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ON EISENSTEIN SERIES AND $\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}$

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ABSTRACT. In this paper, we derive some new identities satisfied by the series $\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}$ using Ramanujan's identities for $L(q)$, $M(q)$ and $N(q)$. Our work is motivated by an attempt to develop a theory of elliptic functions to the septic base.

1. INTRODUCTION

Set, for $|q| < 1$,

$$L(q) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n},$$
$$M(q) = 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n},$$

and

$$N(q) = 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}.$$

In his famous paper "*On certain arithmetical functions*" [6, pp. 136–162], [7] S. Ramanujan established (using elementary methods) many identities involving $L(q)$, $M(q)$ and $N(q)$, three of which are

$$q\frac{dL(q)}{dq} = \frac{L^2(q) - M(q)}{12}, \tag{1.1}$$

$$q\frac{dM(q)}{dq} = \frac{L(q)M(q) - N(q)}{3}, \tag{1.2}$$

and

$$q\frac{dN(q)}{dq} = \frac{L(q)N(q) - M^2(q)}{2}. \tag{1.3}$$

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These identities were employed by K. Venkatachaliengar to derive relations between certain functions x and z which are stated implicitly in Ramanujan's Notebooks. For example, he showed that [9, pp. 93–95] if

$$M(q) = z_3^4(1 + 8x_3) \tag{1.4}$$

and

$$N(q) = z_3^6(1 - 20x_3 - 8x_3^2), \tag{1.5}$$

then

$$z_3 = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x_3\right), \tag{1.6}$$

where ${}_2F_1(a, b; c; z)$ is the ordinary hypergeometric function. The final step in Venkatachaliengar's proof of (1.6), however, requires the knowledge of x_3 and z_3, but these were neither supplied by Ramanujan nor Venkatachaliengar. The mystery was solved recently in a paper by B.C. Berndt, S. Bhargava, and F.G. Garvan[1] [3]. They succeeded in expressing x_3 and z_3 in terms of the Borweins'[2] functions, namely,

$$z_3 = a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}$$

and

$$x_3 = \left(\frac{c(q)}{a(q)}\right)^3,$$

where

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}.$$

As a result, the proof of (1.6) using Venkatachaliengar's idea is now complete [4].

Venkatachaliengar's method of deriving relations between functions x_3 and z_3 from (1.4) and (1.5) can be adopted to deduce new relations between functions x and z satisfying

$$M(q) = z^4F(x) \quad \text{and} \quad N(q) = z^6G(x), \tag{1.7}$$

where $F(x)$ and $G(x)$ are certain rational functions of x. In Section 2, we state two identities found in Ramanujan's Notebooks and establish new parametrizations of $M(q)$ and $N(q)$ of the type given in (1.7). We then invoke Venkatachaliengar's method to derive relations associated with the function

$$\sigma(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}. \tag{1.8}$$

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[1]Professor Garvan informed us recently that J.M. Borwein and P.B. Borwein are the first to have identified x_3 and z_3.

[2]The Borweins here refer to J.M. Borwein and P.B. Borwein.

In Section 3, we use an identity proved in Section 2, together with some of Ramanujan's modular equations to prove the following surprising identity:

$$6\sum_{m,n=-\infty}^{\infty} e^{-\frac{4\pi}{\sqrt{7}}(m^2+mn+2n^2)} = 5\sum_{m,n=-\infty}^{\infty} e^{-\frac{2\pi}{\sqrt{7}}(m^2+mn+2n^2)}. \tag{1.9}$$

Our method of deriving (1.9) depends on the existence of certain types of modular equations. This method can also be modified to derive special values of certain quotients of the classical theta function

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

as well as that of the Borweins' function $a(q)$.

