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SOME VALUES FOR THE ROGERS-RAMANUJAN CONTINUED FRACTION

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ABSTRACT. In his first and lost notebooks, Ramanujan recorded several values for the Rogers-Ramanujan continued fraction. Some of these results have been proved by K. G. Ramanathan, using mostly ideas with which Ramanujan was unfamiliar. In this paper, eight of Ramanujan's values are established; four are proved for the first time, while the remaining four had been previously proved by Ramanathan by entirely different methods. Our proofs employ some of Ramanujan's beautiful eta-function identities, which have not been heretofore used for evaluating continued fractions.

0. **Introduction.** Let, for $|q| < 1$,

$$R(q) = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots$$

and

$$S(q) = -R(-q)$$

denote the famous Rogers-Ramanujan continued fractions. In both his first and second letters to Hardy [11, pp. xxvii, xxviii], Ramanujan communicated theorems about $R(q)$ and $S(q)$. In particular, in his first letter, he asserted that

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} \tag{0.1}$$

and

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}. \tag{0.2}$$

The evaluation (0.1) follows easily from a reciprocity theorem for $R(q)$, which Ramanujan gave in his second letter, and which was first proved by G. N. Watson [14]. The evaluation (0.2) follows from a similar reciprocity theorem for $S(q)$, which apparently Ramanujan did not communicate to Hardy, but which is found in his notebooks [10, p. 204], [2, p. 83]. The latter theorem was first proved by K. G. Ramanathan [5], but (0.2) was first established by Watson [13] in a different manner.

In his second letter, Ramanujan also claimed that

$$R(e^{-2\pi\sqrt{5}}) = \frac{\sqrt{5}}{1 + \left(5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2} - 1\right)^{1/5}} - \frac{\sqrt{5}+1}{2}, \tag{0.3}$$

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which was also first proved in print by Watson [14]. Ramanathan [5] also established (0.3) as well as the companion result

$$S(e^{-\pi\sqrt{5}}) = \frac{\sqrt{5}}{1+\left(5^{3/4}(\frac{\sqrt{5}+1}{2})^{5/2}-1\right)^{1/5}} - \frac{\sqrt{5}-1}{2}.$$

Ramanujan recorded other values for $R(q)$ and $S(q)$ in his first notebook [10] and in his "lost notebook" [12]. Several of these results were proved by Ramanathan [5], [6], [7], [8], who also established further evaluations not claimed by Ramanujan. However, Ramanathan's tools were generally not known to Ramanujan. In particular, Ramanathan skillfully employed Kronecker's limit formula.

In this paper we establish four values for $R(q)$ stated on p. 311 in Ramanujan's first notebook, and four values for $S(q)$ found in the lost notebook [12, pp. 204, 210]. All of our proofs use only theorems and methods known to Ramanujan. In particular, each of our proofs employs an eta-function identity from the unorganized portions of the second notebook [10], [3, Chapter 25]. Ramanathan [6] gave different proofs of Theorems 1, 5, and 6 below; proofs of Theorems 2, 3, and 4 have not been heretofore given. In order to state the first four theorems, we set

$$2c = 1 + \frac{a+b}{a-b}\sqrt{5}, \tag{0.4}$$

where a and b are certain real numbers to be specified below.

THEOREM 1. *Let $a = 5^{1/4}, b = 1$, and c be given by (0.4). Then*

$$R(e^{-4\pi}) = \sqrt{c^2+1} - c. \tag{0.5}$$

THEOREM 2. *Let $a = 3+\sqrt{2}-\sqrt{5}, b = (20)^{1/4}$, and c be given by (0.4). Then*

$$R(e^{-8\pi}) = \sqrt{c^2+1} - c. \tag{0.6}$$

THEOREM 3. *Let $a = 5^{1/4}(4-\sqrt{2}), b = 1+\sqrt{2}+\sqrt{5}-2^{1/4}(3-\sqrt{2}+\sqrt{5}-\sqrt{10})$, and c be given by (0.4). Then*

$$R(e^{-16\pi}) = \sqrt{c^2+1} - c. \tag{0.7}$$

THEOREM 4. *Let $a = (60)^{1/4}, b = 2-\sqrt{3}+\sqrt{5}$, and c be given by (0.4). Then*

$$R(e^{-6\pi}) = \sqrt{c^2+1} - c. \tag{0.8}$$

THEOREM 5. *We have*

$$S(e^{-\pi/\sqrt{35}}) = \left(5\sqrt{5}-7+\sqrt{35(5-2\sqrt{5})}\right)^{1/5}. \tag{0.9}$$

COROLLARY. *We have*

$$S(e^{-\pi\sqrt{7/5}}) = \left(-5\sqrt{5} - 7 + \sqrt{35(5 + 2\sqrt{5})}\right)^{1/5}.$$

THEOREM 6. *We have*

$$S(e^{-\pi/\sqrt{15}}) = \left(\frac{5\sqrt{5} - 3 + \sqrt{30(5 - \sqrt{5})}}{4}\right)^{1/5}. \tag{0.10}$$

COROLLARY. *We have*

$$S(e^{-\pi\sqrt{3/5}}) = \left(\frac{-5\sqrt{5} - 3 + \sqrt{30(5 + \sqrt{5})}}{4}\right)^{1/5}.$$

Theorem 6 and its corollary are not explicitly stated on p. 210 of the lost notebook [12]; Ramanujan merely indicates that he is able to calculate the values of the two continued fractions.

