RAMANUJAN–WEBER CLASS INVARIANT G_n AND WATSON'S EMPIRICAL PROCESS

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Dedicated to Professor J. P. Serre

Abstract

In an attempt to prove all the identities given in Ramanujan's first two letters to G. H. Hardy, G. N. Watson devoted a paper to the evaluation of G_{1353}. At the end of his paper, Watson remarked that his proof was not rigorous as he had assumed certain identities (see (1.1) and (1.2)) which he found empirically. In this paper, we use class field theory, Galois theory and Kronecker's limit formula to justify Watson's assumptions. We shall then use our results to compute some new values of G_n.

1. *Introduction*

Let

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1-aq^n)$$

and

$$\chi(q) = (-q;q^2)_\infty \quad \text{for } |q|<1.$$

S. Ramanujan first introduced the class invariant

$$G_n = 2^{-1/4} e^{\pi\sqrt{n}/24} \chi(e^{-\pi\sqrt{n}})$$

in his famous paper 'Modular equations and approximations to π' [**13**; **14**, pp. 23–39]. In H. Weber's notation,

$$G_n = 2^{-1/4} f(\sqrt{-n}).$$

A table consisting of 50 values of G_n was first constructed by Weber [**19**, pp. 721–726]. In the aforementioned paper, Ramanujan added another 31 values to Weber's list. For these obvious reasons, we shall call G_n a *Ramanujan–Weber class invariant*.

In his second letter to G. H. Hardy, Ramanujan recorded the identity [**3**, p. 62, no. 23],

$$\begin{aligned} G_{1353}^2 = {} & \sqrt{(3+\sqrt{11})(5+3\sqrt{3})\left(\frac{11+\sqrt{123}}{2}\right)}\left(\frac{6817+321\sqrt{451}}{4}\right)^{1/6} \\ & \times\left(\sqrt{\frac{25+3\sqrt{33}}{8}}+\sqrt{\frac{17+3\sqrt{33}}{8}}\right) \\ & \times\left(\sqrt{\frac{569+99\sqrt{33}}{8}}+\sqrt{\frac{561+99\sqrt{33}}{8}}\right). \end{aligned}$$

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In an attempt to prove all the identities given in Ramanujan's two letters to Hardy [**3**], G. N. Watson devoted a paper to the evaluation of G_{1353} [**17**]. At the end of his paper [**17**, p. 132], Watson remarked that his proof was not rigorous, as he had assumed that

$$\frac{G_{1353}G_{123/11}}{G_{451/3}G_{41/33}}+\frac{G_{451/3}G_{41/33}}{G_{1353}G_{123/11}} = 15+3\sqrt{33} \tag{1.1}$$

and

$$\frac{G_{1353}G_{451/3}}{G_{41/33}G_{123/11}}+\frac{G_{41/33}G_{123/11}}{G_{1353}G_{451/3}} = \tfrac{1}{2}(21+3\sqrt{33}). \tag{1.2}$$

Despite the lack of rigour in his empirical process, Watson [**18**] employed it further to verify other values of G_n stated explicitly in [**13**; **14**, pp. 23–39]. In particular, he succeeded in verifying the values of G_n, for $n = 465$, 777, 897, 1645, and 1677 based on the assumptions of identities similar to (1.1) and (1.2).

The main purpose of this paper is to provide rigorous proofs of the aforementioned identities for G_n. We achieve this by justifying Watson's assumptions. We shall then use our results to compute some new values of G_n.

To fix notations, we first recall some facts in algebraic number theory. Let $K = \mathbb{Q}(\sqrt{-m})$ (where m is squarefree) be an imaginary quadratic field and $\mathfrak{D}_K$ be its ring of integers. Two non-zero $\mathfrak{D}_K$ ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be equivalent if there exists an $\alpha \in K\backslash\{0\}$ such that $\mathfrak{a} = \alpha\mathfrak{b}$. It is known that the set of equivalence classes forms a group under multiplication of ideals and we call this group, denoted by C_K, the *ideal class group* of K. The order of C_K, denoted by h_K, is known as the *class number* of K. In Section 2, we shall prove the following.

THEOREM 1.1. *Let*

$$\nu = \begin{cases} 4 & \text{if } 3 \nmid pqr, \\ 12 & \text{if } 3 \mid pqr. \end{cases}$$

Suppose that $K = \mathbb{Q}(\sqrt{-pqr})$ *satisfies the conditions*

(i) *p, q and r are distinct primes such that* $pqr \equiv 1 \pmod 4$, *and*

(ii) *the class number* $h_K = 16$.

Then

$$\alpha = \alpha_{p,q,r} = \left(\frac{G_{pqr}G_{pq/r}}{G_{pr/q}G_{qr/p}}\right)^{\nu}+\left(\frac{G_{pqr}G_{pq/r}}{G_{pr/q}G_{qr/p}}\right)^{-\nu},$$

$$\beta = \beta_{p,q,r} = \left(\frac{G_{pqr}G_{pr/q}}{G_{pq/r}G_{qr/p}}\right)^{\nu}+\left(\frac{G_{pqr}G_{pr/q}}{G_{pq/r}G_{qr/p}}\right)^{-\nu},$$

$$\gamma = \gamma_{p,q,r} = \left(\frac{G_{pqr}G_{qr/p}}{G_{pr/q}G_{pq/r}}\right)^{\nu}+\left(\frac{G_{pqr}G_{qr/p}}{G_{pr/q}G_{pq/r}}\right)^{-\nu},$$

and

$$\delta = \delta_{p,q,r} = (G_{pqr}G_{pq/r}G_{pr/q}G_{qr/p})^{\nu}+(G_{pqr}G_{pq/r}G_{pr/q}G_{qr/p})^{-\nu}$$

are algebraic integers belonging to a real quadratic field $R \in \mathfrak{R}_K$, *where*

$$\mathfrak{R}_K \subset \{\mathbb{Q}(\sqrt{m}) | m = p^{\varepsilon_1}q^{\varepsilon_2}r^{\varepsilon_3},\ \varepsilon_i = 0 \quad or \quad 1 \quad for \quad i = 1,2,3\}$$

contains fields such that none of the primes (2), (p), (q) *or* (r) *is inert.*

Theorem 1.1 provides an explanation of the existence of identities such as (1.1) and (1.2).

In Section 3, we shall prove the following.

THEOREM 1.2. *Let $R = \mathbb{Q}(\sqrt{m})$ be the field which contains α, β, γ and δ. If*

$$2\alpha = a_1 + a_2\sqrt{m}, \quad 2\beta = b_1 + b_2\sqrt{m}, \quad 2\gamma = c_1 + c_2\sqrt{m}, \quad 2\delta = d_1 + d_2\sqrt{m},$$

then a_1, a_2, b_1, b_2, c_1, c_2, d_1 and d_2 are positive integers.

Theorem 1.2 implies that the determinations of α, β, γ and δ can be done in a finite number of steps since the number of real quadratic fields to be considered is finite and the set $\{u + v\sqrt{m} | u, v \in \mathbb{N}\}$ is discrete in $\mathbb{Z}[\sqrt{m}]$. Using Theorems 1.1 and 1.2, we conclude that Watson's assumptions are valid. As corollaries to these theorems, we evaluate G_n for $n = 285$, 429, 465, 561, 609, 645, 777, 805, 897, 957, 1005, 1045, 1065, 1105, 1113, 1185, 1353, 1605, 1645, 1653, 1677, 1705, 1885, 2013, 2233 and 2737. A list of the G_n for $n \neq 465$, 777, 897, 1353, 1645, or 1677 is given in Section 4. The number G_{2737} was first evaluated by D. Shanks [**15**, p. 399, (8)] using Epstein Zeta Functions. Readers are encouraged to read Shanks' paper for a different proof of G_{2737}.

