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193

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On Ramanujan's cubic transformation formula for ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ †

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Abstract

The main aim of this paper is to provide two new proofs of Ramanujan's cubic transformation formula for ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; z)$ (see (1.8) below). For our first proof, we have to develop Ramanujan's elliptic functions in the theory of signature 3 using a different approach from that given in a recent paper by Berndt, Bhargava and Garvan. For our second proof, we use two of Goursat's formulas.

1. Introduction

Define

$$a_0 = a, \quad b_0 = b, \quad a \geq b > 0,$$
$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n}.$$

This iteration is known as the arithmetic–geometric mean iteration (AGM) of Gauss and Legendre. One can show that the limits of the sequences $\{a_n\}$ and $\{b_n\}$ exist and that

$$M(a, b) := \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n.$$

The following two properties of $M(a, b)$ are immediate:

$$M(\lambda a, \lambda b) = \lambda M(a, b), \quad \lambda > 0, \tag{1.1}$$

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and

$$M(a, b) = M\left(\frac{a+b}{2}, \sqrt{ab}\right). \quad (1.2)$$

When $a = 1$,

$$M(1, b) = \frac{1+b}{2} M\left(1, \frac{2\sqrt{b}}{1+b}\right), \quad (1.3)$$

by (1.1) and (1.2).

Around 1799, Gauss [9] determined $M(1, b)$ explicitly by showing that

$$\frac{1}{M(1, x)} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2\right), \quad (1.4)$$

where

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

$(u)_n = (u)(u+1)\cdots(u+n-1)$ and $|z| < 1$. From (1.3) and Gauss' identity (1.4), we obtain the transformation formula

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2\right) = \frac{2}{1+x} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \left\{\frac{1-x}{1+x}\right\}^2\right). \quad (1.5)$$

Upon replacing x by $(1-x)/(1+x)$ in (1.5), we arrive at Landen's transformation formula [1, p. 93]

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \left\{\frac{1-x}{1+x}\right\}^2\right) = (1+x) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x^2\right).$$

In [5], J. M. Borwein and P. B. Borwein considered the following cubic analogue of the AGM: Let

$$d_0 = d, \quad e_0 = e, \quad d \geq e > 0,$$

$$d_{n+1} = \frac{d_n + 2e_n}{3} \quad \text{and} \quad e_{n+1} = \sqrt[3]{\frac{e_n(d_n^2 + d_n e_n + e_n^2)}{3}}.$$

The limits of the sequences $\{d_n\}$ and $\{e_n\}$ exist and

$$\text{AG}_3(d, e) := \lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} e_n.$$

The following analogues of (1.1) and (1.3) hold:

$$\text{AG}_3(\lambda d, \lambda e) = \lambda \text{AG}_3(d, e), \quad \lambda > 0,$$

and

$$\text{AG}_3(1, e) = \frac{1+2e}{3} \text{AG}_3\left(1, \sqrt[3]{\frac{9e(1+e+e^2)}{(1+2e)^3}}\right). \quad (1.6)$$

One of the most surprising results in the Borweins' paper is the beautiful analogue of (1.4), namely,

$$\frac{1}{\text{AG}_3(1, x)} = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x^3\right). \quad (1.7)$$

By computations similar to those of the Gauss–Legendre AGM, they concluded from (1.6) and (1.7) that

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1 - \left\{\frac{1-x}{1+2x}\right\}^3\right) = (1+2x) {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x^3\right). \tag{1.8}$$

Identity (1.8) was first recorded without proof on page 258 of Ramanujan's second notebook [12]. The Borweins rediscovered this identity and provided the first proof by verifying (1.7) using a hypergeometric differential equation.

Recently, while studying Ramanujan's theories of elliptic functions to alternative bases, Berndt, Bhargava and Garvan [4] revisited (1.8) and proved it by verifying that both sides of (1.8) satisfy the differential equation

$$2x(1-x)(1+x+x^2)(1+2x)^2y'' - 2(1+2x)(4x^4+4x^3+2x-1)y' - 4(1-x)^2y = 0$$

and coincide at $x = 0$. At the end of their proof, they remarked that neither their proof nor the Borweins' proof is completely satisfactory and a more natural proof would be desirable.

