ISRAEL JOURNAL OF MATHEMATICS 103 (1998), 1–16

RAMANUJAN'S MODULAR EQUATIONS AND ATKIN–LEHNER INVOLUTIONS

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ABSTRACT

In this paper, we explain the existence of certain modular equations discovered by S. Ramanujan via function field theory. We will prove some of these modular equations and indicate how new equations analogous to those found in Ramanujan's notebooks can be constructed.

§1. Introduction

Let

$$ {}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, $$

where $(a)_k = (a)(a+1)\cdots(a+k-1)$, and $|z|<1$. We say that the **modulus** β has degree n over the **modulus** α when

$$ (1.1)\qquad \frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)} = n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}. $$

Received February 19, 1995 and in revised form December 18, 1996

A **modular equation of degree** n is a relation between α and β which is induced by (1.1).

In his notebooks [11], S. Ramanujan recorded many modular equations of various degrees without proofs. Most of these modular equations, together with their proofs, can now be found in [1] and [2]. Among these identities, there is a collection of modular equations which can easily be interpreted as relations between products of Dedekind η-functions, where

$$\eta(\tau) := e^{2\pi i\tau/24}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}). \tag{1.2}$$

For example, the modular equation (of degree 3) [1, p. 231, Entry 5 (xii)]

$$Q_1+\frac{1}{Q_1}+2\sqrt{2}\left(P_1-\frac{1}{P_1}\right)=0, \tag{1.3}$$

with

$$P_1=\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} \quad\text{and}\quad Q_1=\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4},$$

is equivalent (via [1, Chapter 17, Entry 12]) to the identity

$$-Q+\frac{1}{Q}=P+\frac{8}{P}, \tag{1.4}$$

where

$$P=\frac{\eta^3(\tau)\eta^3(3\tau)}{\eta^3(2\tau)\eta^3(6\tau)} \quad\text{and}\quad Q=\frac{\eta^6(\tau)\eta^6(6\tau)}{\eta^6(2\tau)\eta^6(3\tau)}.$$

Many modular equations similar to that of (1.3) can be found in [2, Chapter 25] and they are very useful in the evaluations of q-continued fractions [3] and Ramanujan–Weber class invariants [4].

In Section 2, we present results from function field theory and theory of modular forms which are required for the proofs of Ramanujan's modular equations.

In Section 3, we illustrate our method with two of Ramanujan's modular equations, one of which is (1.4). We then proceed to discuss most of the modular equations of this type found in [2, Chapter 25]. We mention here that the proofs of Entries 3.14–3.18 presented in [2, Chapter 25, pp. 236–244] also involve modular forms but they require the knowledge of the modular equations in advance.

In Section 4, we will first discuss a way to construct new modular identities and identify Ramanujan's modular equations as a special case of such construction. We then indicate some significant features in all the equations discussed in Section 3 and proceed to construct new modular equations analogous to those found by Ramanujan.

§2. Results from function field theory and theory of modular forms

A symbol $\pi = \prod_{t\in I} t^{r_t}$ is a **generalized permutation** if I is a finite subset of $\mathbb{N}$ and $r_t \in \mathbb{Z}$. To each generalized permutation π, we associate a function

$$\eta_\pi(\tau) = \prod_{t\in I} \eta(t\tau)^{r_t}.$$

Throughout this paper, we will write $\prod_{t\in I}$ as $\prod$, keeping in mind that our product is taken over a finite set I.

Next, suppose

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | c \equiv 0 \pmod{N} \right\}$$

where N is a positive integer. If $e \| N$, we call the matrix

$$W_e = \begin{pmatrix} ae & b \\ cN & de \end{pmatrix}, \quad a,b,c,d \in \mathbb{Z},\ \det(W_e) = e \tag{2.1}$$

an **Atkin–Lehner involution** of $\Gamma_0(N)$.

Given a generalized permutation $\pi = \prod t^{r_t}$ and an Atkin–Lehner involution W_e of $\Gamma_0(N)$, set

$$\eta_\pi(\tau)\big|_{W_e} := \prod \eta(tW_e\tau)^{r_t}. \tag{2.2}$$

In order to simplify each term in the product of (2.2), we require the following two results.

