



ELSEVIER

Journal of Computational and Applied Mathematics 105 (1999) 9–24

JOURNAL OF
COMPUTATIONAL AND
APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

The Rogers–Ramanujan continued fraction

Bruce C. Berndt^{a,*}, Heng Huat Chan^b, Sen-Shan Huang^c, Soon-Yi Kang^a, Jaebum Sohn^a,
Seung Hwan Son^d

^a*Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, United States*

^b*Department of Mathematics, National University of Singapore, Kent Ridge, Singapore 119260, Singapore*

^c*Department of Mathematics, National Chang Hua University of Education, Chang Hua City, Taiwan, ROC*

^d*1808 Stearns Hill Road, Waltham, MA 02451, USA*

Received 29 September 1997

Dedicated to Professor Haakon Waadeland on the occasion of his 70th birthday

Abstract

A survey of many theorems on the Rogers–Ramanujan continued fraction is provided. Emphasis is given to results from Ramanujan’s lost notebook that have only recently been proved. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Rogers–Ramanujan continued fraction; Ramanujan’s lost notebook; Theta-function. Modular equation; Explicit evaluation

1. Introduction

The first infinite continued fractions that one likely encounters in a course in elementary number theory are

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} = \frac{\sqrt{5} + 1}{2} \tag{1.1}$$

and

$$1 - \frac{1}{1 + \frac{1}{1 - \frac{1}{1 + \frac{1}{1 - \dots}}} = \frac{\sqrt{5} - 1}{2}, \tag{1.2}$$

* Corresponding author.

E-mail address: berndt@math.uiuc.edu (B.C. Berndt)

where we use the customary notation

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}} := b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}}.$$

In connection with his first proof of the now celebrated, and subsequently named Rogers–Ramanujan identities, Rogers [28] first considered the natural generalization of (1.1) wherein the n th partial numerator of (1.1) is replaced by q^n , $0 \leq n < \infty$. More precisely, for $|q| < 1$, define the “Rogers–Ramanujan continued fraction” by

$$R(q) := \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \dots}}}}$$

and, for later purposes, define

$$S(q) := -R(-q) \tag{1.3}$$

and

$$K(q) := 1/R(q). \tag{1.4}$$

Also, for $|q| < 1$, let

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} \quad \text{and} \quad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n}$$

denote the “Rogers–Ramanujan functions”, where

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k),$$

and where, for the sequel,

$$(a; q)_{\infty} := \lim_{n \rightarrow \infty} (a; q)_n, \quad |q| < 1.$$

Rogers [28] proved the representation

$$q^{-1/5} R(q) = \frac{H(q)}{G(q)} \tag{1.5}$$

and the Rogers–Ramanujan identities

$$G(q) = \frac{1}{(q; q^5)_{\infty} (q^4; q^5)_{\infty}} \quad \text{and} \quad H(q) = \frac{1}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}}. \tag{1.6}$$

By combining (1.5) and (1.6), we obtain the elegant representation

$$R(q) = q^{1/5} \frac{(q; q^5)_{\infty} (q^4; q^5)_{\infty}}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}}. \tag{1.7}$$

About 20 years later, in his first two letters to Hardy [26, pp. xxvii, xxviii] [12, pp. 21–30, 53–62], S. Ramanujan made several claims about $R(q)$. Moreover, in his notebooks [25] and “lost notebook” [27], Ramanujan recorded without proofs many evaluations and theorems about $R(q)$. Ramanujan’s

lost notebook especially contains an enormous amount of material about $R(q)$, and many of these results have only recently been established for the first time.

The Rogers–Ramanujan continued fraction possesses a rich and beautiful theory containing fascinating and surprising results, and so the purpose of this paper is to provide a survey of our present knowledge about $R(q)$, with a modest emphasis on results found in the lost notebook.

By standard theorems (e.g., see Lorentzen and Waadeland’s text [17, pp. 57, 151, 273]), $R(q)$ converges for $|q| < 1$. From (1.1) and (1.2) it is plain that $R(q)$ converges for $q = \pm 1$. In Section 2, we discuss $R(q)$ for other points on the circle $|q| = 1$, and also examine the approximants to $R(q)$ when $|q| > 1$.

Section 3 features perhaps the two primary formulas involving $R(q)$, and further refinements and generalizations. The two main formulas are particularly useful in the precise determination of certain values of $R(q)$. All of the formulas in this section originate in Ramanujan’s second and lost notebooks [25,27].

Ramanujan was keenly interested in determining exact formulas for $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for positive rational values of n . In his first letter to Hardy, dated 16 January 1913 [26, p. xxvii] [12, p. 29] Ramanujan gave the values

$$R(e^{-2\pi}) = \sqrt{\frac{5 + \sqrt{5}}{2}} - \frac{\sqrt{5} + 1}{2}$$

and

$$S(e^{-\pi}) = \sqrt{\frac{5 - \sqrt{5}}{2}} - \frac{\sqrt{5} - 1}{2},$$

first established by Watson [34], and, in his second letter dated 27 February 1913, he offered the value [26, p. xxviii] [12, p. 57]

$$R(e^{-2\pi\sqrt{5}}) = \frac{\sqrt{5}}{1 + \sqrt[5]{5^{3.4}((\sqrt{5} - 1)/2)^{5.2} - 1}} - \frac{\sqrt{5} + 1}{2},$$

also first proved by Watson [35]. Moreover, in both letters, Ramanujan [26, p. xxvii] [12, pp. 29, 57] asserted that

$$R(e^{-\pi\sqrt{n}}) \text{ “can be exactly found if } n \text{ be any positive rational quantity”}. \tag{1.8}$$

In Section 4, we provide a precise interpretation of this statement and state some theorems that provide a means for explicitly determining $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$. Ramanujan evidently was keenly interested in such determinations, for he not only offered several values in his first and lost notebooks, but he also stated in his lost notebook several formulas leading to explicit determinations, although some of them are not easily applied.

