NOTES ON RAMANUJAN'S SINGULAR MODULI

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1. Introduction

Singular moduli arise in the calculation of class invariants, and so we first define the class invariants of Ramanujan and Weber. Set

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1,$$

and

$$\chi(q) = (-q;q^2)_\infty. \tag{1.1}$$

If

$$q = \exp(-\pi\sqrt{n}), \tag{1.2}$$

where n is a positive integer, the *class invariants* G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \qquad \text{and} \qquad g_n := 2^{-1/4}q^{-1/24}\chi(-q). \tag{1.3}$$

In the notation of Weber [8], $G_n =: 2^{-1/4}f(\sqrt{-n})$ and $g_n =: 2^{-1/4}f_1(\sqrt{-n})$.

As usual, in the theory of elliptic functions, let $k := k(q)$, $0<k<1$, denote the modulus. The singular modulus k_n is defined by $k_n := k(e^{-\pi\sqrt{n}})$, where n is a natural number. Following Ramanujan, set $\alpha_n = k_n^2$.

The algebraic natures of G_n, g_n, and α_n are described in the following theorem of Chan and S.–S. Huang [4].

Theorem 1.
(a) If $n \equiv 1 \pmod 4$, then G_n and $2\alpha_n$ are units.
(b) If $n \equiv 3 \pmod 8$, then $2^{-1/12}G_n$ and $2^2\alpha_n$ are units.
(c) If $n \equiv 7 \pmod 8$, then $2^{-1/4}G_n$ and $2^4\alpha_n$ are units.
(d) If $n \equiv 2 \pmod 4$, then g_n and α_n are units.

Lastly, we define a modular equation in the sense of Ramanujan. Let ${}_2F_1(\frac12,\frac12;1;x)$ be the ordinary or Gaussian hypergeometric function, which, for brevity, we denote by ${}_2F_1(x)$. Suppose that, for some positive integer n and for $\alpha = k^2$ and $\beta = \ell^2$, where k and ℓ are moduli,

$$n\frac{{}_2F_1(1-\alpha)}{{}_2F_1(\alpha)} = \frac{{}_2F_1(1-\beta)}{{}_2F_1(\beta)}. \tag{1.4}$$

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Then a *modular equation of degree n* is a relation between α and β which is induced by (1.4). We often say that β has degree n over α. If, as customary, in the theory of elliptic functions,

$$q := \exp(-\pi \, {}_2F_1(1-\alpha)/{}_2F_1(\alpha)) =: F(\alpha), \tag{1.5}$$

then [1, p. 124] $\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}$ and $\chi(-q) = 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}$. It follows from (1.1), (1.3), and (1.5) that

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24} \qquad \text{and} \qquad g_n = \left\{4\alpha_n(1-\alpha_n)^{-2}\right\}^{-1/24}. \tag{1.6}$$

Thus, if G_n or g_n is known, then α_n can be easily determined from (1.6) by solving a quadratic equation. However, usually, the expression for α_n that one obtains in this manner is not very attractive and does not reflect the algebraic nature of α_n described in Theorem 1. Thus, further algorithms which more readily yield elegant representations of α_n are highly desirable.

In his first notebook [7], Ramanujan offers without proofs over 30 values for α_n and briefly indicates some formulas which are useful in calculating α_n, when n is even. The authors and L.–C. Zhang [3] proved these formulas of Ramanujan, developed some new formulas for calculating α_n when n is odd, and consequently established all of Ramanujan's values for α_n.

Ramanujan obscurely describes two further methods for calculating α_n in his first notebook [7].

In the first, Ramanujan indicates that α_{2n} may be calculated by solving a certain type of modular equation of degree n. For several prime values of n, the desired type of modular equation exists; many of these modular equations can be found in Ramanujan's notebooks and are proved in [1]. This very novel method is described in Section 2, and it is the only known method that does not require *a priori* the value of g_{2n}. Thus, the method is a new, valuable tool in the computation of α_{2n}.