2. Two identities associated with $M(q)$ and $N(q)$

Theorem 2.1. *Let* $f(-q) = \prod_{n=1}^{\infty}(1-q^n), \quad |q|<1$. *Then*

$$\begin{aligned} M(q) &= \left(\frac{f^7(-q)}{f(-q^7)} + 5\cdot 7^2 q f^3(-q) f^3(-q^7) + 7^4 q^2 \frac{f^7(-q^7)}{f(-q)}\right) \\ &\quad \times \left(\frac{f^7(-q)}{f(-q^7)} + 13 q f^3(-q) f^3(-q^7) + 49 q^2 \frac{f^7(-q^7)}{f(-q)}\right)^{1/3} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} N(q) &= \left(\frac{f^7(-q)}{f(-q^7)} - 7^2(5+2\sqrt{7}) q f^3(-q) f^3(-q^7) - 7^3(21+8\sqrt{7}) q^2 \frac{f^7(-q^7)}{f(-q)}\right) \\ &\quad \times \left(\frac{f^7(-q)}{f(-q^7)} - 7^2(5-2\sqrt{7}) q f^3(-q) f^3(-q^7) - 7^3(21-8\sqrt{7}) q^2 \frac{f^7(-q^7)}{f(-q)}\right). \end{aligned} \tag{2.2}$$

Two proofs of Theorem 2.1 were given by S. Raghavan and S.S. Rangachari [5]. As indicated in [5], Theorem 2.1 can easily be deduced from F. Klein's formula which expressed the modular j-invariant in terms of the Hauptmodul $q^{-1}f^4(-q)/f^4(-q^7)$ for $\Gamma_0(7)$. One can also verify Theorem 2.1 using modular equations found in Ramanujan's Notebooks [8].

Identities (2.1) and (2.2) are not suitable for the application of Venkatachaliengar's method. Our next task is to transform these two identities into those analogous to (1.4) and (1.5).

Lemma 2.2. *Let* $\sigma(q)$ *be as given in (1.8) and let*

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

Then for $|q|<1$,

$$\begin{aligned} \sigma(q) &= \varphi(q)\varphi(q^7) + 4q^2\psi(q^2)\psi(q^{14}) \\ &= \left(\frac{f^8(-q) + 13 q f^4(-q) f^4(-q^7) + 49 q f^8(-q^7)}{f(-q)f(-q^7)}\right)^{1/3}. \end{aligned}$$

Proof of the first equality in Lemma 2.2. We first observe that the left hand side is essentially $\sum_{x\in\mathfrak{O}_K} q^{\mathcal{N}(x)}$, where $\mathfrak{O}_K$ is the ring of integers in $K := \mathbb{Q}(\sqrt{-7})$ and $\mathcal{N}(x)$ is the norm of the element x. An element in $\mathfrak{O}_K$ takes the form $m+n\frac{1+\sqrt{-7}}{2}$, $m,n\in\mathbb{Z}$. Hence, we may partition $\mathfrak{O}_K$ into two sets

$$\mathfrak{O}_{K,0} := \left\{m+n\frac{1+\sqrt{-7}}{2} \mid n \text{ even}\right\} \quad \text{and} \quad \mathfrak{O}_{K,1} := \left\{m+n\frac{1+\sqrt{-7}}{2} \mid n \text{ odd}\right\},$$

and deduce that

$$\sum_{x\in\mathfrak{O}_K} q^{\mathcal{N}(x)} = \sum_{x\in\mathfrak{O}_{K,0}} q^{\mathcal{N}(x)} + \sum_{x\in\mathfrak{O}_{K,1}} q^{\mathcal{N}(x)}. \tag{2.3}$$

When $x\in\mathfrak{O}_{K,0}$, x is of the form $m+2k\frac{1+\sqrt{-7}}{2} = m+k+k\sqrt{-7}$. Therefore, $\mathcal{N}(x) = (m+k)^2+7k^2$ and

$$\begin{aligned}\sum_{x\in\mathfrak{O}_{K,0}} q^{\mathcal{N}(x)} &= \sum_{m,k=-\infty}^{\infty} q^{(m+k)^2+7k^2} \\ &= \sum_{u,v=-\infty}^{\infty} q^{u^2+7v^2} = \varphi(q)\varphi(q^7).\end{aligned} \tag{2.4}$$