A modular equation for $R(q)$ is an equation involving $R(q)$ and either $R(-q)$ or $R(q^n)$ for some positive integral value(s) of n . Such elegant equations are the focus of Section 5.

In his second notebook, Ramanujan introduced the parameter $k := R(q)R^2(q^2)$, and in his lost notebook, Ramanujan demonstrated the usefulness of k by stating many elegant formulas involving it. Such formulas are also modular equations, and we offer examples in Section 5.

In Section 6, we offer a variety of further results about $R(q)$. As readers will see, some of these formulas are truly surprising and remarkable, and we wonder how Ramanujan was motivated to discover them.

It would be impossible in a paper of moderate length to discuss every property and application of $R(q)$. In particular, we do not discuss here combinatorial applications, such as a series of formulas recorded by Ramanujan in his lost notebook and proved by Andrews [2]. We also do not examine the finite Rogers–Ramanujan continued fraction. Moreover, except for one result examined in Section 6, we refrain from discussing generalizations of $R(q)$, such as the “generalized Rogers–Ramanujan continued fraction”,

$$\frac{1}{1 + \frac{aq}{1 + \frac{aq^2}{1 + \frac{aq^3}{\dots}}}}$$

Furthermore, the Rogers–Ramanujan continued fraction arises from more general continued fractions of quotients of basic hypergeometric series, which we also do not examine in this paper.

Unless otherwise stated, page numbers refer to the lost notebook [27].

In closing our Introduction, we remark that after receiving Ramanujan’s first two letters, Hardy strongly encouraged Ramanujan to prepare some of his results for publication, and, in particular, advised [12, p. 87], “Write it in the form of a paper ‘On the continued fraction

$$\frac{1}{1 + \frac{1}{x + \frac{1}{x^2 + \dots}}},$$

giving a full proof of the principal and most remarkable theorem, viz. that the fraction can be expressed in finite terms when $x = e^{-\pi\sqrt{n}}$, where n is rational”. However, Ramanujan never followed Hardy’s advice. Eventually, in his paper [24] [26, pp. 214–215], in which he gave a proof of the Rogers–Ramanujan identities (1.6), Ramanujan did provide a proof of (1.5), but he never proved in print anything else about $R(q)$. Although Ramanujan offered nearly 200 results on continued fractions in his notebooks, it is unfortunate that this is the only result on continued fractions that he published, except for a few continued fractions that he submitted as problems to the *Journal of the Indian Mathematical Society*.

2. The convergence of $R(q)$

In his third notebook [25, p. 383], Ramanujan claimed that if $u = R(q)$, then $u^2 + u - 1 = 0$ when $q^n = 1$, where n is any positive integer except multiples of 5 in which case u is not definite”. This claim is not quite correct. Indeed, $R(q)$ diverges at n th roots of unity when n is a multiple of 5 and converges at n th roots of unity otherwise, but its value is not a root of $u^2 + u - 1 = 0$. Huang [14] recently found the source of Ramanujan’s error in Ramanujan’s lost notebook [27, p. 57]. On this page, Ramanujan stated several facts about finite generalized Rogers–Ramanujan continued fractions $P_n(a, q)$ at roots of unity. At the bottom of the page, Ramanujan composed a table of values for $P_n(a, q)$, which unfortunately contains some mistakes. If we set $a = 1$ and let n tend to ∞ , the table yields the erroneous values of $R(q)$ cited above.

In 1917, unaware of the work of Rogers and Ramanujan, Schur [30] [31, pp. 117–136] also proved (1.5) and (1.6) and furthermore examined $R(q)$ at roots of unity. We now state Schur’s theorem. Recall that $K(q)$ is defined by (1.4).

Theorem 2.1. *Let q be a primitive n th root of unity. If n is a multiple of 5, $K(q)$ diverges. When n is not a multiple of 5, let $\lambda = (\frac{n}{5})$, the Legendre symbol. Furthermore, let ρ denote the least positive residue of n modulo 5. Then for $n \not\equiv 0 \pmod{5}$,*

$$K(q) = \lambda q^{(1-\lambda\rho n)/5} K(\lambda).$$

Recall that $K(\lambda)$ is given by (1.1) or (1.2). As an example, let $n=3$. Then $K(q) = -q^2(\sqrt{5}-1)/2$. If $|q|=1$ and q is not a root of unity, it is not known if $R(q)$ converges or diverges.

In his third notebook [25, pp. 374, 382], Ramanujan also considered the approximants of $R(q)$ for $|q| > 1$. Recall that $S(q)$ is defined by (1.3).

Theorem 2.2. *Let $|q| > 1$. Then the odd approximants of $R(q)$ tend to $1/S(-1/q)$, while the even approximants of $R(q)$ tend to $R(1/q^4)$.*

It is remarkable that $R(q)$ reappears in the limits of the even and odd approximants; we know of no other instance of this type of behavior. Both Theorems 2.1 and 2.2 can be found in [3, pp. 62, 67] and [6, Ch. 32, Entry 11].