We use Ramanujan's notation in our proofs. Thus, set

$$f(-q) := (q;q)_\infty, \quad |q| < 1, \tag{0.11}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n).$$

We shall need two related transformation formulas for f [2, p. 43, Entry 27(iii), (iv)]. If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then

$$e^{-\alpha/12}\alpha^{1/4}f(-e^{-2\alpha}) = e^{-\beta/12}\beta^{1/4}f(-e^{-2\beta}) \tag{0.12}$$

and

$$e^{-\alpha/24}\alpha^{1/4}f(e^{-\alpha}) = e^{-\beta/24}\beta^{1/4}f(e^{-\beta}). \tag{0.13}$$

Following Ramanujan [9], [11, p. 23], we define the two invariants

$$G_n = 2^{-1/4}q^{-1/24}(-q;q^2)_\infty \quad \text{and} \quad g_n = 2^{-1/4}q^{-1/24}(q;q^2)_\infty, \tag{0.14}$$

where $n > 0$ and $q = e^{-\pi\sqrt{n}}$. We establish two simple relations for these invariants. From the elementary identity

$$(q^2;q^4)_\infty = (q;q^2)_\infty(-q;q^2)_\infty, \tag{0.15}$$

we easily deduce that

$$g_{4n} = 2^{1/4} g_n G_n, \tag{0.16}$$

where $q = e^{-\pi\sqrt{n}}$. (This is also found in Ramanujan's paper [9, equation (5)], [11, p. 23].) Secondly, recall Jacobi's identity for fourth powers of theta-functions [2, p. 40, Entry 25(vii)], which can be written in the form [16, p. 470]

$$(-q;q^2)_\infty^8 - (q;q^2)_\infty^8 = 16q(-q^2;q^2)_\infty^8 = \frac{16q}{(q^2;q^4)_\infty^8}, \tag{0.17}$$

where we used Euler's identity

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}.$$

By (0.15), we can write (0.17) in the form

$$2^{-2}q^{-1/3}(q;q^2)_\infty^8 2^{-2}q^{-1/3}(-q;q^2)_\infty^8 \left(2^{-2}q^{-1/3}(-q;q^2)_\infty^8 - 2^{-2}q^{-1/3}(q;q^2)_\infty^8\right) = \frac{1}{4},$$

i.e., when $q = e^{-\pi\sqrt{n}}$,

$$(g_n G_n)^8 (G_n^8 - g_n^8) = \frac{1}{4}. \tag{0.18}$$

(This identity is also found in Ramanujan's paper [9, equation (7)], [11, p. 23].)

1. **Proofs of Theorem 1.**

FIRST PROOF. Recall that [2, p. 84, equation (39.1)]

$$\frac{1}{R(e^{-\alpha})} - R(e^{-\alpha}) - 1 = e^{-\alpha/5} \frac{f(-e^{-\alpha/5})}{f(-e^{-5\alpha})}, \tag{1.1}$$

where $\alpha > 0$ and f is defined by (0.11). After some elementary algebraic manipulation, we find that (0.5) is equivalent to the identity

$$\frac{1}{R(e^{-\alpha})} - R(e^{-\alpha}) = 2c, \tag{1.2}$$

with $\alpha = 4\pi$. Thus, from (1.1), (1.2), and (0.4), we must prove that

$$e^{4\pi/5} \frac{f(-e^{-4\pi/5})}{f(-e^{-20\pi})} = \frac{a+b}{a-b}\sqrt{5}, \tag{1.3}$$

where a and b are as stated in Theorem 1.

We shall employ Entry 58 of Chapter 25 of [3, pp. 212–213]. Let

$$P = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)} \quad \text{and} \quad Q = \frac{f(-q^{2/5})}{q^{2/5} f(-q^{10})}.$$

Then

$$(PQ)^2 + 5PQ = P^3 - 2P^2Q - 2PQ^2 + Q^3. \tag{1.4}$$

Let $q = e^{-2\pi}$. Then, by (0.12),

$$P = \sqrt{5} \tag{1.5}$$

Using (1.5) in (1.4), we find that

$$5Q^2 + 5\sqrt{5}Q = 5\sqrt{5} - 10Q - 2\sqrt{5}Q^2 + Q^3. \tag{1.6}$$

It will be convenient to set $Q = \sqrt{5}T$, so that (1.6) takes the form

$$\sqrt{5}T^2 + \sqrt{5}T = 1 - 2T - 2T^2 + T^3 = (T^2 - 3T + 1)(T + 1).$$

Since clearly $T \neq -1$,

$$\sqrt{5}T = T^2 - 3T + 1.$$

Solving this quadratic equation, we find that

$$T = \frac{3 + \sqrt{5} \pm \sqrt{10 + 6\sqrt{5}}}{2}.$$

If we took the minus sign above, we would find that $Q = \sqrt{5}T < 1$. But clearly $Q > 1$, and so we deduce that

$$Q = \sqrt{5}\left(\frac{3 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}}}{2}\right). \tag{1.7}$$

By (1.3) and (1.7), it remains to show that

$$\frac{3 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}}}{2} = \frac{5^{1/4} + 1}{5^{1/4} - 1}. \tag{1.8}$$

However,

$$\begin{aligned} 3 + \sqrt{5} + \sqrt{10 + 6\sqrt{5}} &= 3 + \sqrt{5} + 5^{1/4}\sqrt{2\sqrt{5} + 6} \\ &= 3 + \sqrt{5} + 5^{1/4}(\sqrt{5} + 1) \\ &= \frac{(3 + 5^{1/4} + \sqrt{5} + 5^{3/4})(5^{1/4} - 1)}{5^{1/4} - 1} \\ &= \frac{2(5^{1/4} + 1)}{5^{1/4} - 1}, \end{aligned}$$

and thus (1.8) has been shown to complete the proof.