Theorems 1.1 and 1.2 are natural extensions of results given in [**4**, Theorems 7.4 and 7.5]. They can be extended to evaluate class invariants G_n whenever the class group of $K = \mathbb{Q}(\sqrt{n})$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$. For example, when $n = 3 \cdot 5 \cdot 7 \cdot 17 = 1785$, we have $C_{\mathbb{Q}(\sqrt{-1785})} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and we find that

$$\begin{aligned} G_{1785}^4 = {} & \left(\frac{23\sqrt{35}+33\sqrt{17}}{\sqrt{2}}\right)^{1/3} (10\sqrt{17}+9\sqrt{21})^{1/3} \left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right) \\ & \times \left(\frac{\sqrt{85}+9}{2}\right)\left(\frac{\sqrt{51}+7}{\sqrt{2}}\right)\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)\left(\sqrt{\frac{20+5\sqrt{15}}{4}}\right. \\ & \left. + \sqrt{\frac{16+5\sqrt{15}}{4}}\right)\left(\sqrt{\frac{732+189\sqrt{15}}{4}} + \sqrt{\frac{728+189\sqrt{15}}{4}}\right) \\ & \times \left(\sqrt{\frac{31+8\sqrt{15}}{2}} + \sqrt{\frac{29+8\sqrt{15}}{2}}\right)\left(\sqrt{\frac{108+27\sqrt{15}}{4}} + \sqrt{\frac{104+27\sqrt{15}}{4}}\right). \end{aligned}$$

In [**9**], S. Chowla proved that the number of imaginary quadratic fields with k classes in the principal genus is finite. Applying this result with $k = 2$ indicates that the number of G_n which we can compute using the methods mentioned in this paper is finite. In fact, D. Buell's table [**7**], which gives a list of those n less than 2^{63} satisfying the hypotheses in the previous paragraph, shows that we can compute six more G_n. The associated imaginary quadratic fields for these six n have the same group structure as that of $\mathbb{Q}(\sqrt{-1785})$. For the sake of completion, we shall list these values in Section 5.

2. *Proof of Theorem* 1.1

From class field theory, we know that there exist an everywhere unramified extension $K^{(1)}$ of K such that

$$\mathrm{Gal}(K^{(1)}|K) \simeq C_K.$$

This field is usually known as the Hilbert class field or the absolute class field of K. One can describe the isomorphism explicitly using the j-invariant function (see Lemma 2.1).

Let $\mathfrak{a} = [\tau_1, \tau_2]$ be a $\mathfrak{O}_K$-ideal. Define

$$j(\mathfrak{a}) = 1728 \frac{g_2^3(\mathfrak{a})}{g_2^3(\mathfrak{a}) - 27 g_3^2(\mathfrak{a})},$$

where

$$g_2(\mathfrak{a}) = 60 \sum_{\substack{m,\, n=-\infty \\ (m,\, n) \neq (0,\, 0)}}^{\infty} \frac{1}{(m\tau_1 + n\tau_2)^4} \quad \text{and} \quad g_3(\mathfrak{a}) = 140 \sum_{\substack{m,\, n=-\infty \\ (m,\, n) \neq (0,\, 0)}}^{\infty} \frac{1}{(m\tau_1 + n\tau_2)^6}.$$

It is clear from the definitions of $g_2([\tau_1, \tau_2])$ and $g_3([\tau_1, \tau_2])$ that

$$j([\tau_1, \tau_2]) = j([1, \tau]) =: j(\tau),$$

where $\tau = \tau_2/\tau_1$. We will also let

$$\gamma_2(\tau) = (j(\tau))^{1/3}$$

with the cube root being real-valued when $j(\mathfrak{a})$ is real.

It is well known that $K^{(1)} = K(j(\mathfrak{O}_K))$ [**10**, Theorem 11.1, p. 220]. If D_K is the discriminant of K and $3 \nmid D_K$, then $K^{(1)} = K(\gamma_2(\tau_K))$ [**10**, Theorem 12.2, p. 249], where

$$\tau_K = \begin{cases} \sqrt{-m} & \text{if } D_K \equiv 0 \text{ (mod 4)}, \\ \dfrac{3 + \sqrt{-m}}{2} & \text{if } D_K \equiv 1 \text{ (mod 4)}. \end{cases}$$

These are the facts which lead to the definition of v given in Theorem 1.1.

We are now ready to state the following lemmas.

LEMMA 2.1. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two* $\mathfrak{O}_K$*-ideals. Define* $\sigma_{\mathfrak{a}}(j(\mathfrak{b}))$ *by*

$$\sigma_{\mathfrak{a}}(j(\mathfrak{b})) = j(\bar{\mathfrak{a}}\mathfrak{b}), \tag{2.1}$$

where $\mathfrak{a}\bar{\mathfrak{a}}$ *is a principal ideal. Then* $\sigma_{\mathfrak{a}}$ *is a well-defined element of* $\text{Gal}(K^{(1)} \mid K)$*, and* $\mathfrak{a} \mapsto \sigma_{\mathfrak{a}}$ *induces an isomorphism*

$$C_K \longrightarrow \text{Gal}(K^{(1)} \mid K).$$

Proof. See [**10**, Corollary 11.37, p. 240].

LEMMA 2.2. *Let* $K = \mathbb{Q}(\sqrt{-pqr})$*, where* p*,* q*, and* r *are three distinct primes satisfying* $pqr \equiv 1 \pmod 4$*, and let* v *be as defined in Theorem* 1.1*. Then* G_{pqr}^v *is a real unit generating the field* $K^{(1)}$*.*

Proof. From [**5**, p. 290], we find that G_{pqr}^{12} is a real unit of $K^{(1)}$. Since [**10**, Theorem 12.17, p. 257]

$$j(\mathfrak{O}_K) = j(\sqrt{-pqr}) = \frac{(16 G_{pqr}^{24} - 4)^3}{G_{pqr}^{24}}, \tag{2.2}$$

we conclude that

$$K^{(1)} = K(G_{pqr}^{12}). \tag{2.3}$$

Next, suppose that $3 \nmid pqr$. Then $3 \nmid D_K$ and $\gamma_2(\tau_K)$ generates $K^{(1)}$. From the equality [**10**, Theorem 12.17, p. 257]

$$\gamma_2(\sqrt{-pqr}) = \frac{16 G_{pqr}^{24} - 4}{G_{pqr}^8}$$

and (2.3), we find that $G_{pqr}^8 \in K^{(1)}$. Hence, $G_{pqr}^4 \in K^{(1)}$, by (2.3).

REMARK. In [**5**, p. 290], B. J. Birch quoted M. Deuring's results [**11**, p. 43], and indicated that G_n is a real unit when $n \in \mathbb{N}$ and $n \equiv 1 \pmod 4$. A more elaborate proof of this statement can be found in [**8**, Corollary 5.2]. In fact, from the treatment given in [**8**], one can show that $G_{pq/r}$, $G_{pr/q}$ and $G_{qr/p}$ are units. These facts will be needed in the proof of Theorem 1.1.

Proof of Theorem 1.1. From the hypotheses, we deduce that $\mathfrak{t} = [2, 1+\sqrt{-pqr}]$, $\mathfrak{p} = [p, \sqrt{-pqr}]$ and $\mathfrak{q} = [q, \sqrt{-pqr}]$ are $\mathfrak{O}_K$-ideals lying in distinct equivalence classes. This implies that C_K contains a group $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ generated by the ideal classes $[\mathfrak{t}]$, $[\mathfrak{q}]$ and $[\mathfrak{p}]$. Using the isomorphism in Lemma 2.1, we conclude that $\mathrm{Gal}(K^{(1)} \mid K)$ contains the group $U = \langle \sigma_{\mathfrak{t}}, \sigma_{\mathfrak{p}}, \sigma_{\mathfrak{q}} \rangle$. To show that α, β, γ and δ belong to a field with degree 2 over K, it suffices to show that $\sigma_{\mathfrak{t}}$, $\sigma_{\mathfrak{p}}$ and $\sigma_{\mathfrak{q}}$ fix these elements. The fact that they are algebraic integers follows from Lemma 2.2 and the remark after the lemma.