3. The primary formulas for $R(q)$

To establish several of the claims on $R(q)$ that Ramanujan made in his first two letters to Hardy, Watson [34] first proved two theorems about $R(q)$ that can be found in Ramanujan’s second notebook [4, pp. 265–267]. To state these theorems, we must first define Ramanujan’s function $f(-q)$. For $|q| < 1$,

$$f(-q) := (q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}. \tag{3.1}$$

The second equality in (3.1) is Euler’s pentagonal number theorem. If $q = \exp(2\pi iz)$, where $\text{Im } z > 0$, then $q^{1/24} f(-q) = \eta(z)$, where $\eta(z)$ denotes Dedekind’s eta-function.

Theorem 3.1. *With $f(-q)$ defined by (3.1),*

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}. \tag{3.2}$$

Theorem 3.2. *With $f(-q)$ defined by (3.1),*

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)}. \tag{3.3}$$

The clever proof that we sketch below can be found in a fragment of an unpublished manuscript of Ramanujan that has been published along with Ramanujan’s lost notebook [27, p. 238]. This fragment is in Watson’s handwriting, and so he apparently copied it from an original source that has been lost. This argument is also the one given by Watson in his paper [34].

Ramanujan and Watson applied (3.1) to the right-hand side of (3.2) to deduce that

$$\frac{f(-q^{1/5})}{f(-q^5)} = J_1 - q^{1/5} + q^{2/5}J_2, \tag{3.4}$$

where J_1 and J_2 are power series in q with integral coefficients. They next employed Jacobi’s identity

$$(q; q)_\infty^3 = \sum_{n=0}^\infty (-1)^n (2n + 1) q^{n(n+1)/2} \tag{3.5}$$

on the right-hand side of (3.2) to deduce that

$$\frac{f^3(-q^{1/5})}{f^3(-q^5)} = J'_1 + q^{1/5}J'_2 + 5q^{3/5}, \tag{3.6}$$

where J'_1 and J'_2 are power series in q with integral coefficients. Cubing (3.4), equating the result with (3.6), and then equating coefficients of $q^{2/5}$ on both sides, we find that $J_2 = -1/J_1$. Thus, from (3.4),

$$\frac{f(-q^{1/5})}{f(-q^5)} = J_1 - q^{1/5} - q^{2/5}/J_1. \tag{3.7}$$

It remains to identify J_1 . Utilizing the quintuple product identity and (1.7), Ramanujan and Watson proved that $J_1 = q^{1/5}/R(q)$, which when substituted in (3.7) gives (3.2).

To prove (3.3), replace $q^{1/5}$ by $\rho q^{1/5}$ in (3.2), where ρ denotes any fifth root of unity. Multiplying the five resulting equalities together and simplifying, we deduce (3.3).

On p. 48 in his lost notebook, Ramanujan stated two formulas for $R(q)$ that are “between” (3.2) and (3.3). The primary ingredient in the proof in [11, Theorem 3.1] is Jacobi’s identity (3.5).

Theorem 3.3. *If $f(-q)$ is defined by (3.1), then*

$$\left(\frac{3}{R^2(q)} + R^3(q)\right) q^{2/5} f^3(-q^5) = \sum_{n=-\infty}^\infty (-1)^n (10n + 3) q^{(5n+3)n/2}$$

and

$$\left(\frac{1}{R^3(q)} - 3R^2(q)\right) q^{3/5} f^3(-q^5) = \sum_{n=-\infty}^\infty (-1)^n (10n + 1) q^{(5n+1)n/2}.$$

Ramanujan had the remarkable insight to see that (3.2) could be “factored” to produce two further identities. Although the left-hand side has a natural factorization, it is less clear that the right-hand side has a meaningful factorization. That such factorizations can be equated to yield identities is remarkable. Similarly, Ramanujan found that (3.3) can be factored to yield two additional identities.

Theorem 3.4 (p. 206). *Let $t = R(q)$, and set $\alpha = (1 - \sqrt{5})/2$ and $\beta = (1 + \sqrt{5})/2$. Then*

$$\frac{1}{\sqrt{t}} - \alpha\sqrt{t} = \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^\infty \frac{1}{1 + \alpha q^{n/5} + q^{2n/5}}, \tag{3.8}$$

$$\frac{1}{\sqrt{t}} - \beta\sqrt{t} = \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1 + \beta q^{n/5} + q^{2n/5}}, \tag{3.9}$$

$$\left(\frac{1}{\sqrt{t}}\right)^5 - (\alpha\sqrt{t})^5 = \frac{1}{q^{1/2}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{(1 + \alpha q^n + q^{2n})^5}, \tag{3.10}$$

$$\left(\frac{1}{\sqrt{t}}\right)^5 - (\beta\sqrt{t})^5 = \frac{1}{q^{1/2}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{(1 + \beta q^n + q^{2n})^5}. \tag{3.11}$$

The proofs of (3.8) and (3.9) are similar, and so we thereby obtain a new proof of (3.2). Alternatively, one can prove, say, (3.8), and then employ (3.2) to prove (3.9). Similar remarks can be made about (3.10), (3.11), and (3.3). For proofs of (3.8)–(3.11), see [11, Theorem 4.1].

For the next results, we need to define Ramanujan’s general theta function

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Observe, from (3.1), that $f(-q) = f(-q, -q^2)$.