Ramanathan [6] gave a more difficult proof of Theorem 1 in which class invariants were employed. We have also discovered a proof of Theorem 1 that utilizes class invariants. Since our proof is simpler than that of Ramanathan and much different from our proof above, we give it below. Like Ramanathan's proof, our proof requires the value of G_{25}, and so we give a simple derivation of this evaluation next.

LEMMA.

$$G_{25} = \frac{1+\sqrt{5}}{2}.$$

PROOF. We employ a modular equation of degree 5 found in Entry 13(xiv) in Chapter 19 of Ramanujan's second notebook [2, p. 282]. Let

$$P = 2^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}$$

and

$$Q = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8},$$

where α and β have degrees 1 and 5, respectively. Then

$$Q + \frac{1}{Q} + 2\left(P - \frac{1}{P}\right) = 0. \tag{1.9}$$

Recalling the definition of G_n in (0.14), recalling that [2, p. 37] $\chi(q) := (-q;q^2)_\infty$, and using Entry 12(v) in Chapter 17 of Ramanujan's second notebook [2, p. 124], we find that

$$G_{25} = 2^{-1/4}e^{\pi/24}2^{1/6}e^{-\pi/24}\{\beta(1-\beta)\}^{-1/24} = 2^{-1/12}\{\beta(1-\beta)\}^{-1/24}.$$

Since it is well known and easy to prove that $G_1 = 1$, we have, by the same reasoning as above,

$$G_1 = 2^{-1/12}\{\alpha(1-\alpha)\}^{-1/24} = 1.$$

Hence, it follows that

$$P = \frac{1}{G_{25}^2} \quad \text{and} \quad Q = \frac{1}{G_{25}^3}.$$

Therefore, by (1.9), if $x = G_{25}$,

$$\begin{aligned}\frac{1}{x^3} + x^3 + 2\left(\frac{1}{x^2} - x^2\right) &= \left(\frac{1}{x} + x\right)^3 - 3\left(\frac{1}{x} + x\right) + 2\left(\frac{1}{x} + x\right)\left(\frac{1}{x} - x\right)\\ &= \left(\frac{1}{x} + x\right)\left\{\left(\frac{1}{x} - x\right)^2 + 2\left(\frac{1}{x} - x\right) + 1\right\}\\ &= \left(\frac{1}{x} + x\right)\left\{\left(\frac{1}{x} - x\right) + 1\right\}^2\\ &= 0.\end{aligned}$$

Since $x + 1/x \neq 0$ and $x > 0$, we conclude that

$$G_{25} = x = \frac{1+\sqrt{5}}{2},$$

and so the proof is complete.

We remark that the value of G_{25} is given without proof in Ramanujan's paper [9], [11, p. 26]. Ramanathan [7] also gave a short proof of this lemma.

SECOND PROOF. Recall from our first proof that it suffices to prove (1.3).

Set $\alpha = 2\pi/5$ in (0.12), so that $\beta = 5\pi/2$. After some simplification, we find that

$$f(-e^{-4\pi/5}) = \sqrt{\frac{5}{2}}e^{-7\pi/40}f(-e^{-5\pi}).$$

Thus,

$$Q = e^{4\pi/5}\frac{f(-e^{-4\pi/5})}{f(-e^{-20\pi})} = \sqrt{\frac{5}{2}}e^{5\pi/8}\frac{f(-e^{-5\pi})}{f(-e^{-20\pi})}. \tag{1.10}$$

Since

$$\frac{(q;q)_\infty}{(q^4;q^4)_\infty} = (q;q^2)_\infty(q^2;q^4)_\infty,$$

we deduce from (1.10) that

$$Q = \sqrt{5}g_{25}g_{100}, \tag{1.11}$$

where g_n is defined in (0.14). We thus must determine g_{25} and g_{100}. Since G_{25} was computed in the lemma above, we see from (0.16) and (0.18) that if we determine g_{25}, then Q can be calculated.

For brevity, set $x = g_{25}^8$ and $a = G_{25}^8$. Thus, from (0.18),

$$ax^2 - a^2x + 1/4 = 0, \tag{1.12}$$

Since $x > 0$, from (1.12) and the lemma, we deduce that

$$\begin{aligned} x &= \frac{a^2 + \sqrt{a^4 - a}}{2a} \\ &= \frac{1}{2}\left(G_{25}^8 + G_{25}^2\sqrt{G_{25}^{12} - G_{25}^{-12}}\right) \\ &= \frac{1}{4}G_{25}^2\left(\sqrt{G_{25}^6 + G_{25}^{-6}} + \sqrt{G_{25}^6 - G_{25}^{-6}}\right)^2. \end{aligned} \tag{1.13}$$

By the lemma, $G_{25}^6 = (2+\sqrt{5})^2 = 9 + 4\sqrt{5}$, and so $G_{25}^{-6} = 9 - 4\sqrt{5}$. Hence, by (1.13) and the lemma,

$$\begin{aligned} x = g_{25}^8 &= \frac{1}{4}\left(\frac{1+\sqrt{5}}{2}\right)^2\left(\sqrt{18} + \sqrt{8\sqrt{5}}\right)^2 \\ &= \frac{1}{2}\left(\frac{1+\sqrt{5}}{2}\right)^2(3 + 2\cdot 5^{1/4})^2. \end{aligned} \tag{1.14}$$