THEOREM 2.1 ([8, p. 51, Theorem 2]): *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. *Then*

$$\eta(M\tau) = \nu_\eta(M)(c\tau + d)^{1/2}\eta(\tau),$$

where

$$\nu_\eta(M) = \begin{cases} \left(\dfrac{d}{c}\right)^* \exp\left(\dfrac{\pi i}{12}((a+d)c - bd(c^2-1) - 3c)\right) & \text{if } c \text{ is odd,} \\ \left(\dfrac{c}{d}\right)_* \exp\left(\dfrac{\pi i}{12}((a+d)c - bd(c^2-1) + 3d - 3 - 3cd)\right) & \text{if } d \text{ is odd.} \end{cases}$$

Here,

$$\left(\frac{d}{c}\right)^* = \left(\frac{c}{|d|}\right) \quad \text{and} \quad \left(\frac{c}{d}\right)_* = \left(\frac{c}{|d|}\right)(-1)^{(\operatorname{sgn} c-1)(\operatorname{sgn} d-1)/4},$$

where $\left(\frac{\cdot}{\cdot}\right)$ *is the Jacobi symbol.*

From Theorem 2.1, we easily deduce

COROLLARY 2.2: *Let W_e be an Atkin–Lehner involution of $\Gamma_0(N)$. Let $t > 0$ be such that $t|N$. Suppose $\tau \in \mathbb{H}$, where $\mathbb{H}$ is the upper half plane of $\mathbb{C}$. Then*

$$\eta(tW_e\tau) = \eta\left(t\frac{ae\tau + b}{cN\tau + de}\right) = \nu_\eta(M)\left(\frac{cN\tau + de}{\delta}\right)^{1/2}\eta\left(\frac{et}{\delta^2}\tau\right)$$

where $\delta = (e, t)$ and

$$M = \begin{pmatrix} a\delta & bt/\delta \\ cN\delta/et & de/\delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Proof: Simply set

$$M = \begin{pmatrix} a\delta & bt/\delta \\ cN\delta/et & de/\delta \end{pmatrix}$$

in Theorem 2.1.

We will also need the following results in the sequel.

THEOREM 2.3 ([10]): *Let $\pi = \prod t^{r_t}$ be a generalized permutation. Suppose*

(1) $\sum r_t = 0$,
(2) $\sum tr_t \equiv 0 \pmod{24}$, *and*
(3) $\prod t^{|r_t|}$ *is a square.*

Choose N so that

(4) $r_t = 0$ *if* $t \nmid N$, *and*
(5) $\sum \frac{N}{t} r_t \equiv 0 \pmod{24}$.

Then $\eta_\pi(\tau)$ is invariant under $\Gamma_0(N)$.

THEOREM 2.4 ([9]): *Let π be a generalized permutation and let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{Z})$ containing $\Gamma_0(N)$ for some N such that $\Gamma\backslash\mathbb{H}^*$ is of genus 0. Suppose*

(1) $\sum tr_t = -24$,
(2) $\eta_\pi(\tau)$ *is invariant under the action of* Γ,
(3) $\Gamma_\infty = \{\alpha \in \Gamma : \alpha(\infty) = \infty\} = \left\{\begin{pmatrix} \pm 1 & n \\ 0 & \pm 1 \end{pmatrix} \mid n \in \mathbb{Z}\right\}$, *and*
(4) $\tau = \infty$ *is the unique pole of* η_π *among all inequivalent cusps of* Γ.

Then $\eta_\pi(\tau)$ generates the function field of $\Gamma\backslash\mathbb{H}^*$ over $\mathbb{C}$, i.e., η_π is a bijection from $\Gamma\backslash\mathbb{H}^* \longrightarrow \mathbb{C} \cup \{\infty\}$ and every meromorphic function invariant under Γ can be expressed as a rational function of η_π.

THEOREM 2.5 ([6, p. 49, Lemma 3.5]): *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $\pi = \prod t^{r_t}$ be a generalized permutation. Suppose*

$$\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}\begin{pmatrix} \alpha_t & \beta_t \\ 0 & \delta_t \end{pmatrix},$$

where $\alpha_t = (t, c)$ and $\alpha_t \delta_t = t$. Set $h = \mathrm{lcm}_{t \in I}(\delta_t)$. Then

$$\eta_\pi\left(\frac{a\tau + b}{c\tau + d}\right) = C(c\tau + d)^{\sum r_t/2} \exp\left(\frac{\pi i \tau}{12} \sum_t \frac{\alpha_t^2}{t} r_t\right) g(z),$$

where C is a constant and $g(\tau)$ is a holomorphic function of $e^{2\pi i \tau/h}$ in some neighbourhood of $\tau = \infty$.

Since [6] is not readily available, we sketch a proof of Theorem 2.5.