On p. 207 in his lost notebook, Ramanujan listed the three identities

$$P - Q = 1 + \frac{f(-q^{1/5}, -\lambda q^{2/5})}{q^{1/5} f(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \tag{3.12}$$

$$PQ = 1 - \frac{f(-\lambda, -\lambda^4 q^3) f(-\lambda^2 q, -\lambda^3 q^2)}{f^2(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \tag{3.13}$$

and

$$P^5 - Q^5 = 1 + 5PQ + 5P^2Q^2 + \frac{f(-q, -\lambda^5 q^2) f^5(-\lambda^2 q, -\lambda^3 q^2)}{q f^6(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \tag{3.14}$$

without revealing the definitions of P and Q . Son [33] discovered the identities of P and Q and proved the following theorem.

Theorem 3.5. *If*

$$P = \frac{f(-\lambda^{10} q^7, -\lambda^{15} q^8) + \lambda q f(-\lambda^5 q^2, -\lambda^{20} q^{13})}{q^{1/5} f(-\lambda^{10} q^5, -\lambda^{15} q^{10})} \tag{3.15}$$

and

$$Q = \frac{\lambda f(-\lambda^5 q^4, -\lambda^{20} q^{11}) - \lambda^3 q f(-q, -\lambda^{25} q^{14})}{q^{-1/5} f(-\lambda^{10} q^5, -\lambda^{15} q^{10})}, \tag{3.16}$$

then (3.12)–(3.14) hold.

At the top of p. 207, Ramanujan wrote the quintuple product identity in the form

$$f(-\mu^2 x^3, -\mu x^6) + x f(-\mu, -\mu^2 x^9) = \frac{f(-x^2, -\mu x) f(-\mu x^3)}{f(-x, -\mu x^2)}. \tag{3.17}$$

For a proof of the quintuple product identity in the form (3.17), see [4, pp. 80–82], and for a discussion of the equivalence of (3.17) with more standard formulations, see [5, pp. 10–12].

Now let $\lambda = 1$ in (3.15) and (3.16). Then, by (3.17), with $(x, \mu) = (q, q^2)$ and $(x, \mu) = (q^2, q^{-1})$, respectively,

$$P = \frac{f(-q^7, -q^8) + qf(-q^2, -q^{13})}{q^{1/5}f(-q^5)} = \frac{f(-q^2, -q^3)}{q^{1/5}f(-q, -q^4)} = \frac{1}{q^{1/5}R(q)}$$

and

$$Q = \frac{f(-q^4, -q^{11}) - qf(-q, -q^{14})}{q^{-1/5}f(-q^5)} = \frac{q^{1/5}f(-q, -q^4)}{f(-q^2, -q^3)} = q^{1/5}R(q).$$

Since $PQ = 1$, (3.12) and (3.14) reduce to (3.2) and (3.3), respectively. Hence, Theorem 3.5 yields generalizations of both Theorems 3.1 and 3.2. Son [33] therefore obtained new proofs of these two theorems.

4. Explicit values of $R(q)$

The purpose of this section is to provide a meaning for (1.8). We show that Ramanujan's assertion is indeed correct, provided that certain class invariants can be determined.

From (3.2), we see that to evaluate $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$, it suffices to evaluate

$$A := e^{2\pi\sqrt{n}/5} \frac{f(-e^{-2\pi\sqrt{n}/5})}{f(-e^{-10\pi\sqrt{n}})} \quad (4.1)$$

and

$$A_1 := e^{\pi\sqrt{n}/5} \frac{f(e^{-\pi\sqrt{n}/5})}{f(e^{-5\pi\sqrt{n}})}, \quad (4.2)$$

respectively. Alternatively, from (3.3), we see that it suffices to evaluate

$$A' := e^{2\pi\sqrt{n}/6} \frac{f(-e^{-2\pi\sqrt{n}})}{f(-e^{-10\pi\sqrt{n}})} \quad (4.3)$$

and

$$A'_1 := e^{\pi\sqrt{n}/6} \frac{f(e^{-\pi\sqrt{n}})}{f(e^{-5\pi\sqrt{n}})}, \quad (4.4)$$

respectively. In any case, we merely have to solve a quadratic equation to determine $R(e^{-2\pi\sqrt{n}})$ or $S(e^{-\pi\sqrt{n}})$. Berndt et al. [9] showed how to determine (4.1)–(4.4) from the values of appropriate class invariants.

To define the class invariants G_n and g_n of Weber [36] and Ramanujan [10], first set, after Ramanujan,

$$\chi(q) := (-q; q^2)_\infty, \quad |q| < 1. \quad (4.5)$$

Then, for any positive rational number n , define

$$G_n := 2^{-1/4} e^{\pi\sqrt{n}/24} \chi(e^{-\pi\sqrt{n}})$$

and

$$g_n := 2^{-1/4} e^{\pi\sqrt{n}/24} \chi(-e^{-\pi\sqrt{n}}).$$

For positive odd integers n , G_n is algebraic, and for positive even integers n , g_n is algebraic [36, pp. 540, 541]. For accounts and values of class invariants, consult Weber’s book [36], Ramanujan’s paper [23] [26, pp. 23–39], the paper by Berndt et al. [10], and Berndt’s book [6, Ch. 34].

We cite just two of the theorems proved in [9] that enable us to determine explicit values of $R(e^{-2\pi\sqrt{n}})$.