When $x\in\mathfrak{O}_{K,1}$, x is of the form $m+(2k+1)\frac{1+\sqrt{-7}}{2} = m+k+\frac{1}{2}+(k+\frac{1}{2})\sqrt{-7}$. Therefore, $\mathcal{N}(x) = (m+k+\frac{1}{2})^2+7(k+\frac{1}{2})^2 = (m+k)^2+(m+k)+7k(k+1)+2$ and

$$\begin{aligned}\sum_{x\in\mathfrak{O}_{K,1}} q^{\mathcal{N}(x)} &= \sum_{m,k=-\infty}^{\infty} q^{(m+k)(m+k+1)+7k(k+1)+2} \\ &= q^2\sum_{u,v=-\infty}^{\infty} q^{u(u+1)+7v(v+1)} = 4q^2\psi(q^2)\psi(q^{14}).\end{aligned} \tag{2.5}$$

From (2.3), (2.4) and (2.5), the first equality of Lemma 2.2 follows. □

Remark. Suppose $d\equiv 3 \pmod 4$. Then by similar argument as above, we have

$$\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+\frac{d+1}{4}n^2} = \varphi(q)\varphi(q^d)+4q^{\frac{d+1}{4}}\psi(q^2)\psi(q^{2d}).$$

In order to establish the second equality, we need the following modular equations of degree 7. For the definition of modular equation, see [1, p. 213].

Lemma 2.3. *Suppose β has degree 7 over α. Let $m = \varphi(q)/\varphi(q^7)$, and let $P = \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}$ and $Q = \{\beta(1-\beta)/(\alpha(1-\alpha))\}^{1/6}$. Then*

$$(\alpha\beta)^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8} = 1, \tag{2.6}$$

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}-8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3}, \tag{2.7}$$

$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/2}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}-8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3}, \tag{2.8}$$

and

$$Q + \frac{1}{Q} + 7 = 2\sqrt{2}\left(P + \frac{1}{P}\right). \tag{2.9}$$

The proofs of these modular equations can be found in [1, pp. 314 – 324].

Proof of the second equality in Lemma 2.2. Let $t = (\alpha\beta)^{1/8}$. By (2.6), $(1-t) = \{(1-\alpha)(1-\beta)\}^{1/8}$. Hence,

$$t(1-t) = \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \tag{2.10}$$

From [1, p. 473], we find that

$$\varphi(q^2)\varphi(q^{14}) + 4q^4\psi(q^4)\psi(q^{28}) = \sqrt{\varphi(q)\varphi(q^7)}\{1 - t(1-t)\}.$$

Now, from [1, p. 124, Entry 12 (iii)] and (2.10),

$$\sqrt{\varphi(q)\varphi(q^7)} = 2^{2/3}q^{2/3}\frac{f(-q^2)f(-q^{14})}{(t(1-t))^{2/3}}.$$

Hence,

$$\varphi(q^2)\varphi(q^{14}) + 4q^4\psi(q^4)\psi(q^{28}) = 2^{2/3}q^{2/3}f(-q^2)f(-q^{14})\frac{1-t(1-t)}{(t(1-t))^{2/3}}. \tag{2.11}$$

Next, from (2.7), (2.8) and [1, p. 124, Entry 12(iii)],

$$\begin{aligned} \frac{f^4(-q^2)}{q^2f^4(-q^{14})} &= m^2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3} \\ &= \frac{\sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)}}{(\alpha(1-\alpha))^{1/6}(\beta(1-\beta))^{1/3}} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6} - 8 \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} \frac{49q^2f^4(-q^{14})}{f^4(-q^2)} &= \frac{49}{m^2}\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3} \\ &= \frac{\sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)}}{(\alpha(1-\alpha))^{1/3}(\beta(1-\beta))^{1/6}} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6} - 8. \end{aligned} \tag{2.13}$$

Now,

$$\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6} = 4t(1-t) + \frac{2}{t(1-t)} - 7, \tag{2.14}$$

by (2.9) and (2.10). Also,

(2.15)

$$\begin{aligned} &(\sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)}) \\ &\quad \times \left(\frac{1}{(\alpha(1-\alpha))^{1/6}(\beta(1-\beta))^{1/3}} + \frac{1}{(\alpha(1-\alpha))^{1/3}(\beta(1-\beta))^{1/6}}\right) \\ &\quad = \frac{\sqrt{\alpha(1-\beta)} + \sqrt{\beta(1-\alpha)}}{(t(1-t))^{5/2}}\left(\left(\frac{\alpha(1-\alpha)}{t(1-t)}\right)^{1/6} + \left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/6}\right), \end{aligned}$$

