - \sqrt{G^{24}-1} = \left(\sqrt{\frac{7+3\sqrt{5}}{4}} - \sqrt{\frac{3+3\sqrt{5}}{4}}\right)^4 \left(\sqrt{\frac{3+\sqrt{5}}{2}} - \sqrt{\frac{1+\sqrt{5}}{2}}\right)^4. \tag{3.26}$$

The desired evaluation now immediately follows from (1.2), (3.25), and (3.26).

Let $n = 55$. From Weber's treatise [12, p. 723],

$$\begin{aligned} G_{55}^{12} &= 8\big(\sqrt{5}+2\big)^2 \left(\sqrt{\frac{7+\sqrt{5}}{8}} + \sqrt{\frac{\sqrt{5}-1}{8}}\right)^{12} \\ &= 8\big(\sqrt{5}+2\big)^2 \left(\frac{99+45\sqrt{5}}{4} + \sqrt{\left(\frac{99+45\sqrt{5}}{4}\right)^2 - 1}\right). \end{aligned} \tag{3.27}$$

Apply Lemma 3.5 with $u_1 = (99 + 45\sqrt{5})/4$ and $v = 8(\sqrt{5}+2)^2$. Thus,

$$\begin{aligned} a^2 &= 1 + 2 \cdot 8\big(\sqrt{5}+2\big)^2 \frac{1}{4}\big(99 + 45\sqrt{5}\big) + 64\big(\sqrt{5}+2\big)^4 \\ &= 17469 + 7812\sqrt{5} = \big(93 + 42\sqrt{5}\big)^2 \end{aligned}$$

and

$$\begin{aligned} b^2 &= 1 - 2 \cdot 8\big(\sqrt{5}+2\big)^2 \frac{1}{4}\big(99 + 45\sqrt{5}\big) + 64\big(\sqrt{5}+2\big)^4 \\ &= 3141 + 1404\sqrt{5} = \big(39 + 18\sqrt{5}\big)^2. \end{aligned}$$

Thus, $a = 93 + 42\sqrt{5}$ and $b = 39 + 18\sqrt{5}$, so that

$$\sqrt{\frac{a+b+2}{16}} = \sqrt{\frac{67+30\sqrt{5}}{8}}$$

and

$$\sqrt{\frac{a-b+2}{16}} = \sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{3+\sqrt{5}}{2}.$$

Thus, by Lemma 3.5,

$$G^{12} - \sqrt{G^{24}-1} = \left(\sqrt{\sqrt{\frac{67+30\sqrt{5}}{8}}+\frac{1}{2}} - \sqrt{\sqrt{\frac{67+30\sqrt{5}}{8}}-\frac{1}{2}}\right)^4 \times \left(\sqrt{\frac{4+\sqrt{5}}{2}} - \sqrt{\frac{2+\sqrt{5}}{2}}\right)^4. \tag{3.28}$$

Now, by Lemma 3.3,

$$\begin{aligned}\left(\sqrt{\sqrt{\frac{67+30\sqrt{5}}{8}}+\frac{1}{2}} - \sqrt{\sqrt{\frac{67+30\sqrt{5}}{8}}-\frac{1}{2}}\right)^4 &= 66+30\sqrt{5} - \sqrt{\left(66+30\sqrt{5}\right)^2-1} \\ &= 66+30\sqrt{5} - 9\sqrt{55} - 20\sqrt{11} \\ &= \left(10-3\sqrt{11}\right)\left(3\sqrt{5}-2\sqrt{11}\right).\end{aligned} \tag{3.29}$$

Using (3.29) in (3.28) and then (3.27) and (3.28) in (1.2), we complete the proof. □

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