Theorem 4.1. *Let A be defined by (4.1), and let*

$$V := \sqrt{\frac{G_{25n}}{G_{n/25}}}.$$

Then

$$\frac{A}{\sqrt{5}V} - \frac{\sqrt{5}V}{A} = (V - V^{-1})^2 \left(\frac{V - V^{-1}}{\sqrt{5}} + \frac{\sqrt{5}}{V - V^{-1}} \right).$$

Theorem 4.2. *Let A' be defined by (4.3), and let*

$$V' = \frac{G_{25n}}{G_n}.$$

Then

$$\frac{A'^2}{\sqrt{5}V'} - \frac{\sqrt{5}V'}{A'^2} = \frac{1}{\sqrt{5}}(V'^3 - V'^{-3}).$$

In Theorems 4.1 and 4.2, as well as the other theorems of this type proved in [9], to determine A , A' , A_1 , or A'_1 , one merely has to solve a quadratic equation. Often, $V - V^{-1}$, $V'^3 - V'^{-3}$, and similar expressions simplify considerably.

To illustrate Theorem 4.1, first define

$$2c := \frac{60^{1/4} + 2 - \sqrt{3} + \sqrt{5}}{60^{1/4} - 2 + \sqrt{3} - \sqrt{5}} \sqrt{5} + 1.$$

Then [7, Theorem 4], [6, Ch. 32, Entry 10]

$$R(e^{-6\pi}) = \sqrt{c^2 + 1} - c.$$

Readers have undoubtedly noticed that the values of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ are algebraic numbers. Berndt et al. [9, Corollary 6.3], in fact, proved the following stronger result.

Theorem 4.3. *If n is any positive rational number, then $R(e^{-\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ are units.*

Ramanujan was keenly interested in the determinations of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$. His first two letters to Hardy, his notebooks, and his lost notebook contain several explicit determinations. In particular, p. 210 in the lost notebook contains an incomplete table of values. As the lost notebook

emanates from the last year of his life, it is likely that the incompleteness of the table is reflected in Ramanujan's early death. For proofs of these evaluations, see papers of Berndt and Chan [7] and Berndt et al. [9], where the proofs depend on certain modular equations found by Ramanujan and recorded in his notebooks. See also papers by Ramanathan [19–22] for several evaluations of $R(q)$.

The lost notebook contains various formulas for $R(q)$ and theta-function identities yielding further formulas for $R(q)$. However, although these formulas are very interesting by themselves, they generally do not appear amenable to the calculation of elegant values of $R(q)$. Furthermore, the theta-function identities still must be combined with modular equations to explicitly calculate $R(q)$. The aforementioned claims in the lost notebook have been proved by Kang [16], and we offer here two of these theorems.

We first need to define, in the notation of Ramanujan, the two primary theta-functions,

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n-1)/2}. \quad (4.6)$$

Recall that $\chi(q)$ is defined by (4.5) and that $f(-q)$ is defined in (3.1).

Theorem 4.4. *Let*

$$t := q^{1/5} \frac{\chi(-q^{1/5})}{\chi(-q^5)} \quad \text{and} \quad s := \frac{\varphi(-q^{1/5})}{\varphi(-q^5)}.$$

Then

$$\frac{f(-q^{1/5})}{q^{1/5} f(-q^5)} = \frac{s}{t}, \quad \frac{f(-q^{2/5})}{q^{2/5} f(-q^{10})} = \frac{s}{t^2}, \quad \frac{\psi(q^{1/5})}{q^{3/5} \psi(q^5)} = \frac{s}{t^3},$$

and

$$2s = 1 - 2t - 2t^2 + t^3 + \sqrt{1 - 4t - 10t^3 - 4t^5 + t^6}.$$

The last part of Theorem 4.4 and one of Ramanujan's modular equations can be used to derive some of the results in [9].

Theorem 4.5 (p. 208). *Let t be given in Theorem 4.4. Then*

$$R(q) = \frac{1}{4t} \left(\left(1 + t \frac{\sqrt{5} + 1}{2} \right) \sqrt{1-t} - \sqrt{\left(1-t \right) \left(1 + t \frac{\sqrt{5} + 1}{2} \right)^2 - 2t(\sqrt{5} + 1)} \right) \\ \times \left(- \left(1 - t \frac{\sqrt{5} - 1}{2} \right) \sqrt{1-t} + \sqrt{\left(1-t \right) \left(1 - t \frac{\sqrt{5} - 1}{2} \right)^2 + 2t(\sqrt{5} - 1)} \right).$$

Ramanujan derived a similar formula for $R(q^2)$ [16, Theorem 3.2].

5. Modular equations

We begin this section by providing a list of all the known modular equations involving $R(q)$ with exactly two different arguments.

Theorem 5.1. *Let $u = R(-q)$ and $v = R(q)$. Then*

$$uv(u-v)^4 - u^2v^2(u-v)^2 + 2u^3v^3 = (u-v)(1+u^5v^5).$$

Theorem 5.2. *Let $u = R(q)$ and $v = R(q^2)$. Then*

$$\frac{v-u^2}{v+u^2} = uv^2.$$

Theorem 5.3. *Let $u = R(q)$ and $v = R(q^3)$. Then*

$$(v-u^3)(1+uv^3) = 3u^2v^2.$$

Theorem 5.4. *Let $u = R(q)$ and $v = R(q^4)$. Then*

$$(u^5+v^5)(uv-1) + u^5v^5 + uv = 5u^2v^2(uv-1)^2.$$

Theorem 5.5. *Let $u = R(q)$ and $v = R(q^5)$. Then*

$$u^5 = v \frac{1-2v+4v^2-3v^3+v^4}{1+3v+4v^2+2v^3+v^4}.$$

Theorem 5.6. *Let $u = R(q)$ and $v = R(q^{11})$. Then*

$$uv(1-11u^5-u^{10})(1-11v^5-v^{10}) = (u-v)^{12}.$$

Theorem 5.1, which is due to Ramanujan, was first proved by Andrews et al. [3, pp. 28, 29].