by (2.10). It is easy to see that

$$\begin{aligned}\sqrt{\alpha(1-\beta)}+\sqrt{\beta(1-\alpha)} &= \sqrt{1-(\sqrt{\alpha\beta}-\sqrt{(1-\alpha)(1-\beta)})^2} \\ &= \sqrt{1-(t^4-(1-t)^4)^2} \\ &= 2\sqrt{t(1-t)(2t^2-t+1)(2t^2-3t+2)},\end{aligned} \tag{2.16}$$

and from [1, p. 318, (19.12), (19.13)],

$$\left(\frac{\alpha(1-\alpha)}{t(1-t)}\right)^{1/6}+\left(\frac{\beta(1-\beta)}{t(1-t)}\right)^{1/6} = \sqrt{(2t^2-t+1)(2t^2-3t+2)}. \tag{2.17}$$

Hence, using (2.16) and (2.17), we may simplify (2.15) as

$$\begin{aligned}&(\sqrt{\alpha(1-\beta)}+\sqrt{\beta(1-\alpha)}) \\ &\qquad\times\left(\frac{1}{(\alpha(1-\alpha))^{1/6}(\beta(1-\beta))^{1/3}}+\frac{1}{(\alpha(1-\alpha))^{1/3}(\beta(1-\beta))^{1/6}}\right) \\ &\qquad = 2\frac{(2t^2-t+1)(2t^2-3t+2)}{t^2(1-t)^2}.\end{aligned} \tag{2.18}$$

Therefore, by adding (2.12) and (2.13) and simplifying using (2.14) and (2.18), we deduce that

$$\begin{aligned}\frac{f^4(-q^2)}{q^2f^4(-q^{14})} &+ 49\frac{q^2f^4(-q^{14})}{f^4(-q^2)} \\ &= 2\frac{(2t^2-t+1)(2t^2-3t+2)}{t^2(1-t)^2}-4t(1-t)-\frac{2}{t(1-t)}-9 \\ &= 2^2\frac{(1-t(1-t))^3}{t^2(1-t)^2}-13.\end{aligned} \tag{2.19}$$

Combining (2.11) and (2.19), and replacing q^2 by q, we complete the proof of Lemma 2.2. □

Remarks. Lemma 2.2 is due to Ramanujan. The first proof of the second equality which also utilizes Ramanujan's modular equations of degree 7 can be found in [1, p. 472]. One can also prove the second equality by verifying that the function $\sigma^3(q)/(qf^3(-q)f^3(-q^7))$ is invariant under $\Gamma_0(7)$ and therefore can be expressed in terms of $f^4(-q)/(qf^4(-q^7))$.

Next, define $x := \alpha(q)$ as the function satisfying the relation

$$\frac{1-\alpha(q)}{\alpha(q)} = \frac{1}{7q}\left(\frac{f(-q)}{f(-q^7)}\right)^4, \tag{2.20}$$

and let

$$z := \sigma(q).$$

By Lemma 2.2, we may rewrite (2.1) and (2.2) as

$$M(q) = 7z^4\frac{1+33x+15x^2}{7-x+x^2}, \tag{2.21}$$

and

$$N(q) = 49z^6 \frac{1 - 74x - 225x^2 - 18x^3 - 27x^4}{(7 - x + x^2)^2}. \tag{2.22}$$

Note that (2.21) and (2.22) are now analogues of (1.4) and (1.5) and we are in the position to invoke Venkatachaliengar's method.

Theorem 2.4. *Let x and z be as above. Then*

$$q\frac{dx}{dq} = z^2 x(1-x) \tag{2.23}$$

and

$$z''_x x(x-1) + z'_x(2x-1) = \frac{2z(2x^2 - 2x - 7)}{(7 - x + x^2)^2}, \tag{2.24}$$

where

$$z''_x = \frac{d^2 z}{dx^2} \qquad \text{and} \qquad z'_x = \frac{dz}{dx}.$$

Proof of Theorem 2.4. For simplicity, let $L = L(q), M = M(q)$, and $N = N(q)$. From (2.21) and (2.22), we find that

$$M^3 - N^2 = 592704 \frac{z^{12} x(1-x)^7}{(7 - x + x^2)^4} \tag{2.25}$$

and

$$3qN\frac{dM}{dq} - 2qM\frac{dN}{dq} = 592704q \frac{z^{10} x'(x-1)^6}{(7 - x + x^2)^4}, \tag{2.26}$$

where the "prime" means differentiation with respect to q.

