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Dedicated to our advisor, Professor Bruce C. Berndt on his 60^{th} birthday

In this paper, we derive new Ramanujan-type series for $1/\pi$ which belong to "*Ramanujan's theory of elliptic functions to alternative base* 3" developed recently by B.C. Berndt, S. Bhargava, and F.G. Garvan.

1. Introduction.

Let $(a)_0 = 1$ and, for a positive integer m,

$$(a)_m := a(a+1)(a+2)\cdots(a+m-1),$$

and

$${}_2F_1(a,b;c;z) := \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}, \quad |z|<1.$$

In his famous paper "Modular equations and approximations to π" [**10**], S. Ramanujan offered 17 beautiful series representations for $1/\pi$. He then remarked that two of these series

$$\frac{27}{4\pi} = \sum_{m=0}^{\infty}(2+15m)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}\left(\frac{2}{27}\right)^m \tag{1.1}$$

and

$$\frac{15\sqrt{3}}{2\pi} = \sum_{m=0}^{\infty}(4+33m)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}\left(\frac{4}{125}\right)^m \tag{1.2}$$

"belong to the theory of q_2," where

$$q_2 = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-k^2)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;k^2)}\right).$$

Ramanujan did not elaborate on his "theory of q_2," neither did he provide details for his proofs of (1.1) and (1.2).

Ramanujan's formulas (1.1) and (1.2) were first proved by J.M. Borwein and P.B. Borwein in 1987. Motivated by their study of Ramanujan's series for $1/\pi$ associated with the classical theory of elliptic functions, they established the following result:

Theorem 1.1 ([**3**, p. 186])**.** *Let*

$$K(x) := {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; x), \quad \textit{and} \quad \dot{K}(x) := \frac{dK(x)}{dx}.$$

For $n \in \mathbb{Q}^+$, define the cubic singular modulus to be the unique number α_n satisfying

$$\frac{K(1-\alpha_n)}{K(\alpha_n)} = \sqrt{n}. \tag{1.3}$$

Set

$$\epsilon(n) = \frac{3\sqrt{3}}{8\pi}(K(\alpha_n))^{-2} - \sqrt{n}\left(\frac{3}{2}\alpha_n(1-\alpha_n)\frac{\dot{K}(\alpha_n)}{K(\alpha_n)} - \alpha_n\right), \tag{1.4}$$

$$a_n := \frac{8\sqrt{3}}{9}\left(\epsilon(n) - \sqrt{n}\alpha_n\right), \tag{1.5}$$

and

$$b_n := \frac{2\sqrt{3n}}{3}\sqrt{1-H_n}, \tag{1.6}$$

where

$$H_n := 4\alpha_n(1-\alpha_n). \tag{1.7}$$

Then

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}(a_n + b_n m)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}H_n^m. \tag{1.8}$$

Remark. We state this theorem with a different definition of $\epsilon(n)$ than that given in [**3**]. We have avoided using elliptic integrals of the second kind and Legendre's relation.

The Borweins' theorem indicates that for each positive rational number n, we can easily derive a series for $1/\pi$ belonging to the "theory of q_2" if the values of α_n and $\epsilon(n)$ (the rest of the constants can be computed from these) are known. The computation of these constants for any given n, however, is far from trivial.

The Borweins' method of evaluating α_n involves solving a quartic equation. More precisely, they show that when n is an odd positive integer, α_n is the smaller of the two real solutions of the equation

$$\frac{(9-8\alpha_n)^3}{64\alpha_n^3(1-\alpha_n)} = \frac{(4G_{3n}^{24}-1)^3}{27G_{3n}^{24}}, \tag{1.9}$$

where G_n is the classical Ramanujan-Weber class invariant defined by

$$G_n := 2^{-1/4}e^{\pi\sqrt{n}/24}\prod_{m=1}^{\infty}(1+e^{-\pi\sqrt{n}(2m-1)}).$$