Theorem 5.2 was first stated in Ramanujan's second notebook, and was first proved by Rogers [29, p. 391], while a later proof was given by Andrews et al. [3, pp. 31–33].

The only proof of Theorem 5.3 known to us was given by Rogers [29, p. 392]. Theorem 5.3 is in Ramanujan's second notebook [3, p. 27].

Theorem 5.4, given in Ramanujan's second notebook, was first proved in the monograph by Andrews et al. [3, pp. 34, 35].

Theorem 5.5 was communicated by Ramanujan in his first letter to Hardy [26, p. xxvii] [12, p. 29] and is in Ramanujan's second notebook [3, p. 11]. The first proof in print was given by Rogers [29, p. 392]. Later proofs have been given by Watson [34] and Ramanathan [19].

Theorem 5.6 is due to Rogers [29, p. 392], but Rogers' short proof is a consequence of two of Ramanujan's modular equations.

Theorems 5.1–5.5 can also be found in Berndt's book [6, Ch. 32, Entries 1–3, 5, 6].

Two modular equations involving the Rogers–Ramanujan continued fraction at three distinct arguments can be found on p. 205 in Ramanujan's lost notebook. These, given below, are proved along with some other modular equations involving three arguments in the paper of Berndt et al. [11].

Theorem 5.7. *Let*

$$u = R(q), \quad u' = -R(-q), \quad v = R(q^2), \quad \text{and} \quad w = R(q^4).$$

Then

$$uw = \frac{w - u^2v}{w + v^2} \quad \text{and} \quad uu'v^2 = \frac{uu' - v}{u' - u}.$$

The remainder of this section is devoted to modular equations involving the parameter $k = R(q)R^2(q^2)$.

In his second notebook, Ramanujan offered the formulas in the next theorem, first proved in the *Memoir* [3, pp. 31, 32], and also found in [6, Ch. 32, Entry 4].

Theorem 5.8. *For k defined above with $|k| < 1$,*

$$R^5(q) = k \left(\frac{1-k}{1+k} \right)^2 \quad \text{and} \quad R^5(q^2) = k^2 \left(\frac{1+k}{1-k} \right).$$

On p. 56 in his lost notebook, Ramanujan stated formulas for $R(\sqrt{q})$ and $R(q^4)$, given in the next theorem and first proved by Kang [15]. The factor $((1-k)/(1+k))^{1/10}$ below is missing in Ramanujan's formulation.

Theorem 5.9. *For $|k| < (\sqrt{5} - 1)/2$,*

$$R(\sqrt{q}) = \frac{k^{1/10}(1+k)^{4/5}(1-k)^{1/5}}{\sqrt{k} + \sqrt{1+k-k^2}}$$

and

$$R(q^4) = \left(\frac{1-k}{1+k} \right)^{1/10} \frac{2k^{4/5}}{\sqrt{1-k^2} + \sqrt{1-4k-k^2}}.$$

The parameter k also can be utilized to establish elegant formulas for the theta-functions $\varphi(q)$ and $\psi(q)$, defined by (4.6).

Theorem 5.10. *If $k \leq \sqrt{5} - 2$, then*

$$\frac{\varphi^2(-q)}{\varphi^2(-q^5)} = \frac{1-4k-k^2}{1-k^2} \quad \text{and} \quad \frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{1+k-k^2}{k}.$$

These formulas are also found on p. 56 in the lost notebook; these and many other formulas of this sort in the lost notebook have been proved by Kang [15].

6. Miscellaneous results

Our first theorem in this last section is one of Ramanujan's most curious discoveries about $R(q)$.

Theorem 6.1. For $|q| < 1$,

$$R^3(q) = \frac{\sum_{n=0}^{\infty} q^{5n^2+4n}(1+q^{5n+2})/(1-q^{5n-2}) - \sum_{n=0}^{\infty} q^{5n^2+6n+1}(1+q^{5n+3})/(1-q^{5n-3})}{\sum_{n=0}^{\infty} q^{5n^2+2n}(1+q^{5n+1})/(1-q^{5n-1}) - \sum_{n=0}^{\infty} q^{5n^2+8n+3}(1+q^{5n+4})/(1-q^{5n-4})}. \tag{6.1}$$

We have no insight whatsoever about what led Ramanujan to consider the quotient of q -series on the right-hand side of (6.1) or why he thought that it would simplify to a “nice” function such as $R^3(q)$. In his elegant proof of (6.1), Andrews [1] first transformed (6.1) into the equivalent formulation

$$R^3(q) = \frac{\sum_{n=0}^{\infty} q^{2n}/(1-q^{5n+2}) - \sum_{n=0}^{\infty} q^{3n+1}/(1-q^{5n+3})}{\sum_{n=0}^{\infty} q^n/(1-q^{5n+1}) - \sum_{n=0}^{\infty} q^{4n+3}/(1-q^{5n+4})}. \tag{6.2}$$

Andrews next easily combined both the numerator and denominator on the right-hand side of (6.2) into bilateral series, which are then summed by Ramanujan’s famous ${}_1\psi_1$ summation theorem. Amazingly, the 16 infinite products arising from the two applications of the ${}_1\psi_1$ theorem simplify, via (1.7), to $R^3(q)$.

The next theorem is the only one in our survey that concerns a generalization of $R(q)$. This beautiful result appears on p. 46 in Ramanujan’s lost notebook.