In the second, Ramanujan discloses a method for determining α_{3n} arising from the definition of a modular equation of degree n. In Section 3 we give a rigorous formulation of this formula and prove it by using a device the authors and Zhang introduced in [2] to calculate certain class invariants.

Lastly, in Section 4, we follow our methods in Section 3 and derive a similar formula for α_{5n}. Although we utilize results of Ramanujan from his notebooks in our proof, this formula was not given by Ramanujan.

2. The determination of α_{2n} from modular equations

In the middle of page 292 in his first notebook, Ramanujan claims that, "Changing β to $4B/(1+B)^2$ and α to $1-B^2$ we get an equation in $4B(1-B)/(1+B)$ and the value of B^2 is for $e^{-\pi\sqrt{2n}}$." We now state and prove a rigorous formulation of this assertion.

Theorem 2. *Let β have degree n over α, and suppose α and β are related by a modular equation of the form*

$$F\left((\alpha\beta)^r, \{(1-\alpha)(1-\beta)\}^r\right) = 0, \tag{2.1}$$

for some polynomial F and some positive rational number r. If we replace α by $1-x^2$ and β by $4x/(1+x)^2$, then (2.1) becomes an equation of the form

$$G(x) := g\left(\{4x(1-x)/(1+x)\}^r\right) = 0,$$

for some polynomial g. Furthermore, $x = \sqrt{\alpha_{2n}}$ is a root of this equation.

Proof. Under the designated substitutions,

$$\begin{aligned} F\left((\alpha\beta)^r, \{(1-\alpha)(1-\beta)\}^r\right) &= F\left(\left((1-x^2)\frac{4x}{(1+x)^2}\right)^r, \left(x^2\left(1-\frac{4x}{(1+x)^2}\right)\right)^r\right) \\ &= F\left(\left(\frac{4x(1-x)}{1+x}\right)^r, \frac{1}{4^{2r}}\left(\frac{4x(1-x)}{1+x}\right)^{2r}\right) \\ &= 0. \end{aligned}$$

Hence, the first part of Theorem 2 follows.

Next, setting $\beta = 4x/(1+x)^2$, we find that

$$\frac{{}_2F_1(1-\beta)}{{}_2F_1(\beta)} = \frac{{}_2F_1\left(1-\dfrac{4x}{(1+x)^2}\right)}{{}_2F_1\left(\dfrac{4x}{(1+x)^2}\right)} = \frac{1}{2}\frac{{}_2F_1(1-x^2)}{{}_2F_1(x^2)}, \tag{2.2}$$

by a fundamental transformation for $F(x)$ [1, p. 93], which actually arises from a special case of Pfaff's transformation. With α replaced by $1-x^2$, we find from (1.4) and (2.2) that

$$n\frac{{}_2F_1(x^2)}{{}_2F_1(1-x^2)} = \frac{1}{2}\frac{{}_2F_1(1-x^2)}{{}_2F_1(x^2)}.$$

Therefore,

$$\frac{{}_2F_1(1-x^2)}{{}_2F_1(x^2)} = \sqrt{2n}.$$

Now recalling the definition of a singular modulus and (1.5), we deduce that $x^2 = \alpha_{2n}$, and the proof is complete.

In each example below we use the following denesting theorem in our calculations. If $a^2 - db^2 = c^2$, then

$$\sqrt{a \pm b\sqrt{d}} = \sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}}. \tag{2.3}$$

Example 2.1. Let $n = 3$. Then

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1, \tag{2.4}$$

which is originally due to Legendre and was rediscovered by Ramanujan [1, pp. 230, 232]. With Ramanujan's substitutions, (2.4) takes the form

$$u + \frac{1}{2}u^2 = 1, \tag{2.5}$$

where

$$u=\left(4x\frac{1-x}{1+x}\right)^{1/4}. \tag{2.6}$$

Solving (2.5), we find that $u=\sqrt{3}-1$. Then solving (2.6), we find that $x=2\sqrt{3}-3-2\sqrt{2}+\sqrt{6}$. Using two different modes of calculation, we find that

$$\alpha_6=x^2=(2-\sqrt{3})^2(\sqrt{3}-\sqrt{2})^2=\frac{2\sqrt{3}+\sqrt{6}-3-2\sqrt{2}}{2\sqrt{3}+\sqrt{6}+3+2\sqrt{2}}.$$