Sketch of proof [6, p. 49]: We have

$$\begin{aligned}\eta_\pi\left(\frac{a\tau + b}{c\tau + d}\right) &= \prod_t \left\{\eta\left(\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}\begin{pmatrix} \alpha_t & \beta_t \\ 0 & \delta_t \end{pmatrix} z\right)\right\}^{r_t} \\ &= C' \prod_t \left\{\left(c_t \begin{pmatrix} \alpha_t & \beta_t \\ 0 & \delta_t \end{pmatrix} z + d_t\right)^{1/2} \eta\left(\begin{pmatrix} \alpha_t & \beta_t \\ 0 & \delta_t \end{pmatrix} z\right)\right\}^{r_t} \\ &= C''(cz + d)^{\sum_t r_t/2} \cdot \prod_t \left\{\eta\left(\frac{\alpha_t z + \beta_t}{\delta_t}\right)\right\}^{r_t} \\ &= C'''(cz + d)^{\sum_t r_t/2} \exp\left(\frac{\pi i z}{12} \sum_t \frac{(t, c)^2}{t} r_t\right) g(z),\end{aligned}$$

where

$$g(z) = \prod_t \left[\prod_{n=1}^{\infty} \left\{1 - \exp\left(2\pi i n \left(\frac{\alpha_t z + \beta_t}{\delta_t}\right)\right)\right\}\right]^{r_t}$$

and each of the C', C'' and C''' can be worked out explicitly. Now, $g(z)$ is a holomorphic function of $e^{2\pi i z/h}$, with $h = \mathrm{lcm}_{t \in I}(\delta_t)$, in some neighbourhood of $z = \infty$. Hence the result.

§3. Ramanujan's modular equations

We begin our discussion with the functions

$$P = \frac{\eta^3(\tau)\eta^3(3\tau)}{\eta^3(2\tau)\eta^3(6\tau)} \quad \text{and} \quad Q = \frac{\eta^6(\tau)\eta^6(6\tau)}{\eta^6(2\tau)\eta^6(3\tau)}.$$

Note that P/Q and PQ are invariant under $\Gamma_0(6)$, by Theorem 2.3. Using Theorem 2.4, we check that P/Q is a generator for the function field of $\Gamma_0(6)\backslash\mathbb{H}^*$.

Let W_3 be the Atkin–Lehner involution $\begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}$. Then by Theorem 2.2, we find that

$$PQ\big|_{W_3} = -\frac{P}{Q}. \tag{3.1}$$

Hence PQ is also a generator for the function field of $\Gamma_0(6)\backslash\mathbb{H}^*$. Therefore,

$$PQ = \frac{a + bP/Q}{c + dP/Q} \tag{3.2}$$

for some $a, b, c, d \in \mathbb{Z}$. By (3.1) and (3.2), we find that

$$-\frac{P}{Q} = \frac{a - bPQ}{c - dPQ},$$

which implies that

$$PQ = \frac{a + cP/Q}{b + dP/Q}. \tag{3.3}$$

Comparing (3.3) with (3.2), we deduce that $b = c$.

Next, observe that

$$PQ = 1 - 9q + O(q^2)$$

and

$$\frac{P}{Q} = \frac{1}{q} + 3 + O(q).$$

From (3.2), we find that

$$(1 - 9q + O(q^2))\left(c + \frac{d}{q} + 3d + O(q)\right) = a + \frac{b}{q} + 3b + O(q),$$

which implies that $b = d$ and $a = -8d$, after comparing coefficients of q^{-1} and the constant term. Hence

$$PQ = \frac{-8 + P/Q}{1 + P/Q},$$

which gives (1.4) after some simple rearrangements.

Using the above procedures, we are able to establish several modular equations of Ramanujan. We shall state these identities with their proofs omitted.

ENTRY 3.1: *If*

$$P = \frac{\eta^2(\tau)}{\eta^2(3\tau)} \quad \text{and} \quad Q = \frac{\eta^2(2\tau)}{\eta^2(6\tau)},$$

then

$$PQ + \frac{9}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.$$

ENTRY 3.2: *If*

$$P=\frac{\eta(2\tau)}{\eta(3\tau)} \quad \text{and} \quad Q=\frac{\eta(\tau)}{\eta(6\tau)},$$

then

$$(PQ)^2-\frac{9}{(PQ)^2}=\left(\frac{Q}{P}\right)^3-8\left(\frac{P}{Q}\right)^3.$$