Thus, from (1.11), (1.14), and the lemma,

$$\begin{aligned} Q &= \sqrt{5}2^{1/4}g_{25}^2G_{25} \\ &= \sqrt{5}\left(\frac{1+\sqrt{5}}{2}\right)^{3/2}(3+2\cdot 5^{1/4})^{1/2} \\ &= \sqrt{5}\frac{\sqrt{2}}{\sqrt{5}-1}(1+\sqrt{5})^{1/2}(3+2\cdot 5^{1/4})^{1/2} \\ &= \sqrt{5}\frac{\{(1+5^{1/4})^4\}^{1/2}}{\sqrt{5}-1} \\ &= \sqrt{5}\frac{5^{1/4}+1}{5^{1/4}-1}. \end{aligned}$$

Thus, (1.3) is established, and the proof is complete.

2. **Proof of Theorem 2.** We again shall employ Entry 58 of Chapter 25 in [3, p. 212–213], but now with

$$P = e^{4\pi/5}\frac{f(-e^{-4\pi/5})}{f(-e^{-20\pi})} \quad \text{and} \quad Q = e^{8\pi/5}\frac{f(-e^{-8\pi/5})}{f(-e^{-40\pi})}.$$

By the same reasoning as that used in the first proof of (0.5), in order to prove (0.6), it suffices to prove that

$$Q = \frac{a+b}{a-b}\sqrt{5}, \tag{2.1}$$

where a and b are given in the statement of Theorem 2.

Write (1.4) in the form

$$PQ + 5 = \frac{P^2}{Q} - 2P - 2Q + \frac{Q^2}{P}. \tag{2.2}$$

From Theorem 1, $P = (5^{1/4}+1)\sqrt{5}/(5^{1/4}-1)$. Putting this in (2.2) and setting $Q = \sqrt{5}T$, we find that

$$\frac{5^{1/4}+1}{5^{1/4}-1}\sqrt{5}T + \sqrt{5} = \left(\frac{5^{1/4}+1}{5^{1/4}-1}\right)^2\frac{1}{T} - 2\frac{5^{1/4}+1}{5^{1/4}-1} - 2T + \frac{5^{1/4}-1}{5^{1/4}+1}T^2. \tag{2.3}$$

By an elementary verification, it is easily checked that $T = 1$ is a root of (2.3). Since clearly $Q > \sqrt{5}$, this root is not the one that we seek. Writing (2.3) in the form $a_3T^3 + a_2T^2 + a_1T + a_0 = 0$, and dividing by $T - 1$, we find that

$$(5^{1/4}-1)^2T^2 - 2(1+5^{1/4}+\sqrt{5}+5^{3/4})T - (9+6\cdot 5^{1/4}+3\sqrt{5}+2\cdot 5^{3/4}) = 0. \tag{2.4}$$

Now set

$$T = \frac{a+b}{a-b}$$

in (2.4) to deduce, with the help of *Mathematica,* that

$$-(5+3\sqrt{5})a^2+(6\cdot 5^{1/4}+2\cdot 5^{3/4})ab+(3-3\sqrt{5})b^2=0.$$

Solving for a, we find that

$$a=\frac{5^{1/4}b\pm\sqrt{(7\sqrt{5}-15)b^2}}{\sqrt{5}}.$$

We now set $b=5^{1/4}\sqrt{2}$ and choose the plus sign above, because if we had chosen the minus sign, we would find that $T<0$, which is impossible. Hence,

$$\begin{aligned} a&=\sqrt{2}+\frac{\sqrt{70-30\sqrt{5}}}{\sqrt{5}}\\ &=\sqrt{2}+\sqrt{14-6\sqrt{5}}\\ &=\sqrt{2}+\sqrt{(3-\sqrt{5})^2}\\ &=\sqrt{2}+3-\sqrt{5}. \end{aligned}$$

Hence,

$$Q=T\sqrt{5}=\frac{a+b}{a-b}\sqrt{5}=\frac{\sqrt{2}+3-\sqrt{5}+5^{1/4}\sqrt{2}}{\sqrt{2}+3-\sqrt{5}-5^{1/4}\sqrt{2}}\sqrt{5},$$

and so (2.1) has been shown to complete the proof.

3. **Proof of Theorem 3.** We again employ Entry 58 of Chapter 25 in [3], but now we set

$$P=e^{8\pi/5}\frac{f(-e^{-8\pi/5})}{f(-e^{-40\pi})}\quad\text{and}\quad Q=e^{16\pi/5}\frac{f(-e^{-16\pi/5})}{f(-e^{-80\pi})}.$$

By the same argument that we used in the proofs of Theorems 1 and 2, to prove (0.7), it suffices to prove that

$$Q=\frac{a+b}{a-b}\sqrt{5}, \tag{3.1}$$

where a and b are prescribed in the statement of Theorem 3.