The main purpose of this paper is to provide new proofs of identity (1.8) which will shed some light on its origin. It turns out that, for our first proof, we have to develop Ramanujan's elliptic functions in the theory of signature 3 using a different approach from that given in [4]. In Section 2, we define the Borweins' cubic theta functions and recall some of their properties. In Section 3, we discuss the relations between Eisenstein Series and the Borweins' functions. In Section 4, we derive the cubic analogue of Jacobi's classical inversion formula for the Borweins' functions. Our treatment in this section follows closely that found in [13]. In Section 5, we complete the first proof of (1.8). In our concluding section, we give a second proof of (1.8) using Goursat's formulas [10].

2. *Some facts about the Borweins' cubic theta functions*

For $|q| < 1$, set

$$f(-q) = \prod_{n=1}^{\infty} (1 - q^n),$$

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{n-m} q^{n^2+mn+m^2},$$

and

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(n+\frac{1}{3})^2+(n+\frac{1}{3})(m+\frac{1}{3})+(m+\frac{1}{3})^2},$$

where ω is a primitive cube root of unity. In terms of infinite products,

$$a(q) = \frac{f^3(-q^{\frac{1}{3}}) + 3q^{\frac{1}{3}}f^3(-q^3)}{f(-q)}, \tag{2.1}$$

$$b(q) = \frac{f^3(-q)}{f(-q^3)} \tag{2.2}$$

and

$$c(q) = 3q^{\frac{1}{3}} \frac{f^3(-q^3)}{f(-q)}. \quad (2.3)$$

Identities (2.2) and (2.3) were first discovered by the Borweins and elementary proofs can be found in [6]. Identity (2.1) is due to Ramanujan [2, p. 346]. We will also need the following important result of Ramanujan [2, p. 346]:

$$\left(1 + 9q \frac{f^3(-q^9)}{f^3(-q)}\right) = \left(1 + 27q \frac{f^{12}(-q^3)}{f^{12}(-q)}\right)^{\frac{1}{3}}. \quad (2.4)$$

From (2.1) to (2.4), we deduce the Borweins' cubic analogue of Jacobi's identity [7]

$$a^3(q) = b^3(q) + c^3(q). \quad (2.5)$$

Next, set

$$\alpha(q) = \frac{c^3(q)}{a^3(q)}.$$

From (2.1) and (2.3), we deduce that

$$\frac{1 - \alpha^{\frac{1}{3}}(q^3)}{\alpha^{\frac{1}{3}}(q^3)} = \frac{1}{3q} \left(\frac{f(-q)}{f(-q^9)}\right)^3. \quad (2.6)$$

On the other hand, from (2.1) to (2.3) and (2.5), we have

$$\frac{1 - \alpha(q)}{\alpha(q)} = \frac{1}{27q} \left(\frac{f(-q)}{f(-q^3)}\right)^{12}. \quad (2.7)$$

Hence, using (2.4), (2.6) and (2.7), we conclude that

$$\alpha(q) = 1 - \left(\frac{1 - \alpha^{\frac{1}{3}}(q^3)}{1 + 2\alpha^{\frac{1}{3}}(q^3)}\right)^3. \quad (2.8)$$

Identity (2.8) is a modular equation of degree 3 in the theory of signature 3 [4], and it plays a crucial role in the proof of (1.8). Finally, we record the identity

$$27qf^{24}(-q) = a^{12}(q)(1 - \alpha(q))^3\alpha(q), \quad (2.9)$$

which follows from (2.2), (2.3) and (2.5).