ENTRY 3.3: *If*

$$P=\frac{\eta^2(\tau)\eta^2(5\tau)}{\eta^2(2\tau)\eta^2(10\tau)} \quad \text{and} \quad Q=\frac{\eta^3(2\tau)\eta^3(5\tau)}{\eta^3(\tau)\eta^3(10\tau)},$$

then

$$Q-\frac{1}{Q}=P+\frac{4}{P}.$$

ENTRY 3.4: *If*

$$P=\frac{\eta(\tau)}{\eta(5\tau)} \quad \text{and} \quad Q=\frac{\eta(2\tau)}{\eta(10\tau)},$$

then

$$PQ+\frac{5}{PQ}=\left(\frac{Q}{P}\right)^3+\left(\frac{P}{Q}\right)^3.$$

ENTRY 3.5: *If*

$$P=\frac{\eta(2\tau)}{\eta(5\tau)} \quad \text{and} \quad Q=\frac{\eta(\tau)}{\eta(10\tau)},$$

then

$$PQ-\frac{5}{PQ}=\left(\frac{Q}{P}\right)^2-4\left(\frac{P}{Q}\right)^2.$$

Remark: The groups involved in the first two and the last three modular equations are $\Gamma_0(6)$ and $\Gamma_0(10)$, respectively. Note that both the Riemann surfaces associated with the groups $\Gamma_0(6)$ and $\Gamma_0(10)$ have genus 0 (see [6, p. 107] or Section 5).

Next, consider the following modular equation of degree 7 [1, Entry 19 (ix)]. The proof of this identity requires more work.

ENTRY 3.6: *If*

$$P=\frac{\eta^3(\tau)\eta^3(7\tau)}{\eta^3(2\tau)\eta^3(14\tau)} \quad \text{and} \quad Q=\frac{\eta^4(2\tau)\eta^4(7\tau)}{\eta^4(\tau)\eta^4(14\tau)},$$

then

$$Q+\frac{1}{Q}=P+\frac{8}{P}+7.$$

Proof: By Theorem 2.3, we check that P and Q are both invariant under $\Gamma_0(14)$.

Let W_7 be the Atkin–Lehner involution $\begin{pmatrix} 7 & 3 \\ 14 & 7 \end{pmatrix}$. By Theorem 2.2, we check that $Q\big|_{W_7} = Q^{-1}$ and $P\big|_{W_7} = P$. Hence $Q + Q^{-1}$ and P are invariant under the group $\Gamma := < \Gamma_0(14), \overline{W}_7 >$, where

$$\overline{W}_e := \begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ cN/\sqrt{e} & d\sqrt{e} \end{pmatrix},$$

since $P|_{W_7} = P|_{\overline{W}_7}$. Moreover, P is a generator for the function field of $\Gamma\backslash\mathbb{H}^*$ (by Theorem 2.4). Hence,

$$Q + Q^{-1} = \frac{n(P)}{d(P)},$$

where $n(P)$ and $d(P)$ are polynomials in P.

Now, the cusps of Γ are (1, 0) and (14, 1), and the values $p_1 = 1/14$ and $p_2 = 0/1$ are both poles of Q. Furthermore, Q has no zero in the upper half plane. It follows that the only poles of $Q + Q^{-1}$ are p_1 and p_2. Since P has a pole at p_1 and a zero at p_2, we conclude that $d(P) = P^e$ for some e. Now, by Theorem 2.5,

$$Q\big|_{\tau=p_2} = cq^{-1/7} + \cdots \tag{3.4}$$

and

$$P\big|_{\tau=p_2} = dq^{1/7} + \cdots, \tag{3.5}$$

where c and d are some constants. From (3.4) and (3.5), we deduce that P has a zero at p_2, $e = 1$, and

$$Q + \frac{1}{Q} = \frac{n(P)}{P}.$$

We will next show that the degree of the polynomial $n(P)$ is 2. Suppose $n(P) = \sum_{i=0}^m b_i P^i$ and let $W_2 = \begin{pmatrix} 8 & 1 \\ 14 & 2 \end{pmatrix}$. Then by Theorem 2.2, we find that $P\big|_{W_2} = 8/P$ and $Q\big|_{W_2} = 1/Q$. Hence, under the transformation of W_2, we find that

$$Q + \frac{1}{Q} = \frac{\sum_{i=0}^m b_i P^i}{P} \tag{3.6}$$

becomes

$$\begin{aligned} Q + \frac{1}{Q} &= \frac{\sum_{i=0}^m b_i (8/P)^i}{8/P} \\ &= \frac{P^2 b_0/8 + b_1 P + 8b_2 + \sum_{i>2} b_i (8/P)^{i-1}}{P}. \end{aligned} \tag{3.7}$$

Again, from (3.4) and (3.5), we conclude that $b_i = 0$ when $i > 2$. From (3.6) and (3.7), we also deduce that $b_0 = 8b_2$.