Next, if we multiply (1.2) by $3N$ and (1.3) by $2M$ and eliminate LMN from the resulting equations, we find that the left hand sides of (2.25) and (2.26) are equal. Hence, we conclude that

$$qx' = z^2 x(1-x),$$

which is (2.23).

Now, from (2.21) and (1.2), we have

$$LM - N = 168qx'z^4 \frac{29 + 26x - 6x^2}{(7 - x + x^2)^2} + 84qz'z^3 \frac{1 + 33x + 15x^2}{7 - x + x^2},$$

which implies that

$$L = z^2 \frac{7 - 53x - 3x^2}{7 - x + x^2} + 12q\frac{z'}{z}, \tag{2.27}$$

by (2.21)–(2.23). From (2.27) and (2.21), we deduce that

$$\begin{aligned} q\frac{dL}{dq} =& 12q^2 \frac{z''}{z} + 12q\frac{z'}{z} - 12\left(q\frac{z'}{z}\right)^2 \\ &+ 28z^4 x(1-x)\frac{-13 - 2x + 2x^2}{(7 - x + x^2)^2} + 2qz'z\frac{7 - 53x - 3x^2}{7 - x + x^2} \end{aligned} \tag{2.28}$$

and

$$\frac{L^2 - M}{12} = 12\left(q\frac{z'}{z}\right)^2 - 4z^4 x(1-x)\frac{49 + 2x - 2x^2}{(7 - x + x^2)^2} + 2qz'z\frac{7 - 53x - 3x^2}{7 - x + x^2}. \tag{2.29}$$

By (1.1), (2.28) and (2.29), we find that

$$q^2\frac{z''}{z}+q\frac{z'}{z}-2\left(q\frac{z'}{z}\right)^2=-2z^4x(1-x)\frac{2x^2-2x-7}{(7-x+x^2)^2}. \tag{2.30}$$

Finally, let

$$z'_x=\frac{dz}{dx} \quad \text{and} \quad z''_x=\frac{d^2z}{dx^2}.$$

Then

$$q^2\frac{z''}{z}+q\frac{z'}{z}-2\left(q\frac{z'}{z}\right)^2=z''_xz^3x^2(1-x)^2+z'_xz^3x(1-x)(1-2x),$$

and (2.24) follows from (2.30). □

Remarks. 1. Part of our computations given in this section are done with the aid of MAPLE V.

2. We have not been able to express the solutions of the second order differential equation (2.24) in terms of known functions.

3. Identity (2.23) is not new. It is equivalent to Ramanujan's identity

$$\sigma^2(q)=1-28\sum_{n=1}^{\infty}\frac{nq^{7n}}{1-q^{7n}}+4\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n},$$

which is first proved by Berndt [1, p. 467, Entry 5 (i)] using modular equations of degree 7.

4. The relation (2.23) is also satisfied by other pairs of functions. For example [3, p. 4178],

$$q\frac{dx_3}{dq}=z_3^2x_3(1-x_3),$$

where x_3 and z_3 are discussed in Section 1, and [1, p. 120, Entry 9(i)]

$$q\frac{dx_2}{dq}=z_2^2x_2(1-x_2),$$

where

$$z_2:=\varphi^2(q) \quad \text{and} \quad x_2:=16q\frac{\psi^4(q^2)}{\varphi^4(q)}.$$

3. Evaluation of $\sigma(q)/\sigma(q^2)$ at $q=e^{-2\pi/\sqrt{7}}$

In this final section, we give a method of deriving special values of $\sigma(q)/\sigma(q^k)$, $k\in\mathbb{N}$. We illustrate this method by giving a proof of (1.9).