Set $A=3+\sqrt{2}-\sqrt{5}$ and $B=(20)^{1/4}$. As in the last proof, let $Q=\sqrt{5}T$. Thus, by Theorem 2 and (2.2), we know that

$$\frac{A+B}{A-B}\sqrt{5}T+\sqrt{5}=\left(\frac{A+B}{A-B}\right)^2\frac{1}{T}-2\frac{A+B}{A-B}-2T+\frac{A-B}{A+B}T^2. \tag{3.2}$$

Let

$$T=\frac{a+b}{a-b}$$

in (3.2). Clearing fractions and simplifying with the help of *Mathematica*, we find that

$$\begin{aligned} &(-10-7\sqrt{2}+4\sqrt{5}+2\sqrt{10})a^3+5^{1/4}(8+9\sqrt{2}-2\sqrt{5}-2\sqrt{10})a^2b\\ &\quad+(-20-15\sqrt{2}+6\sqrt{5}+4\sqrt{10})ab^2+5^{1/4}(10+15\sqrt{2}-4\sqrt{5}-6\sqrt{10})b^3=0. \end{aligned}$$

Let $a = 5^{1/4}d$, cancel $5^{3/4}\sqrt{2}$, and simplify to deduce that

$$\begin{aligned}(-7-5\sqrt{2}+2\sqrt{5}+2\sqrt{10})d^3+(9+4\sqrt{2}-2\sqrt{5}-\sqrt{10})bd^2 \\ +(4+3\sqrt{2}-3\sqrt{5}-2\sqrt{10})b^2d+(-6-2\sqrt{2}+3\sqrt{5}+\sqrt{10})b^3=0.\end{aligned} \tag{3.3}$$

Observe that $d = b$ is a root of (3.3). If this were the root that we are seeking, then Q would equal $(5^{1/4}+1)\sqrt{5}/(5^{1/4}-1)$. Thus, with P and Q interchanged, we have the same solutions to (2.2) that we had in the proof of Theorem 2. Clearly, this is not the solution that we want. Hence, dividing (3.3) by $(d-b)$, we find that

$$(4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10})b^2+2(-1+\sqrt{2}+\sqrt{5})bd+(-10-7\sqrt{2}+4\sqrt{5}+2\sqrt{10})d^2=0.$$

Solving for b, we find that

$$b=\frac{2(1-\sqrt{2}-\sqrt{5})d\pm 2\sqrt{2}\sqrt{(116+83\sqrt{2}-52\sqrt{5}-37\sqrt{10})d^2}}{2(4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10})}. \tag{3.4}$$

Since

$$\begin{aligned}\frac{2(1-\sqrt{2}-\sqrt{5})}{2(4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10})}&=\frac{(1-\sqrt{2}-\sqrt{5})}{(2+3\sqrt{2})(2-\sqrt{5})}\\ &=\frac{(6+5\sqrt{2}+4\sqrt{5}+\sqrt{10})}{14}\\ &=\frac{(1+\sqrt{2}+\sqrt{5})(4+\sqrt{2})}{14},\end{aligned}$$

we are motivated to set $d = 4-\sqrt{2}$. Therefore, $a = 5^{1/4}(4-\sqrt{2})$ in agreement with what Ramanujan claimed. Thus, by (3.4),

$$b=\frac{6-5\sqrt{2}-4\sqrt{5}+\sqrt{10}\pm 2^{5/4}\sqrt{283+190\sqrt{2}-125\sqrt{5}-86\sqrt{10}}}{4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10}}. \tag{3.5}$$

Observe that

$$\frac{6-5\sqrt{2}-4\sqrt{5}+\sqrt{10}}{4+6\sqrt{2}-2\sqrt{5}-3\sqrt{10}}=1+\sqrt{2}+\sqrt{5}. \tag{3.6}$$

We next wish to write

$$4(283+190\sqrt{2}-125\sqrt{5}-86\sqrt{10})=(w+x\sqrt{2}+y\sqrt{5}+z\sqrt{10})^2,$$

for certain rational integers w, x, y, and z. Thus,

$$\begin{aligned}w^2+2x^2+5y^2+10z^2&=1132,\\ wx+5yz&=380,\\ wy+2xz&=-250,\\ wz+xy&=-172.\end{aligned} \tag{3.7}$$

David Bradley kindly wrote a program to determine the 24 positive solutions to (3.7). We then found the unique solution of the system of four diophantine equations to be

$$w = 20, \quad x = 9, \quad y = -8, \quad \text{and} \quad z = -5.$$

Thus,

$$\frac{2\sqrt{283 + 190\sqrt{2} - 125\sqrt{5} - 86\sqrt{10}}}{4 + 6\sqrt{2} - 2\sqrt{5} - 3\sqrt{10}} = \frac{20 + 9\sqrt{2} - 8\sqrt{5} - 5\sqrt{10}}{4 + 6\sqrt{2} - 2\sqrt{5} - 3\sqrt{10}} = 3 - \sqrt{2} + \sqrt{5} - \sqrt{10}. \tag{3.8}$$

Putting (3.6) and (3.8) together, we find that

$$b = 1 + \sqrt{2} + \sqrt{5} \pm 2^{1/4}(3 - \sqrt{2} + \sqrt{5} - \sqrt{10}).$$

If we choose the plus sign above, we would find that $a - b < 0$ and $T < 0$, which is impossible. Thus, we conclude that

$$b = 1 + \sqrt{2} + \sqrt{5} - 2^{1/4}(3 - \sqrt{2} + \sqrt{5} - \sqrt{10}),$$

which is what Ramanujan asserted. Thus, (3.1) is proved, and the proof of Theorem 3 is complete.

4. **Proof of Theorem 4.** By the same reasoning in the proofs of Theorems 1–3, to prove (0.8), it suffices to prove that

$$e^{6\pi/5}\frac{f(-e^{-6\pi/5})}{f(-e^{-30\pi})} = \frac{a+b}{a-b}\sqrt{5}, \tag{4.1}$$

where a and b are specified in the statement of Theorem 4.