Next, observe that

$$Q = \frac{1}{q}(1 + 4q + O(q^2)), \qquad Q^{-1} = q(1 - 4q + O(q^2)),$$
$$P = \frac{1}{q}(1 - 3q + O(q^2)), \quad \text{and} \quad P^{-1} = q(1 + 3q + O(q^2)).$$

Comparing the coefficients of q^{-1}, and the constant term after substituting the above expansions into (3.7), we deduce that $b_2 = 1$ and $b_1 = 7$. Hence

$$Q + \frac{1}{Q} = \frac{P^2 + 7P + 8}{P},$$

which is Entry 3.6.

Ramanujan possessed several modular equations similar to that of Entry 3.6. We state these modular equations and briefly mention the groups that are involved.

ENTRY 3.7: *If*

$$P = \frac{\eta(\tau)}{\eta(25\tau)} \quad \text{and} \quad Q = \frac{\eta(2\tau)}{\eta(50\tau)},$$

then

$$PQ + \frac{25}{PQ} = \left(\frac{Q}{P}\right)^3 - 4\left(\frac{Q}{P}\right)^2 - 4\left(\frac{P}{Q}\right)^2 + \left(\frac{P}{Q}\right)^3.$$

ENTRY 3.8: *If*

$$P = \frac{\eta(3\tau)\eta(5\tau)}{\eta(\tau)\eta(15\tau)} \quad \text{and} \quad Q = \frac{\eta(6\tau)\eta(10\tau)}{\eta(2\tau)\eta(30\tau)},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3 + 4.$$

ENTRY 3.9: *If*

$$P = \frac{\eta(6\tau)\eta(5\tau)}{\eta(2\tau)\eta(15\tau)} \quad \text{and} \quad Q = \frac{\eta(3\tau)\eta(10\tau)}{\eta(\tau)\eta(30\tau)},$$

then

$$PQ + \frac{1}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 1.$$

ENTRY 3.10: *If*

$$P = \frac{\eta(\tau)}{\eta(3\tau)} \quad \text{and} \quad Q = \frac{\eta(5\tau)}{\eta(15\tau)},$$

then

$$(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3 - 5.$$

ENTRY 3.11: *If*

$$P = \frac{\eta^3(\tau)}{\eta^3(5\tau)} \quad \text{and} \quad Q = \frac{\eta^3(3\tau)}{\eta^3(15\tau)},$$

then

$$PQ + \frac{125}{PQ} = \left(\frac{Q}{P}\right)^2 - 9\frac{Q}{P} - 9\frac{P}{Q} - \left(\frac{P}{Q}\right)^2.$$

ENTRY 3.12: *If*

$$P = \frac{\eta(3\tau)}{\eta(5\tau)} \quad \text{and} \quad Q = \frac{\eta(\tau)}{\eta(15\tau)},$$

then

$$(PQ)^3 - \frac{125}{(PQ)^3} = \left(\frac{Q}{P}\right)^4 + \left(\frac{Q}{P}\right)^2 - 9\left(\frac{P}{Q}\right)^2 - 81\left(\frac{P}{Q}\right)^4.$$

ENTRY 3.13: *If*

$$P = \frac{\eta(\tau)\eta(2\tau)}{\eta(5\tau)\eta(10\tau)} \quad \text{and} \quad Q = \frac{\eta(3\tau)\eta(6\tau)}{\eta(15\tau)\eta(30\tau)},$$

then

$$PQ + \frac{25}{PQ} = \left(\frac{Q}{P}\right)^2 - 3\frac{Q}{P} - 3\frac{P}{Q} + \left(\frac{P}{Q}\right)^2 - 6.$$

ENTRY 3.14: *If*

$$P = \frac{\eta(\tau)}{\eta(7\tau)} \quad \text{and} \quad Q = \frac{\eta(3\tau)}{\eta(21\tau)},$$

then

$$PQ + \frac{7}{PQ} = \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q}\right)^2 - 3.$$