Now, assume that $3 \mid pqr$. From [**10**, Theorem 12.17, p. 257], we have

$$j(\mathfrak{p}) = j(\sqrt{-qr/p}) = \frac{(16G_{qr/p}^{24} - 4)^3}{G_{qr/p}^{24}}. \tag{2.4}$$

By Lemma 2.1, we find that

$$\sigma_{\mathfrak{p}}(j(\mathfrak{O}_K)) = j(\mathfrak{p}). \tag{2.5}$$

From (2.2), (2.4) and (2.5), we find that

$$\frac{(16\sigma_{\mathfrak{p}}^2(G_{pqr}^{12}) - 4)^3}{\sigma_{\mathfrak{p}}^2(G_{pqr}^{12})} = \frac{(16G_{qr/p}^{24} - 4)^3}{G_{qr/p}^{24}}. \tag{2.6}$$

Simplifying (2.6), we deduce that

$$(a-b)(a+b)\{64(a^2+b^2)a^2b^2 - 48a^2b^2 + 1\} = 0,$$

where $a = \sigma_{\mathfrak{p}}(G_{pqr}^{12})$ and $b = G_{qr/p}^{12}$. But

$$64(a^2+b^2)a^2b^2 - 48a^2b^2 + 1 \neq 0,$$

for otherwise it would contradict the fact that a and b are algebraic integers. Thus,

$$\sigma_{\mathfrak{p}}(G_{pqr}^{12}) = \pm G_{qr/p}^{12}. \tag{2.7}$$

By a similar argument, we have

$$\sigma_{\mathfrak{q}}(G_{pqr}^{12}) = \pm G_{pr/q}^{12}. \tag{2.8}$$

From (2.8) and (2.7), we deduce that both $\sigma_{\mathfrak{p}}(G_{pr/q}^{12})$ and $\sigma_{\mathfrak{q}}(G_{qr/p}^{12})$ are defined, respectively. Using the same argument as in the proof of (2.6), we find that

$$\sigma_{\mathfrak{p}}(G_{pr/q}^{12}) = \pm G_{qp/r}^{12} \tag{2.9}$$

and

$$\sigma_{\mathfrak{q}}(G_{qr/p}^{12}) = \pm G_{pq/r}^{12}. \tag{2.10}$$

By (2.7), (2.8), (2.9) and (2.10), we conclude that $\sigma_{\mathfrak{p}}$ and $\sigma_{\mathfrak{q}}$ fix α, β, γ and δ.

Next, from [**10**, p. 263], we have

$$j(\mathfrak{t}) = j\left(\frac{3+\sqrt{-pqr}}{2}\right) = G_{pqr}^{24}\left(\frac{16}{G_{pqr}^{24}} - 4\right)^3. \tag{2.11}$$

Applying Lemma 2.1 again, we deduce that

$$\sigma_{\mathfrak{t}}(j(\mathfrak{O}_K)) = j(\mathfrak{t}). \tag{2.12}$$

By (2.2), (2.11) and a similar argument to that in the proof of (2.6), we find that

$$\sigma_{\mathfrak{t}}(G_{pqr}^{12}) = \pm G_{pqr}^{-12}. \tag{2.13}$$

Using $\mathfrak{p}$ instead of $\mathfrak{D}_K$ in (2.12) and the corresponding expression for $j(\mathfrak{t}\mathfrak{p})$, we find that

$$\sigma_{\mathfrak{t}}(G_{qr/p}^{12}) = \pm G_{qr/p}^{-12} \quad \text{or} \quad \mp G_{qr/p}^{-12},$$

that is, $\sigma_{\mathfrak{t}}(G_{qr/p}^{12})$ may have the same or opposite sign as $\sigma_{\mathfrak{t}}(G_{pqr}^{12})$. We shall show that the latter case is inadmissible. If

$$\sigma_{\mathfrak{p}}(G_{qr/p}^{12}) = \pm G_{pqr}^{12}, \quad \sigma_{\mathfrak{t}}(G_{qr/p}^{12}) = \mp G_{qr/p}^{-12}$$

and (2.13) holds, then

$$\sigma_{\mathfrak{p}}\sigma_{\mathfrak{t}}(G_{pqr}^{12}) = \pm G_{qr/p}^{-12} \quad \text{and} \quad \sigma_{\mathfrak{t}}\sigma_{\mathfrak{p}}(G_{pqr}^{12}) = \mp G_{qr/p}^{-12}.$$

This clearly contradicts the fact that $\sigma_{\mathfrak{p}}\sigma_{\mathfrak{t}} = \sigma_{\mathfrak{t}}\sigma_{\mathfrak{p}}$. Similar arguments show that, corresponding to (2.13), we have

$$\sigma_{\mathfrak{t}}(G_{pr/q}^{12}) = \pm G_{pr/q}^{-12}, \quad \text{and} \quad \sigma_{\mathfrak{t}}(G_{pq/r}^{12}) = \pm G_{pq/r}^{-12}.$$

Collecting our results, we conclude that $\sigma_{\mathfrak{t}}$ fixes α, β, γ and δ. This proves the first part of Theorem 1.1 when $3 \mid pqr$.

The proof when $3 \nmid pqr$ is similar. In this case, G_{pqr}^4 generates $K^{(1)}$ and $\sigma_{\mathfrak{u}}(G_{pqr}^4)$ is well defined for $\mathfrak{u} = \mathfrak{p}, \mathfrak{q}$, and $\mathfrak{t}$. Hence, we may deduce from (2.6) that

$$16\sigma_{\mathfrak{p}}^2(G_{pqr}^8) - \frac{4}{\sigma_{\mathfrak{p}}^2(G_{pqr}^8)} = 16G_{qr/p}^{16} - \frac{4}{G_{qr/p}^8}. \tag{2.14}$$

Simplifying (2.14), we have $(a-b)(4a^2b+4ab^2+1) = 0$, where $a = \sigma_{\mathfrak{p}}(G_{pqr}^8)$ and $b = G_{qr/p}^8$. But $4a^2b+4ab^2+1 \neq 0$, for otherwise it would contradict the fact that a and b are algebraic integers. Hence, we deduce that

$$\sigma_{\mathfrak{p}}(G_{pqr}^8) = G_{qr/p}^8.$$

Now, since $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(K^{(1)} \mid K)$ and G_{pqr}^4 generates $K^{(1)}$, we find that

$$\sigma_{\mathfrak{p}}(G_{pqr}^4) = \pm G_{qr/p}^4.$$

The rest of the arguments are analogous to those of the previous case, and we shall omit them.

We have already seen that α, β, γ and δ lie in a real quadratic field R. Now, if one of the primes, say $(u) = (2)$, (p), (q), or (r) is inert in R, then the corresponding Frobenius automorphism $\sigma_{\mathfrak{u}} \in \mathrm{Gal}(K^{(1)} \mid K)$, where $\mathfrak{u}^2 = (u)$ in K, is trivial. This contradicts the fact that $\sigma_{\mathfrak{u}}$ has order 2. For more details, see [**4**, Theorem 7.3]. This gives the necessary condition for R and completes the proof of Theorem 1.1.