Using known values for G_{3n}, they derive α_n for $n = 3$ and 5 from (1.9). For example, from (see [**1**, p. 190])

$$G_{15}^{12} = 8\left(\frac{\sqrt{5}+1}{2}\right)^4,$$

they deduce that

$$\alpha_5 = \frac{1}{2} - \frac{11\sqrt{5}}{50}.$$

When n is an even positive integer, the corresponding formula between α_n and g_{3n} is

$$\frac{(9-8\alpha_n)^3}{64\alpha_n^3(1-\alpha_n)} = \frac{(4g_{3n}^{24}+1)^3}{27g_{3n}^{24}}, \tag{1.10}$$

where g_n is the other Ramanujan-Weber class invariant defined by

$$g_n := 2^{-1/4}e^{\pi\sqrt{n}/24}\prod_{m=1}^{\infty}(1-e^{-\pi\sqrt{n}(2m-1)}).$$

Using (1.10) and known values of g_{3n}, they compute α_n for $n = 2$, 4, and 6. Together with the values of $\epsilon(n)$ for $n = 2$, 3, 4, 5, and 6 [**3**, p. 190, Problem 20], they obtained five series for $1/\pi$. Ramanujan's series (1.1) and (1.2) then correspond to $n = 4$ and 5, respectively. At the end of [**3**, Chapter 5, Section 5], the Borweins remark that their explanation of Ramanujan's series (1.1) and (1.2) is "a bit disappointing" as they only have "well-concealed analogues of the original theory for K."

In a recent paper, B. C. Berndt, S. Bhargava, and F. G. Garvan [**2**] succeeded in developing Ramanujan's "corresponding theories" mentioned in [**10**]. One of these theories is Ramanujan's "theory of q_2" and its discovery has motivated us to revisit Ramanujan's series (1.1) and (1.2). This theory is now known as "*Ramanujan's theory of elliptic functions to alternative base* 3" or "*Ramanujan's elliptic functions in the theory of signature* 3."

In this article, we derive some new formulas from the "theory of q_2" which will facilitate the computations of α_n and $\epsilon(n)$. With the aid of cubic

Russell-type modular equations (see [**6**]) and Kronecker's Limit Formula, we discover new Ramanujan-type series for $1/\pi$ belonging to the "theory of q_2." An example of these series, which corresponds to $n = 59$, is

(1.11)
$$\frac{2153559\sqrt{3}}{\pi} = \sum_{m=0}^{\infty}(a+bm)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}\left(\frac{73-40\sqrt{3}}{2^{1/3}\cdot 23^2(4+5\sqrt{3})}\right)^{3m},$$

where

$$a := 1028358\sqrt{3} - 593849 \quad \text{and} \quad b := 19101285\sqrt{3} - 795.$$

Each term in this series gives approximately 10 decimal places of π.

In Section 2, we recall some important results proved in [**2**] and establish new formulas satisfied by $\epsilon(n)$ which lead to a new formula for a_n. In Section 3, we describe our strategy for computing a_n. In Section 4, we indicate that if $3n$ is an *Euler convenient number*, then α_n, as well as other related cubic singular moduli, can be computed explicitly via Kronecker's Limit Formula. These values are used to derive the constants a_n, b_n, and H_n listed in our final section.

2. Ramanujan's elliptic functions in the theory of signature 3 (Ramanujan's "theory of q_2").

Define

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

Theorem 2.1. *If*

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{K(1-\alpha)}{K(\alpha)}\right), \tag{2.1}$$

then

$$\alpha = \frac{c^3(q)}{a^3(q)}. \tag{2.2}$$

Theorem 2.2 (Borweins' Inversion Formula)**.** *We have*

$$a(q) = K\left(\frac{c^3(q)}{a^3(q)}\right) = K(\alpha), \tag{2.3}$$

where $K(\cdot)$ is defined in Theorem 1.1.

Theorem 2.1 and Theorem 2.2 are important results in Ramanujan's theory of elliptic functions in the signature 3 which can be found in [**2**] as Lemma 2.9 and Lemma 2.6, respectively.