ENTRY 3.15: *If*

$$P = \frac{\eta(\tau)}{\eta(3\tau)} \quad \text{and} \quad Q = \frac{\eta(7\tau)}{\eta(21\tau)},$$

then

$$(PQ)^3 + \frac{27}{(PQ)^3} = \left(\frac{Q}{P}\right)^4 - 7\left(\frac{Q}{P}\right)^2 + 7\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^4.$$

ENTRY 3.16: *If*

$$P=\frac{\eta(3\tau)}{\eta(7\tau)} \quad \text{and} \quad Q=\frac{\eta(\tau)}{\eta(21\tau)},$$

then

$$\left(\frac{Q}{P}\right)^3-27\left(\frac{P}{Q}\right)^3=(PQ)^2-PQ+\frac{7}{PQ}-\frac{49}{(PQ)^2}.$$

ENTRY 3.17: *If*

$$P=\frac{\eta(\tau)}{\eta(7\tau)} \quad \text{and} \quad Q=\frac{\eta(5\tau)}{\eta(35\tau)},$$

then

$$(PQ)^2-5+\frac{49}{(PQ)^2}=\left(\frac{Q}{P}\right)^3-5\left(\frac{Q}{P}\right)^2-5\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^3.$$

ENTRY 3.18: *If*

$$P=\frac{\eta(\tau)}{\eta(13\tau)} \quad \text{and} \quad Q=\frac{\eta(3\tau)}{\eta(39\tau)},$$

then

$$PQ+\frac{13}{PQ}=\left(\frac{Q}{P}\right)^2-3\frac{Q}{P}-3-3\frac{P}{Q}+\left(\frac{P}{Q}\right)^2.$$

Remarks: The groups involved in the above entries are, respectively,

$$\langle\Gamma_0(50),\overline{W}_{50}\rangle,\quad \langle\Gamma_0(30),\overline{W}_{15}\rangle,\quad \langle\Gamma_0(30),\overline{W}_{30}\rangle,\quad \langle\Gamma_0(15),\overline{W}_{15}\rangle,$$
$$\langle\Gamma_0(15),\overline{W}_{15}\rangle,\quad \langle\Gamma_0(15),\overline{W}_{5}\rangle,\quad \langle\Gamma_0(30),\overline{W}_{30}\rangle,\quad \langle\Gamma_0(21),\overline{W}_{21}\rangle,$$
$$\langle\Gamma_0(21),\overline{W}_{21}\rangle,\quad \langle\Gamma_0(21),\overline{W}_{3}\rangle,\quad \langle\Gamma_0(35),\overline{W}_{35}\rangle,\quad \text{and}\quad \langle\Gamma_0(39),\overline{W}_{39}\rangle.$$

All the Riemann surfaces associated with the above groups have genus 0 (see [6, pp. 107–109] or Section 5).

§4. Constructions of modular equations

Let σ be a modular function with invariance group A such that $g(A\backslash\mathbb{H}^*)\neq 0$. If there exists a group G such that $m:=[G:A]<\infty$, $g(G\backslash\mathbb{H}^*)=0$, and $A\triangleleft G$, then we always have an identity of the form

$$\sum_{i=1}^{m}\sigma\Big|_{g_i}=\frac{n(f)}{d(f)}, \tag{4.1}$$

where $G=\bigcup_{i=1}^{m} g_iA$, $n(x)$ and $d(x)$ are both polynomials in x, and f generates the function field of $G\backslash\mathbb{H}^*$. To determine $d(f)$, we first set

$$\wp:=\left\{\text{poles of }\sum_{i=1}^{m}\sigma\Big|_{g_i}\right\}\setminus\{\text{poles of } f\}.$$

Since by assumption, f is a bijection from $G\backslash\mathbb{H} \to \mathbb{C} \cup \{\infty\}$, we conclude that

$$d(x) = \prod_{p \in \wp} (x - f(p))^{e_p},$$

where

$$e_p = -\frac{\text{order of } (\sum_{i=1}^m \sigma\big|_{g_i}) \text{ at } p}{\text{order of } (f - f(p)) \text{ at } p}.$$

Note that $d(x)$ is defined to be 1 if $\wp = \phi$. The polynomial $n(x)$ can then be determined by comparing the Fourier expansions of $d(f)\sum_{i=1}^m \sigma\big|_{g_i}$ and f at ∞.