Theorem 6.2. Let $k \geq 0$, $\alpha = (1 + \sqrt{1 + 4k})/2$, and $\beta = (-1 + \sqrt{1 + 4k})/2$. Then, for $|q| < 1$ and $\operatorname{Re} q > 0$,

$$\frac{1}{1 +} \frac{k + q}{1 +} \frac{k + q^2}{1 +} \frac{k + q^3}{1 +} + \dots = \frac{1}{\alpha +} \frac{q}{\alpha + \beta q +} \frac{q^2}{\alpha + \beta q^2 +} \frac{q^3}{\alpha + \beta q^3 +} + \dots. \tag{6.3}$$

The cases $q = 0$ and 1 in Theorem 6.2 yield elementary results, and the case $k = 0$ reduces to the tautology $R(q) = R(q)$. The following elegant instance ($k = 2$) of Theorem 6.2 is also found (with a misprint) on p. 46 of the lost notebook.

Corollary 6.3. For $|q| < 1$,

$$\frac{1}{1 +} \frac{2 + q}{1 +} \frac{2 + q^2}{1 +} \frac{2 + q^3}{1 +} + \dots = \frac{1}{2 +} \frac{q}{2 + q +} \frac{q^2}{2 + q^2 +} \frac{q^3}{2 + q^3 +} + \dots.$$

Lorentzen and Waadeland [17, pp. 77–80] used the Bauer–Muir transformation to prove a special case of Theorem 6.2 and to discuss the rapidity of convergence of the transformed continued fraction. Berndt et al. [11] also used the Bauer–Muir transformation to prove Theorem 6.2.

For the next theorem, recall that $f(-q)$ is defined in (3.1).

Theorem 6.4. For $0 < q < 1$,

$$R(q) = \frac{\sqrt{5} - 1}{2} \exp\left(-\frac{1}{5} \int_q^1 \frac{f^5(-t) dt}{f(-t^5) t}\right) \tag{6.4}$$

and

$$R(q) = \frac{\sqrt{5} - 1}{2} - \frac{\sqrt{5}}{1 + \frac{3 + \sqrt{5}}{2} \exp\left(\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5}) t^{4/5}} dt\right)}. \quad (6.5)$$

These integral representations are somewhat less strange than they appear at a first examination. The former was first proved by Andrews [1], while the latter was first proved by Son [32]. To prove (6.4), we employ a famous identity of Ramanujan [4, p. 257],

$$\frac{f^5(-q)}{f(-q^5)} = 1 - 5 \sum_{n=1}^{\infty} \binom{n}{5} \frac{nq^n}{1 - q^n}, \quad |q| < 1, \quad (6.6)$$

where $\binom{n}{5}$ denotes the Legendre symbol. By using (1.7) along with (6.6), we can readily deduce (6.4). To prove (6.5), we employ the following two lemmas and (6.4). See [4, p. 43, Entry 27 (iii); p. 83, Entry 39 (i)].

Lemma 6.5. *If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then*

$$\alpha^{1/4} e^{-\alpha^{1/2}} f(-e^{-2\alpha}) = \beta^{1/4} e^{-\beta^{1/2}} f(-e^{-2\beta}).$$

Lemma 6.6. *Let $\alpha, \beta > 0$, $\alpha\beta = \pi^2$, $q := e^{-2\alpha}$ and $Q := e^{-2\beta}$. Then*

$$\left(\frac{\sqrt{5} + 1}{2} + R(q)\right) \left(\frac{\sqrt{5} + 1}{2} + R(Q)\right) = \frac{5 + \sqrt{5}}{2}.$$

The previous lemma was communicated by Ramanujan in his famous second letter to Hardy [12, p. 57].

The integrand in (6.5) has the representation

$$q \frac{f^5(-q^5)}{f(-q)} = \sum_{n=1}^{\infty} \binom{n}{5} \frac{q^n}{(1 - q^n)^2}, \quad |q| < 1, \quad (6.7)$$

a companion to (6.6). Equality (6.7) leads to an elegant proof of Ramanujan's congruence $p(5n+4) \equiv 0 \pmod{5}$ for the partition function $p(n)$. For a proof of (6.7), the deduction of this congruence from it, and further references to proofs of (6.7), see Chan's paper [13].

We close our survey with one of several identities for incomplete elliptic integrals of the first kind involving $R(q)$ that are found in Ramanujan's lost notebook. These were first proved by Raghavan and Rangachari [18], while later proofs, more in the spirit of Ramanujan's work, were found by Berndt et al. [8].

Theorem 6.7. *Let $u = R(q)$ and $\varepsilon = (1 + \sqrt{5})/2$. Then, for $|q| < 1$,*

$$5^{3/4} \int_0^q \frac{f^2(-t) f^2(-t^5)}{\sqrt{t}} dt = 2 \int_{\cos^{-1}(\varepsilon u)^{1/2}}^{\pi/2} \frac{d\phi}{\sqrt{1 - \varepsilon^{-5} 5^{-3/2} \sin^2 \phi}}.$$