3. *Proof of Theorem* 1.2

Let C_K^2 denote the subgroup of squares in C_K, and let G_K be the *genus group* C_K/C_K^2. A group homomorphism $\chi: G_K \to \pm 1$ is known as a genus character. One can show that a genus character arises from a certain decomposition of D_K, where D_K is the discriminant of K. More precisely, if χ is a genus character, then there exist d_1 and d_2 satisfying $D_K = d_1 d_2$ with $d_1 > 0$ and $d_i \equiv 0$ or 1 (mod 4), such that, for any prime ideal $\mathfrak{p}$ in K, we have

$$\chi(\mathfrak{p}) = \begin{cases} \left(\dfrac{d_1}{N(\mathfrak{p})}\right), & \text{if } N(\mathfrak{p}) \nmid d_1, \\[2ex] \left(\dfrac{d_2}{N(\mathfrak{p})}\right), & \text{if } N(\mathfrak{p}) \mid d_1, \end{cases}$$

where $N(\mathfrak{p})$ is the norm of the ideal $\mathfrak{p}$ and $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. If $[\mathfrak{a}]$ is an ideal class in C_K and $\mathfrak{a} = \prod \mathfrak{p}^{\alpha_\mathfrak{p}}$, then we set

$$\chi([\mathfrak{a}]) = \prod \chi(\mathfrak{p})^{\alpha_\mathfrak{p}}.$$

LEMMA 3.1. *Let χ be a genus character arising from the decomposition $D_K = d_1 d_2$. Let h_i be the class number of the field $\mathbb{Q}(\sqrt{d_i})$, and w_2 be the number of roots of unity in $\mathbb{Q}(\sqrt{d_2})$, and ε_1 be the fundamental unit of $\mathbb{Q}(\sqrt{d_1})$. Let*

$$F([\mathfrak{a}]) = (N([1, \tau]))^{1/2} |\eta(\tau)|^2,$$

with

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$$

and

$$\tau = \frac{\tau_2}{\tau_1}, \quad \operatorname{Im} \tau > 0, \quad \textit{where} \quad \mathfrak{a} = [\tau_1, \tau_2].$$

Then

$$\varepsilon_1^{2h_1 h_2/w_2} = \prod_{[\mathfrak{a}] \in C_K} F([\mathfrak{a}])^{-\chi([\mathfrak{a}])}. \tag{3.1}$$

Proof. See [**16**, p. 72, Theorem 6].

From now on, we set $K = \mathbb{Q}(\sqrt{-pqr})$ with $h_K = 16$. In this case,

$$G_K \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Suppose that the principal genus contains $[\mathfrak{pt}]$, that is, $\mathbf{G}_0 = \{[\mathfrak{D}_K], [\mathfrak{pt}]\}$. Corresponding to this assumption, we know that there exist three genera, say $\mathbf{G}_1 = \{[\mathfrak{p}], [\mathfrak{t}]\}$, $\mathbf{G}_2 = \{[\mathfrak{q}], [\mathfrak{tpq}]\}$, and $\mathbf{G}_3 = \{[\mathfrak{pq}], [\mathfrak{tq}]\}$. Note that $\mathfrak{G} = \mathbf{G}_0 \cup \mathbf{G}_1 \cup \mathbf{G}_2 \cup \mathbf{G}_3$ forms a subgroup of index 2 in C_K.

Let $\mathfrak{C}$ be the set of genus characters such that $\chi(\mathbf{G}_1) = -1$, $\chi(\mathbf{G}_2) = 1$ and $\chi(\mathbf{G}_3) = -1$. Note that $|\mathfrak{C}| = 2$. For any ideal class $[\mathfrak{a}] \notin \mathfrak{G}$, we have

$$\sum_{\chi \in \mathfrak{C}} \chi([\mathfrak{a}]) = 0,$$

which implies that

$$\prod_{\chi \in \mathfrak{C}} F([\mathfrak{a}])^{-\chi([\mathfrak{a}])} = 1.$$

Hence, by (3.1), we conclude that

$$\begin{aligned} \prod_{\chi \in \mathfrak{G}} \varepsilon_1^{2h_1 h_2/w_2} &= \prod_{\chi \in \mathfrak{C}} \prod_{[\mathfrak{a}] \in C_K} F([\mathfrak{a}])^{-\chi([\mathfrak{a}])} = \prod_{[\mathfrak{a}] \in \mathfrak{G}} \prod_{\chi \in \mathfrak{C}} F([\mathfrak{a}])^{-\chi([\mathfrak{a}])} \\ &= \prod_{[\mathfrak{a}] \in \mathfrak{G}} F([\mathfrak{a}])^{-2\chi([\mathfrak{a}])}, \end{aligned} \tag{3.2}$$

since $|\mathfrak{C}| = 2$. If we substitute $F([\mathfrak{a}])$ by its infinite product representation and use the definition of G_n (see (1.2)), we can rewrite (3.2) as

$$\left(\frac{G_{pqr} G_{qp/r}}{G_{qr/p} G_{pr/q}} \right)^4 = \prod_{\chi \in \mathfrak{C}} \varepsilon_1^{2h_1 h_2/w_2}. \tag{3.3}$$

Next, let $\mathfrak{C}'$ be the set of genus characters such that $\chi(\mathbf{G}_1) = -1$, $\chi(\mathbf{G}_2) = -1$ and $\chi(\mathbf{G}_3) = 1$, then by following the previous argument, we conclude that

$$\left(\frac{G_{pqr}\,G_{pr/q}}{G_{qr/p}\,G_{pq/r}}\right)^4 = \prod_{\chi\in\mathfrak{C}'} \varepsilon_1^{2h_1h_2/w_2}. \tag{3.4}$$

Since

$$w_2 = \begin{cases} 2 \text{ or } 4 & \text{if } 3 \nmid pqr, \\ 2, 4 \text{ or } 6 & \text{if } 3 \mid pqr, \end{cases}$$

we deduce from (3.3) and (3.4) the following lemma.

LEMMA 3.2. *Suppose that* $[\mathfrak{t}\mathfrak{p}]$ *is in the principal genus. Then*

$$\left(\frac{G_{pqr}\,G_{pq/r}}{G_{qr/p}\,G_{pr/q}}\right)^v = \varepsilon_1^{e_1/2}\varepsilon_2^{e_2/2} \quad \textit{and} \quad \left(\frac{G_{pqr}\,G_{pr/q}}{G_{qr/p}\,G_{pq/r}}\right)^v = \varepsilon_3^{e_3/2}\varepsilon_4^{e_4/2},$$

where the ε_i *are fundamental units in some real quadratic fields and the* e_i *are positive integers.*

Next, if we assume that $[\mathfrak{p}]$ is in the principal genus, then by using similar arguments to those in the proof of Lemma 3.2, we obtain the following lemma.

LEMMA 3.3. *Suppose that* $[\mathfrak{p}]$ *is in the principal genus. Then*

$$\left(\frac{G_{pqr}\,G_{qr/p}}{G_{pq/r}\,G_{pr/q}}\right)^v = \varepsilon_1^{e_1/2}\varepsilon_2^{e_2/2} \quad \textit{and} \quad (G_{pqr}\,G_{qr/p}\,G_{pq/r}\,G_{pr/q})^v = \varepsilon_3^{e_3/2}\varepsilon_4^{e_4/2}$$

where the ε_i *are fundamental units in some real quadratic fields and the* e_i *are positive integers.*

If we replace $\mathfrak{p}$ in Lemma 3.2 and Lemma 3.3 by $\mathfrak{q}$ or $\mathfrak{p}\mathfrak{q}$, we obtain four further lemmas.