We now return to our discussions of Ramanujan's modular equations. If we identify σ, f, G and A accordingly in the modular equations given in Section 3, we find that these identities are all special cases of (4.1). Our computations in these cases, unlike the general situation, are less cumbersome due to the simplicity of the functions involved. For example, in Entry 3.6, our $\sigma = Q$, $A = \Gamma_0(14)$, and $G =< \Gamma_0(14), W_7 >$. The generator $f = P$ is 0 at the pole $p = 0/1$ of $Q + 1/Q$ and $e_p = 1$. Hence, $d(x) = x$. Moreover, using Atkin–Lehner involution W_2, we only need to expand the functions $Q + 1/Q$ and P up to the constant terms in order to determine $n(x)$. In short, Ramanujan's modular equations are special cases of (4.1) where σ and f are products of η-functions. Furthermore, the invariance group A of σ is usually of the form $\Gamma_0(N)$ and G is of the form $< \Gamma_0(N), \overline{W}_e >$, with $e \| N$. These observations are enough for us to construct new modular equations.

First, we search through the table given in [6, pp. 107–109] or Section 5 for a group G such that $G\backslash\mathbb{H}^*$ has genus 0. We then construct two generalized permutations π_1 and π_2 (using the conditions in Theorems 2.3 and 2.4 as a guide) such that the corresponding functions $P = \eta_{\pi_1}$ and $Q = \eta_{\pi_2}$ satisfy either

1. P and Q are invariant under a subgroup A with $[G : A] = 2$, and
2. P is a generator of the function field of $G\backslash\mathbb{H}^*$,

or

1′. $(PQ)^k$ (for some integer k) and Q/P are invariant under a subgroup A with $[G : A] = 2$, and

2′. Q/P is a generator of the function field of $G\backslash\mathbb{H}^*$.

With the above P and Q, we proceed as in the proof of Entry 3.6 to construct our new modular equation.

We illustrate our method with an example. If we search through the table given in [6, pp. 107–109] or Section 5, we find that $G =< \Gamma_0(33), \overline{W}_{11} >$ is a good candidate. With this choice of G, we set $\pi_1 = 3/11$ and $\pi_2 = 1/33$. Hence

$$P = \frac{\eta(3\tau)}{\eta(11\tau)} \quad \text{and} \quad Q = \frac{\eta(\tau)}{\eta(33\tau)}.$$

Since $(PQ)^3$ and Q/P are invariant under $A = \Gamma_0(33)$, by Theorem 2.3, (1′) is satisfied. Furthermore, by Corollary 2.2, we find that Q/P and $(PQ)^3-(11/PQ)^3$ are invariant under G. Using Theorem 2.4, we check that Q/P is a generator for the function field of $G\backslash\mathbb{H}^*$ and hence (2′) is satisfied. Finally, we follow the steps in the proof of Entry 3.6 to deduce

ENTRY 4.1: *If*

$$P = \frac{\eta(3\tau)}{\eta(11\tau)} \quad \textit{and} \quad Q = \frac{\eta(\tau)}{\eta(33\tau)},$$

then

$$\begin{aligned}(PQ)^3 - \frac{11^3}{(PQ)^3} = {} & \left(\frac{Q}{P}\right)^5 - \left(3\frac{P}{Q}\right)^5 + 2\left(\left(\frac{Q}{P}\right)^4 - \left(3\frac{P}{Q}\right)^4\right) \\ & + 3\left(\left(\frac{Q}{P}\right)^3 - \left(3\frac{P}{Q}\right)^3\right) - 8\left(\left(\frac{Q}{P}\right)^2 - \left(3\frac{P}{Q}\right)^2\right) \\ & + 9\left(\frac{Q}{P} - 3\frac{P}{Q}\right).\end{aligned}$$

Remarks:

(1) If we take Γ as one of the groups in the previous section, we will usually end up with Ramanujan's P's and Q's. This is the reason why we have chosen $\Gamma := <\Gamma_0(33), \overline{W}_{11}>$ for the construction of our new modular equation. It is amazing that Ramanujan has discovered most of the "simple" modular equations without the knowledge of function field theory.

(2) A modular equation similar to Entry 4.1 was first discovered by B. C. Berndt, S. Bhargava and F. G. Garvan [5, (7.33)] when they proved a modular equation of degree 11 to alternative base 3. We thank Professor Berndt for bringing our attention to this modular equation. If we use the permutations $\pi_1 = 33/11$ and $\pi_2 = 1/3$, that is,

$$P = \frac{\eta(33\tau)}{\eta(11\tau)} \quad \text{and} \quad Q = \frac{\eta(\tau)}{\eta(3\tau)},$$

we would have a new proof of this modular equation.