Proof of (1.9). Let $\beta(q):=\alpha(q^2)$. Then from (2.23), we find that

$$\frac{d\beta(q)}{d\alpha(q)}=2\frac{\sigma^2(q^2)}{\sigma^2(q)}\frac{\beta(q)(1-\beta(q))}{\alpha(q)(1-\alpha(q))}. \tag{3.1}$$

In general, we have

$$\frac{d\alpha(q^k)}{d\alpha(q)}=k\frac{\sigma^2(q^k)}{\sigma^2(q)}\frac{\alpha(q^k)(1-\alpha(q^k))}{\alpha(q)(1-\alpha(q))}.$$

Next, recall the following modular equations of degree 7 found in Ramanujan's Notebooks [2, p. 209]:

$$PQ + \frac{49}{PQ} = \left(\frac{Q}{P}\right)^3 - 8\frac{Q}{P} - 8\frac{P}{Q} + \left(\frac{P}{Q}\right)^3, \tag{3.2}$$

where

$$P = P(q) := q^{-1/2}\frac{f^2(-q)}{f^2(-q^7)} \quad \text{and} \quad Q = Q(q) := q^{-1}\frac{f^2(-q^2)}{f^2(-q^{14})}.$$

By (2.20),

$$\frac{1-x}{x} = P^2/7 \quad \text{and} \quad \frac{1-y}{y} = Q^2/7, \tag{3.3}$$

where $x = \alpha(q)$ and $y = \beta(q)$. Substituting (3.3) into (3.2), we obtain a relation between x and y, namely,

$$7xy - 6y^2x - 6x^2y + 3x^2y^2 + 2x^3y + 2y^3x - x^3 - y^3 = 0. \tag{3.4}$$

Implicitly differentiating (3.4) with respect to x, we find that

$$\frac{dy}{dx} = -\frac{7y - 6y^2 - 12xy + 6y^2x + 6x^2y + 2y^3 - 3x^2}{7x - 6x^2 - 12xy + 6x^2y + 6y^2x + 2x^3 - 3y^2}. \tag{3.5}$$

This proves that the derivative in (3.1) can be expressed in terms of $\alpha(q)$ and $\beta(q)$. Hence, in order to prove (1.9), it suffices to compute $\alpha_1 = \alpha(q_1)$ and $\beta_1 = \beta(q_1)$ where $q_1 = e^{-2\pi/\sqrt{7}}$.

By the transformation formula for $q^{1/24}f(-q)$ [1, p. 43, Entry 27(iii)], we find that $P(q_1) = \sqrt{7}$ and therefore, by (3.3), $\alpha_1 = 1/2$. Substituting α_1 into (3.4) yields $\beta_1 = 1/2 - \sqrt{7}/6$. This implies that

$$\left.\frac{dy}{dx}\right|_{q=q_1} = \frac{25}{81},$$

by (3.5). Hence, by (3.1), $\sigma(q_1^2)/\sigma(q_1) = 5/6$, which is (1.9). □

In a similar way, using the modular equation [2, p. 236, Entry 68],

$$P_1Q_1 + \frac{7}{P_1Q_1} = \left(\frac{Q_1}{P_1}\right)^2 - 3 + \left(\frac{P_1}{Q_1}\right)^2,$$

with

$$P_1 = q^{-1/4}\frac{f(-q)}{f(-q^7)} \quad \text{and} \quad Q_1 = q^{-3/4}\frac{f(-q^3)}{f(-q^{21})},$$

we can show that

$$9\sum_{m,n=-\infty}^{\infty} e^{-6\pi/\sqrt{7}(m^2+mn+2n^2)} = \sqrt{27+6\sqrt{21}}\sum_{m,n=-\infty}^{\infty} e^{-2\pi/\sqrt{7}(m^2+mn+2n^2)}. \tag{3.6}$$

Concluding remarks

This paper is motivated by an attempt to develop a theory of elliptic functions to the septic base (see [3] for the definition of elliptic functions to alternative bases). Our investigation here shows that this theory may not exist since we are unable to solve (2.24) in terms of known functions. The failure to do so implies that there are no simple analogues of Jacobi's inversion formula as well as the Borweins' inversion formula in the septic case. The existence of a simple inversion formula is important in developing elliptic functions to alternative bases.

It is interesting to note that proofs of simple identities such as (1.9) and (3.6) involve so many non-trivial identities scattered in Ramanujan's Notebooks. Although Ramanujan possessed all the identities discussed in this paper, (1.9) and (3.6) were not found in his Notebooks.

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