LEMMA 3.4. *Suppose that* $[\mathfrak{t}\mathfrak{q}]$ *is in the principal genus. Then*

$$\left(\frac{G_{pqr}\,G_{pq/r}}{G_{pr/q}\,G_{qr/p}}\right)^v = \varepsilon_1^{e_1/2}\varepsilon_2^{e_2/2} \quad \textit{and} \quad \left(\frac{G_{pqr}\,G_{qr/p}}{G_{pq/r}\,G_{pr/q}}\right)^v = \varepsilon_3^{e_3/2}\varepsilon_4^{e_4/2},$$

where the ε_i *are fundamental units in some real quadratic fields and the* e_i *are positive integers.*

LEMMA 3.5 *Suppose that* $[\mathfrak{q}]$ *is in the principal genus. Then*

$$\left(\frac{G_{pqr}\,G_{pr/q}}{G_{pq/r}\,G_{qr/p}}\right)^v = \varepsilon_1^{e_1/2}\varepsilon_2^{e_2/2} \quad \textit{and} \quad (G_{pqr}\,G_{qr/p}\,G_{pq/r}\,G_{pr/q})^v = \varepsilon_3^{e_3/2}\varepsilon_4^{e_4/2},$$

where the ε_i *are fundamental units in some real quadratic fields and the* e_i *are positive integers.*

LEMMA 3.6. *Suppose that* $[\mathfrak{t}\mathfrak{p}\mathfrak{q}]$ *is in the principal genus. Then*

$$\left(\frac{G_{pqr}\,G_{pr/q}}{G_{pq/r}\,G_{qr/p}}\right)^v = \varepsilon_1^{e_1/2}\varepsilon_2^{e_2/2} \quad \textit{and} \quad \left(\frac{G_{pqr}\,G_{qr/p}}{G_{pq/r}\,G_{pr/q}}\right)^v = \varepsilon_3^{e_3/2}\varepsilon_4^{e_4/2},$$

where the ε_i *are fundamental units in some real quadratic fields and the* e_i *are positive integers.*

LEMMA 3.7. *Suppose that* $[\mathfrak{pq}]$ *is in the principal genus. Then*

$$\left(\frac{G_{pqr}G_{pq/r}}{G_{pr/q}G_{qr/p}}\right)^{\nu} = \varepsilon_1^{e_1/2}\varepsilon_2^{e_2/2} \quad \textit{and} \quad (G_{pqr}G_{qr/p}G_{pq/r}G_{pr/q})^{\nu} = \varepsilon_3^{e_3/2}\varepsilon_4^{e_4/2},$$

where the ε_i *are fundamental units in some real quadratic fields and the* e_i *are positive integers.*

REMARKS. Lemmas 3.2–3.7 are natural extensions of theorems discovered by L. Kronecker [**19**, pp. 525–547], K. G. Ramanathan [**12**] and L.-C. Zhang [**3**, Theorems 3.1, 3.2; **21**]. These authors have used their results explicitly to evaluate certain infinite products related to either the class invariant G_n or the Rogers–Ramanujan continued fraction. Here, we shall use Lemmas 3.2–3.7 implicitly without having to determine the fundamental units involved.

From class field theory, we know that if H is a subgroup of C_K, then there exists an abelian and everywhere unramified extension $L|K$ such that

$$\mathrm{Gal}(K^{(1)}|K) \simeq H.$$

In particular, when $H = C_K^2$, the corresponding field $M|K$ is known as the *genus field* of K. One can show that M is the maximal unramified extension of K which is abelian over $\mathbb{Q}$ [**10**, p. 122].

Proof of Theorem 1.2. Let $[\mathfrak{a}_1]$ and $[\mathfrak{a}_2]$ be two ideal classes in $\mathfrak{G}$, and let $H = \langle[\mathfrak{a}_1],[\mathfrak{a}_2]\rangle$. From the previous paragraph, we know that there exists an abelian and everywhere unramified extension $L|K$ such that $\mathrm{Gal}(K^{(1)}|K) \simeq H$. In fact, from the isomorphism of Lemma 2.1, we find that $L = \mathrm{Fix}(\langle\sigma_{\mathfrak{a}_1},\sigma_{\mathfrak{a}_2}\rangle)$. Since $\mathrm{Gal}(K^{(1)}|K) \simeq \mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_4$, the group $\mathrm{Gal}(L|K)$ is isomorphic to either $\mathbb{Z}_2\oplus\mathbb{Z}_2$ or $\mathbb{Z}_4$. In the former case, L is a subfield of M, the genus field of K, and one of the ideal classes in H must be in the principal genus. As for the latter case, $\mathrm{Gal}(L|\mathbb{Q}) \simeq D_8$, where D_8 is the dihedral group of eight elements, since L is generalized dihedral over $\mathbb{Q}$ [**10**, p. 191]. Hence, $\mathrm{Gal}(L|\mathbb{Q})$ is non-abelian.

Now, let

$$\alpha = \eta+\eta^{-1} \quad \text{where } \eta = \left(\frac{G_{pqr}G_{pq/r}}{G_{pr/q}G_{qr/p}}\right)^{\nu}.$$

Then, by assumption, we have

$$2(\eta+\eta^{-1}) = a_1 + a_2\sqrt{m}. \tag{3.5}$$

Note that $\sigma_{\mathfrak{pq}}$, $\sigma_{\mathfrak{tp}}$ and $\sigma_{\mathfrak{tq}}$ fix η and $\sigma_{\mathfrak{p}}(\eta) = \eta^{-1}$. Therefore, the field $L := \mathrm{Fix}(\langle\sigma_{\mathfrak{tp}},\sigma_{\mathfrak{tq}}\rangle)$ is of degree 4 over K.

Suppose that $\mathrm{Gal}(L|K) \simeq \mathbb{Z}_2\oplus\mathbb{Z}_2$. Then, from the beginning of our proof, we conclude that either $[\mathfrak{tp}]$, $[\mathfrak{tq}]$ or $[\mathfrak{pq}]$ is in the principal genus. Using either Lemma 3.2, Lemma 3.4 or Lemma 3.7, which correspond to each of the possibilities, we deduce that

$$\eta = \varepsilon^{e/2}\varepsilon'^{e'/2},$$

where ε and ε' are fundamental units in certain quadratic fields, and $e, e' \in \mathbb{N}$. It is known that a fundamental unit of a real quadratic field takes the form $u+v\sqrt{d}$ with

$u, v > 0$ [**6**, p. 133]. Furthermore, if $\sqrt{u+v\sqrt{d}} = u'\sqrt{d_1}+v'\sqrt{d_2}$, then $u', v' \geqslant 0$. Collecting these observations, we deduce that η is of the form $u_1+u_2\sqrt{l_1}+u_3\sqrt{l_2}+u_4\sqrt{l_1 l_2}$, where $u_i \geqslant 0$ for each i. Hence, if

$$2\alpha = 2(\eta+\eta^{-1}) = a_1+a_2\sqrt{m},$$

then a_1 and a_2 must be positive integers since the u_i are positive.

Next, suppose that $\mathrm{Gal}(L|K) \simeq \mathbb{Z}_4$. Recall that $\mathrm{Gal}(L|\mathbb{Q}) \simeq D_8$ is non-abelian. We claim that there exists $\sigma \in \mathrm{Gal}(L|K)$ such that $\sigma(\eta)$ is complex. Suppose that the contrary holds. Then $L \cap \mathbb{R} = \mathbb{Q}(\eta)$ would be Galois over $\mathbb{Q}$, and hence $\mathrm{Gal}(L|\mathbb{Q}(\eta))$ is a normal subgroup of $\mathrm{Gal}(L|\mathbb{Q})$. On the other hand, $\mathrm{Gal}(\mathbb{Q}(\eta)|\mathbb{Q}) \simeq \mathrm{Gal}(L|K)$, and is a normal subgroup of $\mathrm{Gal}(L|\mathbb{Q})$ [**10**, p. 191]. Hence, $\mathrm{Gal}(L|\mathbb{Q})$ is isomorphic to the direct sum of $\mathrm{Gal}(L|\mathbb{Q}(\eta))$ and $\mathrm{Gal}(\mathbb{Q}(\eta)|\mathbb{Q})$ and is therefore an abelian group. This contradicts our initial assumption.

Next, we shall show that $\sigma(\sqrt{m}) = -\sqrt{m}$. Suppose that the contrary holds. Then $\sigma(\eta)+\sigma(\eta)^{-1} = \eta+\eta^{-1}$, and therefore $\sigma(\eta)$ is equal to η or η^{-1}. This shows that $\sigma(\eta)$ is real, which contradicts our choice of σ. Now, applying σ to (3.5), we deduce that

$$2(\sigma(\eta)+\sigma(\eta)^{-1}) = a_1-a_2\sqrt{m}. \tag{3.6}$$

From (3.5), (3.6) and the fact that $\sigma(\eta)$ is complex, we find that

$$(a_1+a_2\sqrt{m})^2 \geqslant 16 \quad \text{and} \quad (a_1-a_2\sqrt{m})^2 < 16.$$

This implies that $4a_1 a_2\sqrt{m} > 0$. Since $\eta > 0$, we deduce that a_1 and a_2 are positive. The integrality of a_1 and a_2 follows easily from Theorem 1.1.