It should be remarked that there are two roots of (2.6), namely, $\sqrt{\alpha_6}$ and $\sqrt{\alpha_{2/3}}$. For if we set $\alpha=4x/(1+x)^2$ and $\beta=1-x^2$ at the beginning of our proof of Theorem 2, we find that x still satisfies (2.1). However, in this case,

$$\frac{{}_2F_1(1-x^2)}{{}_2F_1(x^2)}=\sqrt{\frac{2}{n}}.$$

That we have selected the correct root can be verified by a numerical check.

The former representation for α_6, and two other representations as well, can be found in Ramanujan's first notebook, and they were proved by the authors and Zhang in [3, Theorem 2.1].

Example 2.2. Let $n=5$. Then we have Ramanujan's modular equation of degree 5 [1, p. 280],

$$(\alpha\beta)^{1/2}+\{(1-\alpha)(1-\beta)\}^{1/2}+2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}=1. \tag{2.7}$$

Using Ramanujan's substituions, we find that (2.7) can be put in the shape

$$u+\frac{1}{4}u^2+2u=\frac{1}{4}u^2+3u=1, \tag{2.8}$$

where

$$u=\left(4x\frac{1-x}{1+x}\right)^{1/2}. \tag{2.9}$$

Solving (2.8), we deduce that $u=-6+2\sqrt{10}$. Next, solving (2.9), we find that $x=3\sqrt{10}-9-4\sqrt{5}+6\sqrt{2}$. Lastly, by two distinct routes for calculation,

$$\alpha_{10}=x^2=(\sqrt{10}-3)^2(3-2\sqrt{2})^2=\frac{3\sqrt{10}+6\sqrt{2}-9-4\sqrt{5}}{3\sqrt{10}+6\sqrt{2}+9+4\sqrt{5}}.$$

The former representation was given by Ramanujan in his first notebook and proved by the authors and Zhang in [3, Theorem 2.1].

Example 2.3. Let $n=11$. The modular equation

$$(\alpha\beta)^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}+2\{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}=1 \tag{2.10}$$

is independently due to Schröter and Ramanujan [1, pp. 363–364]. With Ramanujan's substitutions, (2.10) can be put in the form

$$u^2 + 6u - 2 = 0, \tag{2.11}$$

where

$$u = \left(4x\frac{1-x}{1+x}\right)^{1/4}. \tag{2.12}$$

Solving (2.11), we find that $u = -3 + \sqrt{11}$. Then solving (2.12), we find that $x = 30\sqrt{11} - 99 - 70\sqrt{2} + 21\sqrt{22}$. Hence,

$$\alpha_{22} = x^2 = (10 - 3\sqrt{11})^2(3\sqrt{11} - 7\sqrt{2})^2 = \frac{30\sqrt{11} + 21\sqrt{22} - 99 - 70\sqrt{2}}{30\sqrt{11} + 21\sqrt{22} + 99 + 70\sqrt{2}}.$$

In his first notebook Ramanujan offered the former value, which was proved in [3, Theorem 2.1].

3. A formula for α_{3n}

Let q be given by (1.2), and suppose that β has degree n over α. Thus, (1.4) holds. Now suppose also that β has degree 3 over $1 - \alpha =: \alpha'$. Then, by (1.4),

$$3\frac{{}_2F_1(\alpha)}{{}_2F_1(1-\alpha)} = \frac{{}_2F_1(1-\beta)}{{}_2F_1(\beta)} = n\frac{{}_2F_1(1-\alpha)}{{}_2F_1(\alpha)}. \tag{3.1}$$