3. Ramanujan's $L(q)$, $M(q)$ and $N(q)$

Set, for $|q| < 1$,

$$L(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n},$$

$$M(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n},$$

and

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}.$$

In his famous paper On certain Arithmetical Functions [11], Ramanujan established (using elementary methods) many identities involving $L(q)$, $M(q)$ and $N(q)$. We quote four of these identities which will be needed in what follows:

$$M^3(q) - N^2(q) = 1728qf^{24}(-q), \tag{3-1}$$

$$q \frac{dL(q)}{dq} = \frac{L^2(q) - M(q)}{12}, \tag{3-2}$$

$$q \frac{dM(q)}{dq} = \frac{L(q)M(q) - N(q)}{3} \tag{3-3}$$

and

$$q \frac{dN(q)}{dq} = \frac{L(q)N(q) - M^2(q)}{2}. \tag{3-4}$$

Our aim in this section is to establish some relations between the Eisenstein Series $M(q)$ and $N(q)$ and the Borweins' cubic functions.

THEOREM 3-1. *Let $z_3 = a(q)$ and $x_3 = \alpha(q)$. Then*

$$M(q) = z_3^4(1 + 8x_3) \tag{3-5}$$

and

$$M(q^3) = \frac{z_3^4}{9}(9 - 8x_3). \tag{3-6}$$

Identities (3-5) and (3-6) were first proved by Berndt, Bhargava, and Garvan [4, theorems 4-2 and 4-4] using (1-8) and the inversion formula of the Borweins' functions. To avoid circular argument, we deduce these identities from the classical theory of elliptic functions and modular equations of degree 3.

Proof of Theorem 3.1. Let

$$\sqrt{z_2} = \varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{3-7}$$

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

and

$$x_2 = 16q \frac{\psi^4(q^2)}{\varphi^4(q)}. \tag{3-8}$$

From the classical theory of elliptic functions, we know that [2, p. 126, entry 13(i)]

$$M(q^2) = z_2^4(1 - x_2 + x_2^2). \tag{3-9}$$

We will establish (3-5) from (3-9) via Ramanujan's modular equations of degree 3. Suppose the modulus β has degree 3 over the modulus α . Then from [2, p. 230], we find that

$$x_2 = \alpha = p \frac{(2+p)^3}{(1+2p)^3}, \tag{3-10}$$

and

$$\beta = p^3 \frac{(2+p)}{(1+2p)}, \tag{3-11}$$

where p is given by

$$\frac{\varphi^2(q)}{\varphi^2(q^3)} =: m =: 1 + 2p. \quad (3.12)$$

Next, from [2, p. 460] we find that

$$\left. \begin{aligned} a(q^2) &= \frac{\varphi^3(q)}{4\varphi(q^3)} + \frac{3\varphi^3(q^3)}{4\varphi(q)} \\ &= \varphi^2(q) \frac{m^2 + 3}{4m^{\frac{3}{2}}} \\ &= \frac{(p^2 + p + 1)}{(1 + 2p)^{\frac{3}{2}}}. \end{aligned} \right\} \quad (3.13)$$

Hence

$$a^4(q^2) = z_2^4 \frac{(p^2 + p + 1)^4}{(1 + 2p)^6}. \quad (3.14)$$

From (2.3), [2, p. 124, entry 12(iii)], and [2, p. 232, (5.1)], we deduce that

$$\left. \begin{aligned} c^3(q^2) &= \left(3q^{\frac{2}{3}} \frac{f^3(-q^6)}{f(-q^2)} \right)^3 \\ &= \frac{27}{4} \frac{\varphi^9(q^3)}{\varphi^3(q)} \left(\frac{\beta^3(1 - \beta)^3}{\alpha(1 - \alpha)} \right)^{\frac{1}{4}} \\ &= \frac{27}{4} \frac{\varphi^9(q^3)}{\varphi^3(q)} \frac{(m^2 - 1)^2}{16}. \end{aligned} \right\} \quad (3.15)$$

Dividing (3.15) by the cube of (3.13), and using (3.12), we conclude that

$$\frac{c^3(q^2)}{a^3(q^2)} = \frac{27}{4} \frac{p^2(p + 1)^2}{(1 + p + p^2)^3}. \quad (3.16)$$