In a similar way, we can show that b_1, b_2, c_1, c_2, d_1, and d_2 are positive integers. Of course, we have to use Lemmas 3.2–3.7, as appropriate.

REMARK. The argument given here for the case when $\mathrm{Gal}(L|\mathbb{Q})$ is non-abelian is due to Weber [**10**, p. 269].

4. *Some values of* G_n

In this section, we list explicit values of G_n which are mentioned in the Introduction. Our computations are done with the aid of *Mathematica* and MAPLE V.

$$G_{285}^2 = \sqrt{\frac{\sqrt{5}+\sqrt{3}}{\sqrt{2}}}\sqrt{\frac{\sqrt{5}+1}{2}}\left(\frac{\sqrt{19}+\sqrt{15}}{2}\right)^{1/4}(2\sqrt{5}+\sqrt{19})^{1/6}$$
$$\times\left(\sqrt{\frac{8+\sqrt{19}}{4}}+\sqrt{\frac{4+\sqrt{19}}{4}}\right)\left(\sqrt{\frac{11+2\sqrt{19}}{4}}+\sqrt{\frac{7+2\sqrt{19}}{4}}\right)$$

$$G_{429}^2 = \sqrt{\frac{\sqrt{13}+3}{2}}\sqrt{\sqrt{13}+2\sqrt{3}}\left(\frac{\sqrt{13}+\sqrt{11}}{\sqrt{2}}\right)^{1/6}\sqrt{73\sqrt{143}+504\sqrt{3}}$$
$$\times\left(\sqrt{\frac{5+2\sqrt{3}}{4}}+\sqrt{\frac{1+2\sqrt{3}}{4}}\right)\left(\sqrt{\frac{16+9\sqrt{3}}{4}}+\sqrt{\frac{12+9\sqrt{3}}{4}}\right)$$

$$G_{561}^2=\left(\frac{99\sqrt{51}+707}{\sqrt{2}}\right)^{1/6}\left(\frac{3\sqrt{187}+41}{\sqrt{2}}\right)^{1/6}\sqrt{\frac{\sqrt{3}+1}{\sqrt{2}}}\sqrt{\frac{\sqrt{11}+3}{\sqrt{2}}}$$
$$\times\left(\sqrt{\frac{13+\sqrt{33}}{8}}+\sqrt{\frac{5+\sqrt{33}}{8}}\right)\left(\sqrt{\frac{53+9\sqrt{33}}{8}}+\sqrt{\frac{45+9\sqrt{33}}{8}}\right)$$

$$G_{609}^2=\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\sqrt{3\sqrt{3}+2\sqrt{7}}(2\sqrt{7}+\sqrt{29})^{1/6}(13\sqrt{29}+70)^{1/6}$$
$$\times\left(\sqrt{\frac{6+\sqrt{7}}{4}}+\sqrt{\frac{2+\sqrt{7}}{4}}\right)\left(\sqrt{\frac{38+13\sqrt{7}}{4}}+\sqrt{\frac{34+13\sqrt{7}}{4}}\right)$$

$$G_{645}^2=\left(\frac{3\sqrt{5}+\sqrt{43}}{\sqrt{2}}\right)^{1/6}\left(\frac{9\sqrt{3}+7\sqrt{5}}{\sqrt{2}}\right)^{1/6}(17\sqrt{5}+38)^{1/6}(320\sqrt{5}+63\sqrt{129})^{1/12}$$
$$\times\left(\sqrt{\frac{39+3\sqrt{129}}{8}}+\sqrt{\frac{31+3\sqrt{129}}{8}}\right)$$
$$\times\left(\sqrt{\frac{107+9\sqrt{129}}{8}}+\sqrt{\frac{99+9\sqrt{129}}{8}}\right)$$

$$G_{805}^2=\sqrt{\frac{\sqrt{23}+5}{\sqrt{2}}}\sqrt{\frac{\sqrt{7}+3}{\sqrt{2}}}\left(\frac{21\sqrt{5}+47}{2}\right)^{1/4}\left(\frac{17\sqrt{5}+3\sqrt{161}}{2}\right)^{1/4}$$
$$\times\left(\sqrt{\frac{91+7\sqrt{161}}{8}}+\sqrt{\frac{83+7\sqrt{161}}{8}}\right)$$
$$\times\left(\sqrt{\frac{19+\sqrt{161}}{8}}+\sqrt{\frac{11+\sqrt{161}}{8}}\right)$$

$$G_{957}^2=\left(\frac{1083\sqrt{11}+667\sqrt{29}}{\sqrt{2}}\right)^{1/6}\left(\frac{55\sqrt{29}+171\sqrt{3}}{\sqrt{2}}\right)^{1/6}(13\sqrt{29}+70)^{1/6}$$
$$\times(16\sqrt{29}+15\sqrt{33})^{1/12}\left(\sqrt{\frac{23+3\sqrt{33}}{8}}+\sqrt{\frac{15+3\sqrt{33}}{8}}\right)$$
$$\times\left(\sqrt{\frac{195+33\sqrt{33}}{8}}+\sqrt{\frac{187+33\sqrt{33}}{8}}\right)$$

$$G_{1005}=\left(\frac{7\sqrt{5}+9\sqrt{3}}{\sqrt{2}}\right)^{1/6}\left(\frac{11\sqrt{5}+3\sqrt{67}}{\sqrt{2}}\right)^{1/6}(17\sqrt{5}+38)^{1/6}$$
$$\times(17176\sqrt{5}+2709\sqrt{201})^{1/12}$$
$$\times\left(\sqrt{\frac{43+3\sqrt{201}}{4}}+\sqrt{\frac{39+3\sqrt{201}}{4}}\right)$$
$$\times\left(\sqrt{\frac{387+27\sqrt{201}}{8}}+\sqrt{\frac{379+27\sqrt{201}}{8}}\right)$$

$$G_{1045}^2 = \sqrt{\frac{3\sqrt{11}+\sqrt{95}}{2}}\,\sqrt{2\sqrt{5}+\sqrt{19}}\,\sqrt{\frac{5\sqrt{5}+11}{2}}\,\sqrt{3\sqrt{5}+2\sqrt{11}}$$
$$\times\left(\sqrt{\frac{20+5\sqrt{11}}{4}}+\sqrt{\frac{16+5\sqrt{11}}{4}}\right)\left(\sqrt{\frac{7+2\sqrt{11}}{2}}+\sqrt{\frac{5+2\sqrt{11}}{2}}\right)$$

$$G_{1065}^2 = \sqrt{\sqrt{3}+2}\,\sqrt{\frac{\sqrt{5}+3}{2}}\,(82\sqrt{71}+309\sqrt{5})^{1/6}\,(37\sqrt{71}+180\sqrt{3})^{1/6}$$
$$\times\left(\sqrt{\frac{17+4\sqrt{15}}{4}}+\sqrt{\frac{13+4\sqrt{15}}{4}}\right)$$
$$\times\left(\sqrt{\frac{235+60\sqrt{15}}{4}}+\sqrt{\frac{231+60\sqrt{15}}{4}}\right)$$

$$G_{1105}^2 = \left(\sqrt{\frac{31+\sqrt{1105}}{8}}+\sqrt{\frac{23+\sqrt{1105}}{8}}\right)\left(\sqrt{\frac{41+\sqrt{1105}}{8}}\right.$$
$$\left.+\sqrt{\frac{33+\sqrt{1105}}{8}}\right)\left(\sqrt{\frac{109+3\sqrt{1105}}{8}}+\sqrt{\frac{101+3\sqrt{1105}}{8}}\right)$$
$$\times\left(\sqrt{\frac{173+5\sqrt{1105}}{8}}+\sqrt{\frac{165+5\sqrt{1105}}{8}}\right)$$