Apply the transformation formula (0.12) with $\alpha = 3\pi/5$ and $\beta = 5\pi/3$. After some simplification,

$$f(-e^{-6\pi/5}) = \sqrt{\frac{5}{3}}e^{-4\pi/45}f(-e^{-10\pi/3}).$$

Thus,

$$e^{6\pi/5}\frac{f(-e^{-6\pi/5})}{f(-e^{-30\pi})} = \sqrt{\frac{5}{3}}e^{10\pi/9}\frac{f(-e^{-10\pi/3})}{f(-e^{-30\pi})} =: \sqrt{\frac{5}{3}}A. \tag{4.2}$$

Because $30 = 9 \cdot 10/3$, we are led to Ramanujan's cubic continued fraction

$$G(q) := \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots, \quad |q| < 1.$$

From [2, p. 345, Entry 1(i)],

$$G(-q) = -q^{1/3}\frac{\chi(q)}{\chi^3(q^3)},$$

where

$$\chi(q) = (-q; q^2)_\infty.$$

In particular,

$$G(-e^{-5\pi}) = -e^{-5\pi/3}\frac{\chi(e^{-5\pi})}{\chi^3(e^{-15\pi})} = -\frac{G_{25}}{\sqrt{2}G_{225}^3}, \tag{4.3}$$

by (0.14). Recalling the value $G_{25} = (1+\sqrt{5})/2$ from the lemma of Section 1 and the value

$$G_{225} = \frac{1}{4}(1+\sqrt{5})(2+\sqrt{3})^{1/3}\left\{\sqrt{4+\sqrt{15}} + (15)^{1/4}\right\}$$

from Ramanujan's paper [9], [11, p. 28], but apparently first proved in print by Watson [15], we find that, from (4.3),

$$\begin{aligned} G(-e^{-5\pi}) &= -\frac{16\sqrt{2}}{(1+\sqrt{5})^2(2+\sqrt{3})\left\{\sqrt{4+\sqrt{15}}+(15)^{1/4}\right\}} \\ &= -\frac{2\sqrt{2}(2-\sqrt{3})(3-\sqrt{5})}{(4+\sqrt{15})^{3/2}+3(4+\sqrt{15})(15)^{1/4}+3\sqrt{4+\sqrt{15}}\sqrt{15}+(15)^{3/4}} \\ &= -\frac{(2-\sqrt{3})(3-\sqrt{5})}{\sqrt{2}\left(\sqrt{4+\sqrt{15}}(1+\sqrt{15})+(15)^{1/4}(3+\sqrt{15})\right)}. \end{aligned}$$

Now

$$\begin{aligned} \sqrt{4+\sqrt{15}}(1+\sqrt{15}) &= \sqrt{4+\sqrt{15}(1+\sqrt{15})^2} \\ &= \sqrt{94+24\sqrt{15}} \\ &= \sqrt{2}\sqrt{\frac{1}{4}(6\sqrt{3}+4\sqrt{5})^2} \\ &= \frac{1}{\sqrt{2}}(6\sqrt{3}+4\sqrt{5}). \end{aligned}$$

Thus,

$$\begin{aligned} G(-e^{-5\pi}) &= -\frac{(2-\sqrt{3})(3-\sqrt{5})}{6\sqrt{3}+4\sqrt{5}+(60)^{1/4}(3+\sqrt{5})} \\ &= -\frac{(2-\sqrt{3})(3-\sqrt{5})\left(6\sqrt{3}+4\sqrt{5}-3(60)^{1/4}-\sqrt{15}(60)^{1/4}\right)}{8}. \end{aligned} \tag{4.4}$$

Now Chan [4, Theorem 1] has shown that $G(q)$ satisfies the modular equation

$$G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0.$$

Replacing q by $-q$ and solving for $G(q^2)$, we find that

$$G(q^2) = \frac{1-\sqrt{1-8G^3(-q)}}{4G(-q)}. \tag{4.5}$$

Set $q = e^{-5\pi}$, as above, and $v = G(e^{-10\pi})$. Recall the definition of A from (4.2). Then Entry 1(iv) from Chapter 20 in Ramanujan's second notebook [2, p. 345] can be written in the form

$$3 + A^3 = \frac{1}{v} + 4v^2. \tag{4.6}$$

Thus, by (4.6) and (4.5), with $w := G(-e^{-5\pi})$ given by (4.4),

$$\begin{aligned} A^3 &= 4\left(\frac{1-\sqrt{1-8w^3}}{4w}\right)^2 + \frac{4w}{1-\sqrt{1-8w^3}} - 3 \\ &= -\frac{(w+1)^2(2w-1)}{w^2}, \end{aligned} \tag{4.7}$$

by a somewhat lengthy, but straightforward, calculation.

Hence, by (4.1), (4.2), and (4.7), it remains to show that

$$\left(\frac{(w+1)^2(1-2w)}{w^2}\right)^{1/3} = \frac{a+b}{a-b}\sqrt{3}, \tag{4.8}$$

where a and b are specified in the statement of Theorem 4. We used *Mathematica* to verify (4.8) and complete the proof.