Next, substituting (3.10) into (3.9), we find that

$$\begin{aligned} M(q^2) &= z_2^4 \frac{(p^2 + p + 1)(p^6 + 3p^5 + 60p^4 + 115p^3 + 60p^2 + 3p + 1)}{(1 + 2p)^6} \\ &= a^4(q^2) \frac{(p^6 + 3p^5 + 60p^4 + 115p^3 + 60p^2 + 3p + 1)}{(p^2 + p + 1)^3} \quad \text{by (3.14)} \\ &= a^4(q^2) \left(1 + 54 \frac{p^2(p + 1)^2}{(p^2 + p + 1)^3} \right) \\ &= a^4(q^2) \left(1 + 8 \frac{c^3(q^2)}{a^3(q^2)} \right) \quad \text{by (3.16)}. \end{aligned}$$

Replacing q^2 by q , we deduce (3.5). For the proof of (3.6), we observe that since β has degree 3 over α ,

$$M(q^6) = \varphi^8(q^3)(1 - \beta + \beta^2), \quad (3.17)$$

by (3.9). Using (3.11) and (3.12), we may rewrite (3.17) as

$$\left. \begin{aligned} M(q^6) &= \varphi^8(q^3) \frac{(p^2 + p + 1)(p^6 + 3p^5 - 5p^3 + 3p + 1)}{(1 + 2p)^2} \\ &= a^4(q^2) \frac{(p^6 + 3p^5 - 5p^3 + 3p + 1)}{(p^2 + p + 1)^3} \quad \text{by (3.14)} \\ &= a^4(q^2) \left(1 - 6 \frac{p^2(p + 1)^2}{(p^2 + p + 1)^3} \right) \\ &= a^4(q^2) \left(1 - \frac{8}{9} \frac{c^3(q^2)}{a^3(q^2)} \right), \quad \text{by (3.16)}. \end{aligned} \right\} \quad (3.18)$$

Replacing q^2 by q , we deduce (3.6). This completes the proof of Theorem 3.1.

From (2.9) and (3.1), we find that

$$M^3(q) - N^2(q) = 64z_3^{12}(1 - x_3)^3x_3. \quad (3.19)$$

Hence, by (3.5), we deduce that

$$N(q) = \sqrt{M^3(q) - 64z_3^{12}(1 - x_3)^3x_3} = z_3^6(1 - 20x_3 - 8x_3^2). \quad (3.20)$$

4. *Venkatachaliengar's derivation of the inversion formula for the Borweins' functions*

In [13, pp. 93–95], Venkatachaliengar applied Ramanujan's differential equations (3.2)–(3.4) to show that, if

$$M(q) = z^4(1 + 8x) \quad (4.1)$$

and

$$N(q) = z^6(1 - 20x - 8x^2), \quad (4.2)$$

then

$$z = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right). \quad (4.3)$$

In the previous section, we showed that (4.1) and (4.2) are satisfied when $z = z_3$ and $x = x_3$, and so, from (4.3), we deduce the inversion formula for the Borweins' functions, namely,

$$a(q) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha(q)\right). \quad (4.4)$$

In this section, we will follow Venkatachaliengar's derivations of Ramanujan's cubic inversion formula (4.3). The only difference is that we shall use our knowledge of z and x in the final step of our argument. For simplicity, let $L = L(q)$, $M = M(q)$, and $N = N(q)$. From (3.3) and (3.5), we have

$$LM - N = q\{12(1 + 8x)z^3z' + 24x'z^4\}, \quad (4.5)$$

where the 'prime' means differentiation with respect to q . From (3.4) and (3.20), we find that

$$LN - M^2 = q\{12z^5(1 - 20x - 8x^2)z' - (40 + 32x)x'z^6\}. \quad (4.6)$$

Next, multiply (4.5) and (4.6) by N and M , respectively. Solving the resulting simultaneous equations using (3.19), we deduce that

$$\begin{aligned} &q\{-M\{12(1 - 20x - 8x^2)z^5z' - (40 + 32x)x'z^6\} + N\{12(1 + 8x)z^3z' + 24x'z^4\}\} \\ &= 64x(1 - x)^3z^{12}. \end{aligned}$$