$$G_{1113}^2 = (\sqrt{3}+2)\,\sqrt{3\sqrt{3}+2\sqrt{7}}\,(25\sqrt{53}+182)^{1/6}\,(11\sqrt{7}+4\sqrt{53})^{1/6}$$
$$\times\left(\sqrt{\frac{9+2\sqrt{7}}{4}}+\sqrt{\frac{5+2\sqrt{7}}{4}}\right)\left(\sqrt{\frac{197+74\sqrt{7}}{4}}+\sqrt{\frac{193+74\sqrt{7}}{4}}\right)$$

$$G_{1185}^2 = (\sqrt{3}+2)\,(17\sqrt{5}+38)^{1/6}\,(4\sqrt{5}+\sqrt{79})^{1/6}\,\sqrt{\frac{5\sqrt{3}+\sqrt{79}}{2}}$$
$$\times\left(\sqrt{\frac{21+2\sqrt{79}}{4}}+\sqrt{\frac{17+2\sqrt{79}}{4}}\right)\left(\sqrt{\frac{341+38\sqrt{79}}{4}}\right.$$
$$\left.+\sqrt{\frac{337+38\sqrt{79}}{4}}\right)$$

$$G_{1605}^2 = (305\sqrt{5}+682)^{1/6}\,(1281\sqrt{321}+10264\sqrt{5})^{1/12}$$
$$\times\left(\frac{3409\sqrt{5}+4401\sqrt{3}}{\sqrt{2}}\right)^{1/6}\left(\frac{569\sqrt{5}+123\sqrt{107}}{\sqrt{2}}\right)^{1/6}\left(\sqrt{\frac{19+\sqrt{321}}{8}}\right.$$
$$\left.+\sqrt{\frac{11+\sqrt{321}}{8}}\right)\left(\sqrt{\frac{2207+123\sqrt{321}}{8}}+\sqrt{\frac{2199+123\sqrt{321}}{8}}\right)$$

$$G_{1653}^2 = (4984\sqrt{29}+3555\sqrt{57})^{1/12}\,(13\sqrt{29}+70)^{1/6}\left(\frac{21\sqrt{19}+17\sqrt{29}}{\sqrt{2}}\right)^{1/6}$$
$$\times\left(\frac{171\sqrt{3}+55\sqrt{29}}{\sqrt{2}}\right)^{1/6}\left(\sqrt{\frac{755+99\sqrt{57}}{8}}+\sqrt{\frac{747+99\sqrt{57}}{8}}\right)$$
$$\times\left(\sqrt{\frac{975+129\sqrt{57}}{8}}+\sqrt{\frac{967+129\sqrt{57}}{8}}\right)$$

$$\begin{aligned}
G_{1705}^2 = {} & \sqrt{\frac{5\sqrt{11}+3\sqrt{31}}{2}}\left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)\sqrt{5\sqrt{5}+2\sqrt{31}}\sqrt{\frac{13\sqrt{5}+29}{2}} \\
& \times\left(\sqrt{\frac{30+5\sqrt{31}}{4}}+\sqrt{\frac{26+5\sqrt{31}}{4}}\right) \\
& \times\left(\sqrt{\frac{54+9\sqrt{31}}{4}}+\sqrt{\frac{50+9\sqrt{31}}{4}}\right)
\end{aligned}$$

$$\begin{aligned}
G_{1885}^2 = {} & \left(\frac{\sqrt{13}+3}{2}\right)\left(\frac{\sqrt{5}+3}{2}\right)\left(\frac{5\sqrt{29}+27}{2}\right)(3\sqrt{29}+2\sqrt{65}) \\
& \times\left(\sqrt{\frac{242+30\sqrt{65}}{4}}+\sqrt{\frac{238+30\sqrt{65}}{4}}\right) \\
& \times\left(\sqrt{\frac{121+15\sqrt{65}}{8}}+\sqrt{\frac{113+15\sqrt{65}}{8}}\right)
\end{aligned}$$

$$\begin{aligned}
G_{2013}^2 = {} & (5040\sqrt{3}+337\sqrt{671})^{1/12}\left(\frac{31\sqrt{61}+73\sqrt{11}}{\sqrt{2}}\right)^{1/6}\sqrt{\frac{5\sqrt{61}+39}{2}} \\
& \times\sqrt{2\sqrt{61}+9\sqrt{3}}\left(\sqrt{\frac{38+20\sqrt{3}}{4}}+\sqrt{\frac{34+20\sqrt{3}}{4}}\right) \\
& \times\left(\sqrt{\frac{1303+750\sqrt{3}}{4}}+\sqrt{\frac{1299+750\sqrt{3}}{4}}\right)
\end{aligned}$$

$$\begin{aligned}
G_{2233}^2 = {} & \sqrt{5\sqrt{7}+4\sqrt{11}}\sqrt{\sqrt{29}+2\sqrt{7}}\left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)\left(\frac{\sqrt{29}+5}{2}\right) \\
& \times\left(\sqrt{\frac{73+4\sqrt{319}}{4}}+\sqrt{\frac{69+4\sqrt{319}}{4}}\right) \\
& \times\left(\sqrt{\frac{643+36\sqrt{319}}{4}}+\sqrt{\frac{638+36\sqrt{319}}{4}}\right)
\end{aligned}$$

$$\begin{aligned}
G_{2737}^2 = {} & \left(\sqrt{\frac{325+25\sqrt{161}}{8}}+\sqrt{\frac{317+25\sqrt{161}}{8}}\right)\left(\sqrt{\frac{397+31\sqrt{161}}{8}}\right. \\
& \left.+\sqrt{\frac{389+31\sqrt{161}}{8}}\right)\left(\sqrt{\frac{623+49\sqrt{161}}{4}}+\sqrt{\frac{619+49\sqrt{161}}{4}}\right) \\
& \times\left(\sqrt{\frac{2533+199\sqrt{161}}{8}}+\sqrt{\frac{2525+199\sqrt{161}}{8}}\right)
\end{aligned}$$

5. *Values of* G_n *with* $C_{\mathbb{Q}(\sqrt{-n})} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$

From Buell's table [7], we find that for $n < 2^{63}$, $C_{\mathbb{Q}(\sqrt{-n})} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$ when $n = 1785$, 2145, 3045, 3705, 4305, 4845, and 5005. Note that $n = pqrs$ where p, q, r, and s are distinct primes. To compute the G_{pqrs} for these seven n, we consider the eight expressions analogous to α, β, γ and δ defined in Theorem 1.1. One such example is

$$\left(\frac{G_{pqrs}G_{pq/rs}G_{pr/qs}G_{ps/qr}}{G_{pqr/s}G_{prs/q}G_{pqs/r}G_{qrs/p}}\right)^v + \left(\frac{G_{pqr/s}G_{prs/q}G_{pqs/r}G_{qrs/p}}{G_{pqrs}G_{pq/rs}G_{pr/qs}G_{ps/qr}}\right)^v. \tag{5.1}$$