Hence,

$$\frac{{}_2F_1(\alpha)}{{}_2F_1(1-\alpha)} = \sqrt{\frac{n}{3}}, \tag{3.2}$$

and, from (3.1) and (3.2),

$$\frac{{}_2F_1(1-\beta)}{{}_2F_1(\beta)} = 3\sqrt{\frac{n}{3}} = \sqrt{3n}. \tag{3.3}$$

Next, from [1, p. 237], since β has degree 3 over α', we have the parametrizations

$$\alpha' = p\left(\frac{2+p}{1+2p}\right)^3 \qquad \text{and} \qquad \beta = p^3\left(\frac{2+p}{1+2p}\right), \tag{3.4}$$

where $0 < p < 1$. It follows from (3.4) that

$$(1-\alpha')\beta = \frac{(1-p)^3(1+p)}{(1+2p)^3}p^3\left(\frac{2+p}{1+2p}\right) = \left(p\frac{1-p}{1+2p}\right)^3\frac{(1+p)(2+p)}{1+2p} \tag{3.5}$$

and

$$(1-\beta)\alpha' = \frac{(1-p)(1+p)^3}{(1+2p)}p\left(\frac{2+p}{1+2p}\right)^3 = p\frac{1-p}{1+2p}\left(\frac{(1+p)(2+p)}{1+2p}\right)^3. \tag{3.6}$$

Next, set

$$2t = p\frac{1-p}{1+2p} \tag{3.7}$$

and observe that

$$2(1-t) = \frac{(1+p)(2+p)}{1+2p}. \tag{3.8}$$

It follows from (3.5)–(3.8) that

$$\alpha\beta = (1-\alpha^{'})\beta = 16t^3(1-t) \tag{3.9}$$

and

$$(1-\alpha)(1-\beta) = (1-\beta)\alpha^{'} = 16t(1-t)^3. \tag{3.10}$$

Now, set

$$k = 4t(1-t). \tag{3.11}$$

Observe from (3.9), (3.10), (1.6), (3.2), and (3.3) that

$$k = \left(2G_{n/3}^6 G_{3n}^6\right)^{-1}. \tag{3.12}$$

We determine β (α_{3n}) as a function of k. From (3.11), we find that, with no loss of generality in choosing the minus sign in the first equality below,

$$t = \frac{1-\sqrt{1-k}}{2} \qquad \text{and} \qquad 1-t = \frac{1+\sqrt{1-k}}{2}.$$

Thus, by (3.9) and (3.10), we find that

$$(1-\alpha^{'})\beta = (1-\sqrt{1-k})^3(1+\sqrt{1-k}) = k(1-\sqrt{1-k})^2 \tag{3.13}$$

and

$$(1-\beta)\alpha^{'} = (1-\sqrt{1-k})(1+\sqrt{1-k})^3 = k(1+\sqrt{1-k})^2. \tag{3.14}$$

Subtracting (3.13) from (3.14), we find that

$$\alpha^{'} = 4k\sqrt{1-k} + \beta. \tag{3.15}$$

Substituting (3.15) into (3.14), we deduce that

$$\beta^2 + \beta(4k\sqrt{1-k} - 1) + k(1-\sqrt{1-k})^2 = 0.$$

Thus,

$$\begin{aligned} \alpha_{3n} = \beta &= \frac{1-4k\sqrt{1-k} \pm \sqrt{(4k\sqrt{1-k}-1)^2 - 4k(1-\sqrt{1-k})^2}}{2} \\ &= \frac{1-4k\sqrt{1-k} \pm (1-2k)\sqrt{1-4k}}{2} \\ &= \frac{1-\sqrt{1-4k\left(1-2k\pm\sqrt{(1-k)(1-4k)}\right)^2}}{2}. \end{aligned} \tag{3.16}$$

This last formulation was that given by Ramanujan.