Therefore, by (4.1) and (4.2),

$$q \left\{ -12(1+8x)(1-20x-8x^2)z^9z' + z^{10}(40+32x)(1+8x)x' \right. \\ \left. + 12(1+8x)(1-20x-8x^2)z^9z' + 24z^{10}(1-20x-8x^2)x' \right\} = 64x(1-x)^3z^{12}.$$

This implies that

$$qx' = z^2x(1-x). \quad (4.7)$$

Now, substituting (4.7) into (4.5), we have

$$LM = 12q(1+8x)z^3z' + 24x(1-x)z^6 + (1-20x-8x^2)z^6 \\ = 12q(1+8x)z^3z' + (1-4x)(1+8x)z^6.$$

Therefore, by (4.1),

$$L = 12q \frac{z'}{z} + z^2(1-4x). \quad (4.8)$$

So,

$$\left. \begin{aligned} qL' &= 12q^2 \frac{z''}{z} - 12q^2 \frac{z'^2}{z^2} + 12q \frac{z'}{z} + 2q(1-4x)zz' - 4qx'z^2 \\ &= 12q^2 \frac{z''}{z} - 12q^2 \frac{z'^2}{z^2} + 12q \frac{z'}{z} + 2q(1-4x)zz' - 4x(1-x)z^4, \end{aligned} \right\} \quad (4.9)$$

by (4.7). Using (3.2), (4.8) and (4.9), we deduce that

$$12q^2 \frac{z''}{z} - 12q^2 \frac{z'^2}{z^2} + 12q \frac{z'}{z} + 2q(1-4x)zz' - 4x(1-x)z^4 \\ = 12q^2 \frac{z'^2}{z^2} + \frac{(1-4x)^2z^4}{12} + 2q(1-4x)zz' - \frac{(1+8x)z^4}{12},$$

which implies that

$$12q^2 \frac{z''}{z} - 24q^2 \frac{z'^2}{z^2} + 12q \frac{z'}{z} = \frac{8}{3}z^4x(1-x). \quad (4.10)$$

Let

$$z'_x := \frac{dz}{dx} \quad \text{and} \quad z''_x := \frac{d^2z}{dx^2}.$$

Then, by (4.7),

$$qz' = q \frac{dz}{dx} \frac{dx}{dq} = z'_x z^2 x(1-x),$$

and

$$qz' + q^2z'' = z''_x z^4 x^2 (1-x)^2 + 2zz'_x(1-x)z'_x q + z'_x z^4 x(1-x)(1-2x).$$

This implies that

$$qz' + q^2z'' - 2z'q^2 \frac{z'}{z} = z''_x z^4 x^2 (1-x)^2 + z'_x z^4 x(1-x)(1-2x). \quad (4.11)$$

Substituting (4.11) into (4.10), we conclude that

$$x(1-x)z''_x + (1-2x)z'_x = \frac{2}{9}z.$$

This is an example of a hypergeometric differential equation [8, p. 246] and the solutions of this differential equation are linear combination of

$$F(x) := {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right) \quad \text{and} \quad G(x) := {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x\right).$$

Hence, for some constants a_1 and a_2 ,

$$z = a_1F(x) + a_2G(x).$$

From our knowledge of z and x , we know that $z - a_1F(x)$ is analytic at $x = 0$. This implies that $G(x)$ is analytic at $x = 0$ if $a_2 \neq 0$, which is clearly a contradiction [1, example 2, p. 81]. Hence, from the q -expansions of z and x , we conclude that $a_1 = 1$, and that

$$z = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right),$$

which is (4.3).

Identity (4.3) is a cubic analogue of the classical identity [2, chapter 17, p. 98, entry 3]

$$z_2 = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x_2\right), \tag{4.12}$$

where z_2 and x_2 are defined as in (3.7) and (3.8), respectively.