Let α be given as in (2.2). Then it is known that (see [**2**, (4.4)] and [**5**, (4.7)])

$$q\frac{d\alpha}{dq} = K^2(\alpha)\alpha(1-\alpha). \tag{2.4}$$

The *modulus* β is said to have *degree* n over the *modulus* α when there is a relation

$$\frac{K(1-\beta)}{K(\beta)} = n\frac{K(1-\alpha)}{K(\alpha)}. \tag{2.5}$$

Hence, when q satisfies (2.1),

$$q^n = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{K(1-\beta)}{K(\beta)}\right),$$

and applying (2.4) with q and α replaced by q^n and β, respectively, we deduce that

$$q\frac{d\beta}{dq} = nK^2(\beta)\beta(1-\beta). \tag{2.6}$$

Combining (2.6) and (2.4), we arrive at:

Theorem 2.3. *If β has degree n over α, then*

$$m^2(\alpha,\beta)\frac{d\beta}{d\alpha} = n\frac{\beta(1-\beta)}{\alpha(1-\alpha)}, \tag{2.7}$$

where

$$m(\alpha,\beta) := \frac{K(\alpha)}{K(\beta)}. \tag{2.8}$$

We call the quantity $m(\alpha,\beta)$ the *multiplier of degree n in the theory of signature* 3. We are now ready to derive new formulas satisfied by $\epsilon(n)$.

Theorem 2.4. *Let $\epsilon(r)$ be defined as in* (1.4)*. Then*

$$\epsilon\left(\frac{1}{r}\right) = \frac{\sqrt{r}-\epsilon(r)}{r}.$$

Proof. Set

$$\tau = \frac{K(1-\alpha)}{K(\alpha)}.$$

Then

$$\frac{d\tau}{d\alpha}K(\alpha) + \dot{K}(\alpha)\tau = -\dot{K}(1-\alpha).$$

From (2.1) and (2.4), we deduce that

$$\frac{d\alpha}{d\tau} = \frac{-2\pi}{\sqrt{3}} K^2(\alpha)\alpha(1-\alpha).$$

Hence,

$$\dot{K}(1-\alpha) = \frac{\sqrt{3}}{2\pi}\frac{1}{K(\alpha)\alpha(1-\alpha)} - \dot{K}(\alpha)\tau. \tag{2.9}$$

Next, note that from (1.3)

$$\alpha_{1/r} = 1 - \alpha_r. \tag{2.10}$$

Therefore, by (1.4) and (2.9) with $\tau = \sqrt{r}$,

$$\begin{aligned}
\epsilon\left(\frac{1}{r}\right) &= \frac{3\sqrt{3}}{8\pi}K^{-2}(1-\alpha_r) - \sqrt{\frac{1}{r}}\left(\frac{3\alpha_r(1-\alpha_r)\dot{K}(1-\alpha_r)}{2K(1-\alpha_r)} - (1-\alpha_r)\right) \\
&= \frac{3\sqrt{3}}{8\pi}\frac{1}{K^2(1-\alpha_r)} - \frac{3\sqrt{3}}{4\pi r K^2(\alpha_r)} + \frac{3\alpha_r(1-\alpha_r)\dot{K}(\alpha_r)}{2\sqrt{r}K(\alpha_r)} + \frac{1}{\sqrt{r}} - \frac{\alpha_r}{\sqrt{r}} \\
&= \frac{\sqrt{r} - \epsilon(r)}{r}.
\end{aligned}$$

□

Theorem 2.5. *Let*

$$m^* := m(\alpha_r, \alpha_{n^2r}) \quad \textit{and} \quad \dot{m}^* := \frac{dm}{d\alpha}(\alpha_r, \alpha_{n^2r}).$$

Then

$$\epsilon(n^2r) = m^{*2}\left(\epsilon(r) - \sqrt{r}\left(\alpha_r - \frac{3}{2}m^{*-1}\alpha_r(1-\alpha_r)\dot{m}^* - \frac{n\alpha_{n^2r}}{m^{*2}}\right)\right). \tag{2.11}$$