Using similar reasonings as in Theorems 1.1 and 1.2, we conclude that the eight expressions, one of which is given in (5.1), are of the form $a+b\sqrt{m}$ with $a, b > 0$. We then proceed to compute G_{pqrs}. The following list contains the remaining six values of G_n:

$$\begin{aligned}
G_{2145}^4 = &\left(\frac{23\sqrt{143}+123\sqrt{5}}{\sqrt{2}}\right)^{1/3}(5\sqrt{13}+18)^{1/3}\left(\frac{\sqrt{15}+\sqrt{13}}{\sqrt{2}}\right)\left(\frac{7\sqrt{3}+\sqrt{143}}{2}\right)\\
&\times\left(\frac{\sqrt{5}+3}{2}\right)\left(\frac{\sqrt{11}+3}{\sqrt{2}}\right)\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\left(\frac{\sqrt{15}+\sqrt{11}}{2}\right)\left(\sqrt{\frac{8+\sqrt{55}}{4}}\right.\\
&\left.+\sqrt{\frac{4+\sqrt{55}}{4}}\right)\left(\sqrt{\frac{8212+1107\sqrt{55}}{4}}+\sqrt{\frac{8208+1107\sqrt{55}}{4}}\right)\\
&\times\left(\sqrt{\frac{62+8\sqrt{55}}{4}}+\sqrt{\frac{58+8\sqrt{55}}{4}}\right)\\
&\times\left(\sqrt{\frac{358+48\sqrt{55}}{4}}+\sqrt{\frac{354+48\sqrt{55}}{4}}\right)
\end{aligned}$$

$$\begin{aligned}
G_{3045}^4 = &(63\sqrt{15}+244)^{1/3}(78\sqrt{29}+71\sqrt{35})^{1/3}\sqrt{\frac{11\sqrt{5}+\sqrt{609}}{2}}\left(\frac{\sqrt{5}+3}{2}\right)\\
&\times\left(\frac{\sqrt{21}+5}{2}\right)\left(\frac{\sqrt{29}+5}{2}\right)\left(\frac{3\sqrt{3}+\sqrt{29}}{\sqrt{2}}\right)\left(\frac{\sqrt{7}+3}{\sqrt{2}}\right)\\
&\times\left(\sqrt{\frac{25+\sqrt{609}}{8}}+\sqrt{\frac{17+\sqrt{609}}{8}}\right)\\
&\times\left(\sqrt{\frac{8292+336\sqrt{609}}{4}}+\sqrt{\frac{8288+336\sqrt{609}}{4}}\right)\\
&\times\left(\sqrt{\frac{50+2\sqrt{609}}{4}}+\sqrt{\frac{46+2\sqrt{609}}{4}}\right)\\
&\times\left(\sqrt{\frac{164733+6675\sqrt{609}}{8}}+\sqrt{\frac{164725+6675\sqrt{609}}{8}}\right)
\end{aligned}$$

$$\begin{aligned}
G_{3705}^4 = &(8\sqrt{19}+9\sqrt{15})^{1/3}\left(\frac{\sqrt{5}+\sqrt{247}}{\sqrt{2}}\right)^{1/3}\left(\frac{9\sqrt{3}+\sqrt{247}}{2}\right)\left(\frac{3\sqrt{19}+13}{\sqrt{2}}\right)\\
&\times(\sqrt{5}+2)\left(\frac{\sqrt{15}+\sqrt{13}}{\sqrt{2}}\right)\left(\frac{3\sqrt{13}+11}{2}\right)\left(\frac{3\sqrt{3}+5}{\sqrt{2}}\right)\\
&\times\left(\sqrt{\frac{344+55\sqrt{39}}{4}}+\sqrt{\frac{340+55\sqrt{39}}{4}}\right)
\end{aligned}$$

$$\times\left(\sqrt{\frac{1950+312\sqrt{39}}{4}}+\sqrt{\frac{1946+312\sqrt{39}}{4}}\right)$$

$$\times\left(\sqrt{\frac{20+3\sqrt{39}}{4}}+\sqrt{\frac{16+3\sqrt{39}}{4}}\right)$$

$$\times\left(\sqrt{\frac{177552+28431\sqrt{39}}{4}}+\sqrt{\frac{177548+28431\sqrt{39}}{4}}\right)$$

$$G_{4305}^4=\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)\left(\frac{\sqrt{7}+\sqrt{5}}{\sqrt{2}}\right)\left(\frac{243\sqrt{123}+2695}{\sqrt{2}}\right)\left(\frac{59\sqrt{289}+447\sqrt{5}}{\sqrt{2}}\right)$$

$$\times(\sqrt{5}+2)\left(\frac{\sqrt{21}+5}{2}\right)\left(\frac{7\sqrt{21}+5\sqrt{41}}{2}\right)\left(\frac{3\sqrt{205}+43}{2}\right)$$

$$\times\left(\sqrt{\frac{28437+2775\sqrt{105}}{2}}+\sqrt{\frac{28435+2775\sqrt{105}}{3}}\right)$$

$$\times\left(\sqrt{\frac{217+21\sqrt{105}}{8}}+\sqrt{\frac{209+21\sqrt{105}}{8}}\right)$$

$$\times\left(\sqrt{369+36\sqrt{105}}+\sqrt{368+3\sqrt{105}}\right)$$

$$\times\left(\sqrt{\frac{565+55\sqrt{105}}{2}}+\sqrt{\frac{563+55\sqrt{105}}{2}}\right)$$

$$G_{4845}^4=(139\sqrt{85}+294\sqrt{19})^{1/3}(41\sqrt{85}+378)^{1/3}\sqrt{\frac{3\sqrt{255}+11\sqrt{19}}{2}}\left(\frac{\sqrt{17}+\sqrt{15}}{\sqrt{2}}\right)$$

$$\times(\sqrt{5}+2)(2\sqrt{5}+\sqrt{19})(\sqrt{15}+4)\left(\frac{\sqrt{285}+17}{2}\right)$$

$$\times\left(\sqrt{\frac{35+8\sqrt{19}}{4}}+\sqrt{\frac{31+8\sqrt{19}}{4}}\right)$$

$$\times\left(\sqrt{\frac{267289+61320\sqrt{19}}{4}}+\sqrt{\frac{267285+61320\sqrt{19}}{4}}\right)$$

$$\times\left(\sqrt{\frac{125+28\sqrt{19}}{4}}+\sqrt{\frac{121+28\sqrt{19}}{4}}\right)$$

$$\times\left(\sqrt{268753+61656\sqrt{19}}+\sqrt{268752+61656\sqrt{19}}\right)$$

$$G_{5005}^4=\sqrt{\frac{9\sqrt{55}+7\sqrt{91}}{2}}(3\sqrt{5}+2\sqrt{11})\left(\frac{\sqrt{13}+3}{2}\right)\left(\frac{\sqrt{7}+3}{\sqrt{2}}\right)$$

$$\times\left(\frac{3\sqrt{7}+\sqrt{65}}{\sqrt{2}}\right)\left(\frac{5\sqrt{5}+11}{2}\right)\left(\frac{\sqrt{13}+\sqrt{11}}{\sqrt{2}}\right)\left(\frac{\sqrt{77}+9}{2}\right)$$

$$\times\left(\sqrt{\frac{1337+140\sqrt{91}}{2}}+\sqrt{\frac{1335+140\sqrt{91}}{2}}\right)$$

$$\times\left(\sqrt{\frac{24202+2537\sqrt{91}}{4}}+\sqrt{\frac{24198+2537\sqrt{91}}{4}}\right)$$
$$\times\left(\sqrt{\frac{2110+221\sqrt{91}}{4}}+\sqrt{\frac{2106+221\sqrt{91}}{4}}\right)$$
$$\times\left(\sqrt{\frac{139277+14600\sqrt{91}}{2}}+\sqrt{\frac{139275+14600\sqrt{91}}{2}}\right).$$

REMARK. It is interesting to note that our methods can be used to compute G_n whenever the class groups of $\mathbb{Q}(\sqrt{-n})$ has one class per genus. Assuming the generalized Riemann hypothesis, P. J. Weinberger [**20**] showed that $Q(\sqrt{-1365})$ is the only imaginary quadratic field with $C_{\mathbb{Q}(\sqrt{-1365})} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. To compute G_{1365}, we simply follow the computations of G_n in this section. The only difference is that we obtain rational integers for expressions such as (5.1) instead of real quadratic numbers. Of course, the above value can be obtained directly from Kronecker's Limit Formula.

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