On page 258 of his second notebook [12], Ramanujan recorded the identity

$$(1+p+p^2) {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{p^3(2+p)}{1+2p}\right) = \sqrt{1+2p} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right), \tag{4.13}$$

where $0 < p < 1$. Identity (4.13) was first proved by Berndt, Bhargava and Garvan in [4, theorem 5.6]. We conclude this section with a new proof of (4.13). From (4.4) and (3.16), we find that

$$\left. \begin{aligned} a(q^2) &= {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \alpha(q^2)\right) \\ &= {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right), \end{aligned} \right\} \tag{4.14}$$

where p is given by (3.12). Substituting (4.14), (4.12) and (3.10) into (3.13), we find that

$${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right) = \frac{(1+p+p^2)}{(1+2p)^{\frac{3}{2}}} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; p\left(\frac{2+p}{1+2p}\right)^3\right). \tag{4.15}$$

Identity (4.13) now follows from (4.15) and the identity [2, p. 238, entry 6(i)]

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; p\left(\frac{2+p}{1+2p}\right)^3\right) = (1+2p) {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; p^3\left(\frac{2+p}{1+2p}\right)\right).$$

5. Ramanujan's cubic transformation formula

From (4.3), we observe that we may express (3.5) and (3.6) as functions of t^3 , namely,

$$M_1^*(t^3) := M_1(q) := M(q) = F^4(t^3)(1+8t^3) \tag{5.1}$$

and

$$M_3^*(t^3) := M_3(q) := M(q^3) = \frac{F^4(t^3)}{9}(9-8t^3), \tag{5.2}$$

where $t = x_3^{\frac{1}{3}}$, with x_3 given as in Theorem 3.1. Set $t_1 := \alpha(q^{\frac{1}{3}})$. By (2.8), we find that

$$t_1^3 = 1 - \left(\frac{1-t}{1+2t} \right)^3. \tag{5.3}$$

Using (5.2) and (5.3), we deduce that

$$\left. \begin{aligned} M_3^*(t_1^3) &:= M_3^* \left(1 - \left\{ \frac{1-t}{1+2t} \right\}^3 \right) \\ &= \frac{1}{9} F^4 \left(1 - \left\{ \frac{1-t}{1+2t} \right\}^3 \right) \left(1 + 8 \left\{ \frac{1-t}{1+2t} \right\}^3 \right). \end{aligned} \right\} \tag{5.4}$$

Now, dividing (5.4) by (5.1), we obtain

$$\left. \begin{aligned} \frac{M_3^* \left(1 - \left\{ \frac{1-t}{1+2t} \right\}^3 \right)}{M_1^*(t^3)} &= \frac{\left(1 + 8 \left\{ \frac{1-t}{1+2t} \right\}^3 \right) F^4 \left(1 - \left\{ \frac{1-t}{1+2t} \right\}^3 \right)}{9(1+8t^3) F^4(t^3)} \\ &= \frac{1}{(1+2t)^4} \frac{F^4 \left(1 - \left\{ \frac{1-t}{1+2t} \right\}^3 \right)}{F^4(t^3)}. \end{aligned} \right\} \tag{5.5}$$

Now,

$$M_3^* \left(1 - \left\{ \frac{1-t}{1+2t} \right\}^3 \right) = M_3^*(t_1^3) = M_3(q^{\frac{1}{3}}) = M_1(q) = M_1^*(t^3).$$

Hence, (1.8) follows from (5.5).

Remarks. Shortly after the discovery of the above proof of (1.8), the author received a letter from Professor Berndt containing Venkatachaliengar’s ‘proof’ of (1.8). Venkatachaliengar’s ‘proof’ is similar to the author’s proof, but his final argument is circular. More precisely, he showed that identities (5.3) and (1.8) are equivalent but failed to show that (5.3) holds. We have shown here that (5.3) follows from the modular equation (2.8) (via the Borweins’ functions).