5. **Proof of Theorem 5.** We shall employ a result from Chapter 19 of Ramanujan's second notebook [2, pp. 270–271]. (In particular, see equation (12.13) on p. 270 and the definitions of μ and ν given on p. 271.) Thus,

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{qf^6(-q^5)},$$

or

$$\frac{1}{S^5(q)} + 11 - S^5(q) = \frac{f^6(q)}{qf^6(q^5)}. \tag{5.1}$$

Setting

$$2u := 11 - \frac{f^6(q)}{qf^6(q^5)} \tag{5.2}$$

and solving (5.1) for $S(q)$, we readily find that

$$S(q) = (u + \sqrt{u^2+1})^{1/5}, \tag{5.3}$$

where we took the positive root of the quadratic equation in $S^5(q)$, because $S(q) > 0$. If $q = e^{-\pi/\sqrt{35}}$, we thus see from (5.1) and (5.3) that it suffices to determine

$$e^{\pi/\sqrt{35}}\frac{f^6(e^{-\pi/\sqrt{35}})}{f^6(e^{-\pi\sqrt{5/7}})}.$$

To determine the quotient above, we employ another eta-function identity of Ramanujan [3, p. 236, Entry 71]. Let

$$P = \frac{f(q)}{q^{1/4}f(q^7)} \quad \text{and} \quad Q = \frac{f(q^5)}{q^{5/4}f(q^{35})}.$$

Then

$$(5.4) \qquad (PQ)^2 + 5 + \frac{49}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 + 5\left(\frac{Q}{P}\right)^2 + 5\left(\frac{P}{Q}\right)^2 - \left(\frac{P}{Q}\right)^3.$$

If $q = e^{-\pi/\sqrt{35}}$, then

$$P = e^{\pi/(4\sqrt{35})}\frac{f(e^{-\pi/\sqrt{35}})}{f(e^{-\pi\sqrt{7/5}})} \quad \text{and} \quad Q = e^{5\pi/(4\sqrt{35})}\frac{f(e^{-\pi\sqrt{5/7}})}{f(e^{-\pi\sqrt{35}})}.$$

Setting $\alpha = \pi\sqrt{35}$ and then $\alpha = \pi\sqrt{7/5}$ in the transformation formula (0.13), we find that, respectively,

$$(5.5) \qquad f(e^{-\pi\sqrt{35}}) = (35)^{-1/4}e^{17\pi/(12\sqrt{35})}f(e^{-\pi/\sqrt{35}})$$

and

$$(5.6) \qquad f(e^{-\pi\sqrt{7/5}}) = (5/7)^{1/4}e^{\pi/(12\sqrt{35})}f(e^{-\pi\sqrt{5/7}}).$$

Then

$$(5.7) \qquad PQ = \frac{e^{\pi/(4\sqrt{35})}f(e^{-\pi/\sqrt{35}})}{(5/7)^{1/4}e^{\pi/(12\sqrt{35})}f(e^{-\pi\sqrt{5/7}})}\frac{e^{5\pi/(4\sqrt{35})}f(e^{-\pi\sqrt{5/7}})}{(35)^{-1/4}e^{17\pi/(12\sqrt{35})}f(e^{-\pi/\sqrt{35}})} = \sqrt{7}$$

and

$$(5.8) \qquad \frac{Q}{P} = \sqrt{5}\left(e^{-\pi/(6\sqrt{35})}\frac{f(e^{-\pi\sqrt{5/7}})}{f(e^{-\pi/\sqrt{35}})}\right)^2 =: \sqrt{5}A^2.$$

Substituting (5.7) and (5.8) into (5.4), we deduce that

$$19 = (\sqrt{5}A^2)^3 + 5(\sqrt{5}A^2)^2 + 5(\sqrt{5}A^2)^{-2} - (\sqrt{5}A^2)^{-3}.$$

Setting $x = \sqrt{5}A^2 - (\sqrt{5}A^2)^{-1}$, we can rewrite the foregoing equation in the form

$$19 = x^3 + 3x + 5x^2 + 10$$

or

$$(x-1)(x+3)^2 = 0.$$

It is not difficult to see that x is positive. Thus, $x = 1$ is the only viable root. Solving the resulting equation

$$(\sqrt{5}A^2)^2 - (\sqrt{5}A^2) - 1 = 0,$$

we find that

$$A^2 = \frac{5+\sqrt{5}}{10}.$$

Hence, with $q = e^{-\pi/\sqrt{35}}$, it follows that

$$\frac{f^6(q)}{qf^6(q^5)} = \left(\frac{5+\sqrt{5}}{10}\right)^{-3}. \tag{5.9}$$

Thus, by (5.2),

$$2u = 11 - \left(\frac{10}{5+\sqrt{5}}\right)^3 = 11 - \frac{25}{5+2\sqrt{5}} = -14 + 10\sqrt{5}.$$

Using this value for u in (5.3), we conclude that

$$S(e^{-\pi/\sqrt{35}}) = \left(5\sqrt{5} - 7 + \sqrt{(5\sqrt{5}-7)^2 + 1}\right)^{1/5},$$

which, upon simplification, yields (0.9).

Proof of Corollary. Let $q = e^{-\pi\sqrt{7/5}}$. Then from (5.1)–(5.3), we see that it suffices to evaluate

$$e^{\pi\sqrt{7/5}} \frac{f^6(e^{-\pi\sqrt{7/5}})}{f^6(e^{-\pi\sqrt{35}})}.$$

However, from (5.5), (5.6), and (5.9),

$$\begin{aligned} e^{\pi\sqrt{7/5}} \frac{f^6(e^{-\pi\sqrt{7/5}})}{f^6(e^{-\pi\sqrt{35}})} &= 125 e^{-\pi/\sqrt{35}} \frac{f^6(e^{-\pi\sqrt{5/7}})}{f^6(e^{-\pi/\sqrt{35}})} \\ &= 125\left(\frac{10}{5+\sqrt{5}}\right)^3. \end{aligned}$$

Thus, by (5.2),

$$2u = 11 - 125\left(\frac{10}{5+\sqrt{5}}\right)^3 = -14 - 10\sqrt{5}.$$

The remainder of the proof follows in exactly the same way as before.