§5. Genus zero discrete groups Γ of $\mathrm{SL}_2(\mathbb{R})$ such that $\Gamma_0(N) \subset \Gamma \subset \Gamma_0(N)+$

In this section, we give a list of all genus zero discrete groups Γ of $\mathrm{SL}_2(\mathbb{R})$ such that $\Gamma_0(N) \subset \Gamma \subset \Gamma_0(N)+$, where $\Gamma_0(N)+$ refers to groups generated by $\Gamma_0(N)$

Table 1. Table of genus zero discrete groups Γ of $SL_2(\mathbb{R})$ such that $\Gamma_0(N) \subset \Gamma \subset \Gamma_0(N)+$

1,	2, 2 + 2,
3, 3 + 3,	4, 4 + 4,
5, 5 + 5,	6, 6 + 2, 6 + 3,
	6 + 6, 6 + 2 + 3,
7, 7 + 7,	8, 8 + 8,
9, 9 + 9,	10, 10 + 2, 10 + 5,
	10 + 10, 10 + 2 + 5,
11 + 11,	12, 12 + 4, 12 + 3,
	12 + 12, 12 + 4 + 3,
13, 13 + 13,	14 + 7, 14 + 14, 14 + 2 + 7
15 + 5, 15 + 15, 15 + 3 + 5,	16, 16 + 16,
17 + 17,	18, 18 + 2, 18 + 9,
	18 + 18, 18 + 2 + 9,
19 + 19,	20 + 4, 20 + 20, 20 + 4 + 5,
21 + 3, 21 + 21, 21 + 3 + 7,	22 + 11, 22 + 2 + 11,
23 + 23,	24 + 8, 24 + 24, 24 + 8 + 3,
25, 25 + 25,	26 + 26, 26 + 2 + 13,
27 + 27,	28 + 7, 28 + 4 + 7,
29 + 29,	30 + 15, 30 + 30, 30 + 2 + 15,
	30 + 3 + 5, 30 + 6 + 10, 30 + 2 + 3 + 5,
31 + 31,	32 + 32,
33 + 11, 33 + 3 + 11,	34 + 2 + 17,
35 + 35, 35 + 5 + 7,	36 + 4, 36 + 36, 36 + 4 + 9,
38 + 2 + 19,	39 + 39, 39 + 3 + 13,
41 + 41,	42 + 3 + 14, 42 + 6 + 14, 42 + 2 + 3 + 7,
44 + 4 + 11,	45 + 9 + 5
46 + 23,	46 + 2 + 23,
47 + 47,	49 + 49,
50 + 50, 50 + 2 + 25,	51 + 3 + 17,
54 + 2 + 27,	55 + 5 + 11,
56 + 8 + 7,	59 + 59
60 + 4 + 15	60 + 20 + 15, 60 + 4 + 3 + 5,
62 + 2 + 31,	66 + 11 + 6, 66 + 2 + 3 + 11,
69 + 2 + 23,	70 + 10 + 14, 70 + 2 + 5 + 7,
71 + 71,	78 + 26 + 39, 78 + 2 + 3 + 13,
87 + 3 + 29,	92 + 4 + 23,
94 + 2 + 47,	95 + 5 + 19,
105 + 3 + 5 + 7,	110 + 2 + 5 + 11,
119 + 7 + 17.	

and all its Atkin–Lehner involutions $\overline{W}_e, e||N$. This list (Table 1) is compiled by the second author using Kluit's results [7] and it can also be found in [6, pp. 107–109]. The first number N represent the group $\Gamma_0(N)$ and the subsequent numbers e (after the "+" sign) represent the corresponding Atkin–Lehner involutions $\overline{W}_e$. For example, the symbol "$33+11$" represents the group $< \Gamma_0(33), \overline{W}_{11} >$.

§6. Conclusion

The modular equations discussed in this paper are just a small collection of those discovered by Ramanujan. We have shown that these identities can be constructed with the knowledge of function field theory and the theory of modular forms. It will be of great interest to devise other techniques which will facilitate uniform treatments of other types of modular equations found in Ramanujan's notebooks and Lost Notebook.

ACKNOWLEDGEMENT: We would like to thank B. C. Berndt for reading the preliminary version of this work and giving valuable suggestions. We would also like to thank the referee for his valuable comments and for pointing out errors and misprints in an earlier version of this paper. The first author would like to thank the Department of Mathematics at the National University of Singapore for the facilities provided during his stay in Singapore.

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