We now resolve the sign ambiguity in (3.16). From (3.13) and (3.14), it is clear that both α' and β satisfy the equation

$$y^2 + y(4k\sqrt{1-k} - 1) + k(1 - \sqrt{1-k})^2 = 0. \tag{3.17}$$

Since $\alpha' = 1 - \alpha$, it follows from (3.2) and (3.3) that the solutions of (3.17) are α_{3n} and $\alpha_{3/n}$. Thus, it suffices to show that

$$\alpha_{3/n} > \alpha_{3n}, \tag{3.18}$$

for $n > 1$. If

$$\varphi(q) = \sum_{k=-\infty}^{\infty} q^{k^2},$$

then clearly

$$\varphi^2(e^{-\pi\sqrt{3/n}}) > \varphi^2(e^{-\pi\sqrt{3n}}).$$

From Entry 6 in Chapter 17 of Ramanujan's second notebook [1, p. 101], it follows that

$${}_2F_1(\alpha_{3/n}) > {}_2F_1(\alpha_{3n}).$$

Since ${}_2F_1(x)$ is increasing on $(0, 1)$, (3.18) follows.

Thus, we have proved the following theorem.

Theorem 3. *Let q be given by (1.2), suppose that β has degree n over α, let β have the parametrization (3.4), and define t by (3.7). Then, if k is defined by (3.11), α_{3n} has the representations given in (3.16), where the minus sign must be chosen.*

Ramanujan's formulation of Theorem 3 at the bottom of page 310 in his first notebook is a bit different. He first gives (3.9) and (3.10), but with the left sides switched. He then states (3.7), followed by the equality

$$F\left(p^3 \frac{2+p}{1+2p}\right) = e^{-\pi\sqrt{3n}},$$

which is a consequence of (3.3) and (3.4), where F is defined in (1.5). He concludes by defining k in (3.11) and by claiming that a more complicated, somewhat ambiguous version of the right side of (3.16) equals $e^{-\pi\sqrt{3n}}$, i.e., he forgot to write "F" in front of the right side of (3.16). (In recording specific values of $F(\alpha_n)$, Ramanujan frequently omitted parentheses about the arguments.)

Example 3.1. Let $n = 1$. Then from Weber's tables [8, p. 721], $G_3 = 2^{1/12} = G_{1/3}$, since $G_n = G_{1/n}$ [5], [6, p. 23]. Hence, by (3.12), $k = 1/4$, and by (3.16),

$$\alpha_3 = \frac{2 - \sqrt{3}}{4}.$$

See our paper [3, Theorem 3.2] for another proof and further references.

Example 3.2. Let $n = 5$. Then, from Weber's tables [8, p. 721], $G_{15} = 2^{-1/12}(\sqrt{5}+1)^{1/3}$. It can also be verified that $G_{5/3} = 2^{-1/12}(\sqrt{5}-1)^{1/3}$. Hence, it is easily seen from (3.12) that $k = 1/16$. Therefore, from (3.16),

$$\alpha_{15} = \frac{16 - 7\sqrt{3} - \sqrt{15}}{32},$$

which is simpler than the formula

$$\alpha_{15} = \frac{1}{16}\left(\frac{\sqrt{5}-1}{2}\right)^4 (2-\sqrt{3})^2(4-\sqrt{15}),$$

which was given by Ramanujan in his second notebook. For its proof and further references, see [3, Theorem 3.2].

4. A formula for α_{5n}

We proceed as in Section 3; the calculations are easier, however. Let q be given by (1.2), and suppose that β has degree n over α and degree 5 over $1-\alpha =: \alpha'$. Then, by the same argument as in Section 3, but with 3 replaced by 5, we find that

$$\frac{{}_2F_1(\alpha)}{{}_2F_1(1-\alpha)} = \sqrt{\frac{n}{5}} \qquad \text{and} \qquad \frac{{}_2F_1(1-\beta)}{{}_2F_1(\beta)} = \sqrt{5n}. \tag{4.1}$$

Now, from (4.1), (1.6), and Entry 14(ii) in Chapter 19 of Ramanujan's second notebook [1, p. 288], if $p, 0 < p < (5\sqrt{5}-11)/2$, is defined by

$$1 + 2p = m := \frac{{}_2F_1(\alpha)}{{}_2F_1(\beta)}, \tag{4.2}$$

then

$$k := (G_{n/5}G_{5n})^{-4} = \{4\alpha(1-\alpha)4\beta(1-\beta)\}^{1/6} = p\frac{2-p}{1+2p}. \tag{4.3}$$