Ramanathan [8] employed more recondite ideas to determine Theorem 5 and its corollary, although only the corollary is explicitly stated by him.

6. **Proof of Theorem 6.** To prove (0.10), by (5.1), it suffices to determine

$$e^{\pi/\sqrt{15}} \frac{f^6(e^{-\pi/\sqrt{15}})}{f^6(e^{-\pi\sqrt{3/5}})}.$$

To do this, we employ another P–Q identity of Ramanujan [3, p. 221, Entry 62]. If

$$P = \frac{f(q)}{q^{1/12} f(q^3)} \quad \text{and} \quad Q = \frac{f(q^5)}{q^{5/12} f(q^{15})},$$

then

$$(PQ)^2 - 5 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3. \tag{6.1}$$

Set $q = e^{-\pi/\sqrt{15}}$, so that

$$P = e^{\pi/(12\sqrt{15})}\frac{f(e^{-\pi/\sqrt{15}})}{f(e^{-\pi\sqrt{3/5}})} \quad \text{and} \quad Q = e^{5\pi/(12\sqrt{15})}\frac{f(e^{-\pi\sqrt{5/3}})}{f(e^{-\pi\sqrt{15}})}.$$

By (0.13), with $\alpha = \pi\sqrt{15}$ and then $\alpha = \pi\sqrt{3/5}$, we find that, respectively,

$$f(e^{-\pi\sqrt{15}}) = (15)^{-1/4}e^{14\pi/(24\sqrt{15})}f(e^{-\pi/\sqrt{15}}) \tag{6.2}$$

and

$$f(e^{-\pi\sqrt{3/5}}) = (5/3)^{1/4}e^{-2\pi/(24\sqrt{15})}f(e^{-\pi/\sqrt{5/3}}). \tag{6.3}$$

It follows that, upon simplification,

$$PQ = \sqrt{3} \tag{6.4}$$

and

$$\frac{Q}{P} = \sqrt{5}\left(e^{-\pi/(6\sqrt{15})}\frac{f(e^{-\pi\sqrt{5/3}})}{f(e^{-\pi/\sqrt{15}})}\right)^2 =: \sqrt{5}A^2. \tag{6.5}$$

Employing (6.4) and (6.5) in (6.1), we deduce that

$$5\sqrt{5}A^6 - \frac{1}{5\sqrt{5}A^6} = 1.$$

Solving for A^6, we find that

$$A^6 = \frac{1+\sqrt{5}}{10\sqrt{5}}, \tag{6.6}$$

since $A^6 > 0$. Using this value in (5.2), we deduce that

$$2u = 11 - \frac{10\sqrt{5}}{1+\sqrt{5}} = -\frac{3}{2} + \frac{5\sqrt{5}}{2}$$

and

$$u^2 + 1 = \frac{150 - 30\sqrt{5}}{16}.$$

Using these calculations in (5.3), we complete the proof.

PROOF OF COROLLARY. By (5.1), we need to calculate

$$e^{\pi\sqrt{3/5}}\frac{f^6(e^{-\pi\sqrt{3/5}})}{f^6(e^{-\pi\sqrt{15}})}.$$

A brief calculation with the use of (6.2) and (6.3) shows that

$$e^{\pi\sqrt{3/5}}\frac{f^6(e^{-\pi\sqrt{3/5}})}{f^6(e^{-\pi\sqrt{15}})} = 125e^{-\pi\sqrt{15}}\frac{f^6(e^{-\pi\sqrt{5/3}})}{f^6(e^{-\pi/\sqrt{15}})}.$$

Hence, from (5.2) and (6.6),

$$2u = 11 - 125\frac{1+\sqrt{5}}{10\sqrt{5}} = -\frac{3}{2} - \frac{5\sqrt{5}}{2},$$

and the remainder of the proof is exactly the same as that for Theorem 6.

By using Kronecker's limit formula, Ramanathan [6] established both Theorem 6 and its corollary.

7. **Concluding remarks.** Ramanujan claimed on pp. 204 and 210 of his lost notebook [12] to have determined certain other values for the Rogers-Ramanujan continued fraction. Most of these, if not proved by Ramanathan, can be established by Ramanathan's theorems [6], [8]. However, as mentioned earlier, Ramanathan's methods were entirely unknown to Ramanujan, and it would be instructive to construct proofs of all these results using ideas with which Ramanujan was familiar.

One could also evaluate many Rogers-Ramanujan continued fractions by using Ramanujan's modular equations relating $R(q)$ to each of $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$, and $R(q^5)$. These can be found in the unorganized pages of his second notebook, and proofs and/or discussions of all of them can be found in the monograph [1, pp. 11, 27, 28, 31, 34]. In particular, since $R(e^{-2\pi})$ has been evaluated in (0.1), these modular equations can, in principle, be used to prove Theorems 1–4. However, our attempts to use modular equations gave us evaluations that were considerably more complicated than those of Ramanujan, and we had difficulty reconciling the different evaluations.

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