References

- [1] G.E. Andrews, An introduction to Ramanujan's "lost" notebook, *Amer. Math. Mon.* 86 (1979) 89–108.
- [2] G.E. Andrews, Ramanujan's "lost" notebook. III. The Rogers–Ramanujan continued fraction, *Adv. Math.* 41 (1981) 186–208.
- [3] G.E. Andrews, B.C. Berndt, L. Jacobsen, R.L. Lamphere, The continued fractions found in the unorganized portions of Ramanujan's notebooks, *Memoir No. 477, Amer. Math. Soc.* 99 (1992).
- [4] B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991.
- [5] B.C. Berndt, Ramanujan's theory of theta-functions, in: M. Ram Murty (Ed.), *Theta Functions from the Classical to the Modern, CRM Proceedings and Lecture Notes*, vol. 1, American Mathematical Society, Providence, 1993, pp. 1–63.
- [6] B.C. Berndt, *Ramanujan's Notebooks, Part V*, Springer, New York, 1998.
- [7] B.C. Berndt, H.H. Chan, Some values for the Rogers–Ramanujan continued fraction, *Can. J. Math.* 47 (1995) 897–914.
- [8] B.C. Berndt, H.H. Chan, S.-S. Huang, Incomplete elliptic integrals in Ramanujan's lost notebook, in: M.E.H. Ismail, D. Stanton (Eds.), *q-Series: A Current Perspective*, Amer. Math. Soc., Providence, to appear.
- [9] B.C. Berndt, H.H. Chan, L.-C. Zhang, Explicit evaluations of the Rogers–Ramanujan continued fraction, *J. Reine Angew. Math.* 480 (1996) 141–159.
- [10] B.C. Berndt, H.H. Chan, L.-C. Zhang, Ramanujan's class invariants, Kronecker's limit formula, and modular equations, *Trans. Amer. Math. Soc.* 349 (1997) 2125–2173.
- [11] B.C. Berndt, S.-S. Huang, J. Sohn, S.H. Son, Some theorems on the Rogers–Ramanujan continued fraction in Ramanujan's lost notebook, *Trans. Amer. Math. Soc.*, to appear.
- [12] B.C. Berndt, R.A. Rankin, *Ramanujan: Letters and Commentary*, Amer. Math. Soc., Providence, 1995; *London Math. Soc.*, London, 1995.
- [13] H.H. Chan, New proofs of Ramanujan's partition identities for moduli 5 and 7, *J. Number Theory* 53 (1995) 144–158.
- [14] S.-S. Huang, Ramanujan's evaluations of Rogers–Ramanujan type continued fractions at primitive roots of unity, *Acta Arith.* 80 (1997) 49–60.
- [15] S.-Y. Kang, Some theorems on the Rogers–Ramanujan continued fraction and associated theta function identities in Ramanujan's lost notebook, *The Ramanujan J.* 3 (1999) 1–21.
- [16] S.-Y. Kang, Ramanujan's formulas for the explicit evaluation of the Rogers–Ramanujan continued fraction and theta-functions, *Acta Arith.*, to appear.
- [17] L. Lorentzen, H. Waadeland, *Continued Fractions with Applications*. North-Holland, Amsterdam, 1992.
- [18] S. Raghavan, S.S. Rangachari, *On Ramanujan's elliptic integrals and modular identities, Number Theory and Related Topics*, Oxford University Press, Bombay, 1989, pp. 119–149.
- [19] K.G. Ramanathan, On Ramanujan's continued fraction, *Acta Arith.* 43 (1984) 209–226.
- [20] K.G. Ramanathan, On the Rogers–Ramanujan continued fraction, *Proc. Indian Acad. Sci. (Math. Sci.)* 93 (1984) 67–77.
- [21] K.G. Ramanathan, Ramanujan's continued fraction, *Indian J. Pure Appl. Math.* 16 (1985) 695–724.
- [22] K.G. Ramanathan, Some applications of Kronecker's limit formula, *J. Indian Math. Soc.* 52 (1987) 71–89.
- [23] S. Ramanujan, Modular equations and approximations to π , *Q. J. Math. (Oxford)* 45 (1914) 350–372.
- [24] S. Ramanujan, Proof of certain identities in combinatory analysis, *Proc. Cambridge Philos. Soc.* 19 (1919) 214–216.
- [25] S. Ramanujan, *Notebooks*, 2 vols, Tata Institute of Fundamental Research, Bombay, 1957.
- [26] S. Ramanujan, *Collected Papers*, Chelsea, New York, 1962.
- [27] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
- [28] L.J. Rogers, Second memoir on the expansion of certain infinite products, *Proc. London Math. Soc.* 25 (1894) 318–343.
- [29] L.J. Rogers, On a type of modular relation, *Proc. London Math. Soc.* 19 (1920) 387–397.
- [30] I.J. Schur, Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, *S.-B. Preuss. Akad. Wiss. Phys.-Math. Kl.* (1917) 302–321.
- [31] I.J. Schur, *Gesammelte Abhandlungen*, vol. 2, Springer, Berlin, 1973.

- [32] S.H. Son, Some integrals of theta functions in Ramanujan's lost notebook, in: R. Gupta, K.S. Williams (Eds.), Proc. Canad. No. Thy. Assoc. No. 5, Amer. Math. Soc., Providence, 1999, pp. 329–339.
- [33] S.H. Son, Some theta function identities related to the Rogers–Ramanujan continued fraction, Proc. Amer. Math. Soc. 126 (1998) 2895–2902.
- [34] G.N. Watson, Theorems stated by Ramanujan (VII): theorems on continued fractions, J. London Math. Soc. 4 (1929) 39–48.
- [35] G.N. Watson, Theorems stated by Ramanujan (IX): two continued fractions, J. London Math. Soc. 4 (1929) 231–237.
- [36] H. Weber, Lehrbuch der Algebra, dritter Band, Chelsea, New York, 1961.