Therefore,

$$1 - k = \frac{1+p^2}{1+2p}. \tag{4.4}$$

Solving (4.3) for p, we find that

$$p = 1 - k - \sqrt{k^2 - 3k + 1}. \tag{4.5}$$

Also, from [1, p. 289, Entry 14(iii)],

$$1 - 2\alpha_{5n} = (1 + p - p^2)\left(\frac{1+p^2}{1+2p}\right)^{1/2}. \tag{4.6}$$

Hence, using (4.4) and (4.5) in (4.6), we find that, after a modicum of calculation,

$$\alpha_{5n} = \frac{1}{2} - \sqrt{1-k}\left(2k - k^2 + \left(\frac{1}{2} - k\right)\sqrt{k^2 - 3k + 1}\right). \tag{4.7}$$

Thus, we have established the following theorem.

Theorem 4. *Let q be given by (1.2), let β have degree n over α, define p by (4.2), and define k by (4.3). Then α_{5n} has the representation (4.7).*

Example 4.1. Let $n = 1$. From Weber's tables [8, p. 721],

$$G_5 = \left(\frac{\sqrt{5}+1}{2}\right)^{1/4} = G_{1/5},$$

since $G_n = G_{1/n}$. Thus, from (4.3), $k = (3-\sqrt{5})/2$. Noting that $k^2 - 3k + 1 = 0$, we find from (4.7) that

$$\alpha_5 = \frac{1}{2} - \left(\frac{\sqrt{5}-1}{2}\right)^{3/2},$$

which can be compared with the value

$$\alpha_5 = \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^3 \left(\sqrt{\frac{3+\sqrt{5}}{4}} - \sqrt{\frac{\sqrt{5}-1}{2}}\right)^4,$$

given by Ramanujan in his first notebook and proved by the authors and Zhang in [3, Theorem 3.2].

Example 4.2. Let $n = 5$. From Weber's tables [8, p. 722],

$$G_{25} = \frac{\sqrt{5}+1}{2},$$

and so, since $G_1 = 1$, from (4.3), $k = (7-3\sqrt{5})/2$. Thus,

$$\sqrt{k^2 - 3k + 1} = \sqrt{14 - 6\sqrt{5}} = 3 - \sqrt{5},$$

by an application of (2.3). After a moderate amount of calculation and the use of (2.3), we find that (4.7) yields

$$\alpha_{25} = \tfrac{1}{2} + 6 \cdot 5^{1/4}(4\sqrt{5} - 9).$$

This can be compared with the value

$$\alpha_{25} = \frac{1}{2}(161 - 72\sqrt{5})\left(\sqrt{\frac{5+\sqrt{5}}{4}} - \sqrt{\frac{1+\sqrt{5}}{4}}\right)^8,$$

imprecisely given in Ramanujan's first notebook and established in [3, Theorem 3.2].

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References

1. B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer–Verlag, New York, 1991.
2. B. C. Berndt, H. H. Chan, and L.–C. Zhang, *Ramanujan's class invariants, Kronecker's limit formula, and modular equations*, Trans. Amer. Math. Soc. **349** (1997), 2125–2173.
3. B. C. Berndt, H. H. Chan, and L.–C. Zhang, *Ramanujan's singular moduli*, The Ramanujan Journal **1** (1997), 53–74.
4. H. H. Chan and S.–S. Huang, *On the Ramanujan–Gordon–Göllnitz continued fraction*, The Ramanujan Journal **1** (1997), 75–90.
5. S. Ramanujan, *Modular equations and approximations to* π, Quart. J. Math. (Oxford) **45** (1914), 350–372.
6. S. Ramanujan, *Collected Papers*, Chelsea, New York, 1962.
7. S. Ramanujan, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.
8. H. Weber, *Lehrbuch der Algebra, dritter Band*, Chelsea, New York, 1961.

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