6. Proof of (1.8) using Goursat’s formulas

Let u and v be real variables. In [10, p. 140, (126) and (127)], Goursat recorded the following transformation formulas:

$$\begin{aligned} {}_2F_1 \left(4c, 4c + \frac{1}{3}; 6c + \frac{1}{2}; u \right) &= \left(1 - \frac{8}{9}u \right)^{-3c} \\ &\times {}_2F_1 \left(c, c + \frac{1}{3}; 2c + \frac{5}{6}; 64 \frac{u^3(1-u)}{(9-8u)^3} \right), \quad 0 \leq u < \frac{9}{4} - \frac{3}{4}\sqrt{3}. \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} {}_2F_1 \left(4c, 4c + \frac{1}{3}; 2c + \frac{5}{6}; v \right) &= \left(1 + 8v \right)^{-3c} \\ &\times {}_2F_1 \left(c, c + \frac{1}{3}; 2c + \frac{5}{6}; 64 \frac{v(1-v)^3}{(1+8v)^3} \right), \quad 0 \leq v < -\frac{5}{4} + \frac{3}{4}\sqrt{3}. \end{aligned} \tag{6.2}$$

Set $k := \sqrt[3]{1-u}$ and $l := \sqrt[3]{v}$. Assuming that

$$64 \frac{u^3(1-u)}{(9-8u)^3} = 64 \frac{v(1-v)^3}{(1+8v)^3}, \tag{6.3}$$

we find that

$$k \frac{1-k^3}{1+8k^3} = l \frac{1-l^3}{1+8l^3}.$$

Hence, $k = l$ or

$$(-1+k+l+2kl)(1+k+k^2+(1+4k-2k^2)l+(1-2k+4k^2)l^2) = 0.$$

This implies that the only two solutions to (6.3) are

$$u = 1-v \quad \text{or} \quad u = 1 - \left(\frac{1-\sqrt[3]{v}}{1+2\sqrt[3]{v}} \right)^3,$$

since the discriminant of $(1+k+k^2+(1+4k-2k^2)l+(1-2k+4k^2)l^2)$ is negative.

Now, the solution $u = 1-v$ is inadmissible since $\frac{9}{4} - \frac{3}{4}\sqrt{3} \leq u < 1$ is outside the range of u in (6.1). Hence, by (6.1) and (6.2),

$$\begin{aligned} {}_2F_1 \left(4c, 4c + \frac{1}{3}; 6c + \frac{1}{2}; 1 - \left(\frac{1-\sqrt[3]{v}}{1+2\sqrt[3]{v}} \right)^3 \right) &= (1+2\sqrt[3]{v})^{12c} \\ &\times {}_2F_1 \left(4c, 4c + \frac{1}{3}; 2c + \frac{5}{6}; v \right), \quad 0 \leq v < -\frac{5}{4} + \frac{3}{4}\sqrt{3}. \end{aligned} \tag{6.4}$$

The range of v in (6.4) can be extended to $0 \leq v < 1$. Setting $c = \frac{1}{12}$ and $v = x^3$ in (6.4), we deduce (1.8).

Remarks. Identity (6.4) was first discovered by Berndt, Bhargava and Garvan [4, theorem 2.3]. Contrary to what they have claimed in the paragraph before [4, theorem 2.3], we have shown that Ramanujan's transformation for ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x)$ and one of its generalizations can be deduced from Goursat's results.

The proof of (6.4) given in this section is motivated by two new identities recently discovered by the author and Berndt [3, section 2], namely,

$$j(\tau) = 27 \frac{(1+8x_3)^3}{x_3(1-x_3)^3} \tag{6.5}$$

and

$$j(3\tau) = 27 \frac{(9-8x_3)^3}{(1-x_3)x_3^3}, \tag{6.6}$$

where j is the well-known modular j -invariant and $q = e^{2\pi i\tau}$. Note that the reciprocals of the right hand sides of (6.5) and (6.6) both appear in Goursat's formulas (6.1) and (6.2). Now, one can deduce (6.6) from (6.5) (see [3, section 2]) by using (2.8) and hence, it is not surprising that a solution of (6.3) is of the form

$$u = 1 - \left(\frac{1-\sqrt[3]{v}}{1+2\sqrt[3]{v}} \right)^3.$$

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