Proof. Suppose β has degree n over α. Then from (2.8), we deduce that

$$m\frac{dK(\beta)}{d\alpha} + K(\beta)\frac{dm}{d\alpha} = \frac{dK(\alpha)}{d\alpha}. \tag{2.12}$$

Using (2.7), we may rewrite (2.12) as

$$\frac{n\beta(1-\beta)}{K(\beta)}\frac{dK(\beta)}{d\beta} = \frac{m^2\alpha(1-\alpha)}{K(\alpha)}\frac{dK(\alpha)}{d\alpha} - m\alpha(1-\alpha)\frac{dm}{d\alpha}. \tag{2.13}$$

Next, suppose $\alpha = \alpha_r$. Then $\beta = \alpha_{n^2r}$, and by (1.4), (2.8), and (2.13),

$$\begin{aligned}
\epsilon(n^2r) &= \frac{3\sqrt{3}}{8\pi K^2(\alpha_{n^2r})} - n\sqrt{r}\left(\frac{3\alpha_{n^2r}(1-\alpha_{n^2r})}{2K(\alpha_{n^2r})}\dot{K}(\alpha_{n^2r}) - \alpha_{n^2r}\right) \\
&= \frac{3\sqrt{3}m^{*2}}{8\pi K^2(\alpha_r)} \\
&\quad - \sqrt{r}\left(\frac{3m^{*2}\alpha_r(1-\alpha_r)}{2K(\alpha_r)}\dot{K}(\alpha_r) - \frac{3}{2}m^*\alpha_r(1-\alpha_r)\dot{m}^* - n\alpha_{n^2r}\right)
\end{aligned}$$

$$= m^{*\,2}\left(\epsilon(r) - \sqrt{r}\left(\alpha_r - \frac{3}{2}m^{*\,-1}\alpha_r(1-\alpha_r)\dot{m}^* - \frac{n\alpha_{n^2r}}{m^{*\,2}}\right)\right).$$

□

If we set $r = 1/n$ in (2.11) and use (2.10), we find that

$$\epsilon(n) = n\left(\epsilon\left(\frac{1}{n}\right) - \sqrt{\frac{1}{n}}\left(1 - \alpha_n - \frac{3\alpha_n(1-\alpha_n)}{2\sqrt{n}}\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n) - \alpha_n\right)\right)$$
$$= -\epsilon(n) + 2\alpha_n\sqrt{n} + \frac{3\alpha_n(1-\alpha_n)}{2}\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n).$$

Hence, we have:

Theorem 2.6.

$$\epsilon(n) = \sqrt{n}\alpha_n + \frac{3\alpha_n(1-\alpha_n)}{4}\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n).$$

Corollary 2.7. *With a_n and H_n defined in Theorem* 1.1*, we have*

$$a_n = \frac{H_n}{2\sqrt{3}}\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n).$$

Theorems 2.4, 2.5, and 2.6 are the respective cubic analogues of [**3**, (5.1.5), Theorem 5.2, and (5.2.5)].

3. Computations of a_n.

It is clear from Corollary 2.7 that in order to compute a_n it suffices to compute α_n and $dm/d\alpha$, where m is the multiplier of degree n. We will discuss the computation of the latter in this section. Suppose there is a relation between α and β, where β has degree n over α. Then we can determine $d\beta/d\alpha$ by implicitly differentiating the relation with respect to α. Substituting $d\beta/d\alpha$ into (2.7), we conclude that m can be expressed in terms of α and β. This implies that $dm/d\alpha$ is a function of α and β.

A relation between α and β induced by (2.5) (i.e., when β has degree n over α) is known as a *modular equation of degree n in the theory of signature* 3. (We sometimes call these *cubic modular equations*.) Our discussion in the previous paragraph indicates that our computations of $dm/d\alpha$ depend on the existence of such modular equations.

The first few modular equations in the theory of signature 3 are given by Ramanujan in his notebooks. One of these is the following modular equation of degree 2:

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} = 1. \tag{3.1}$$

Proofs of Ramanujan's modular equations in the theory of signature 3 are now available in [**2**] and [**6**].

Recently, we showed that [**6**] if p is a prime, then there is a relation between $x := (\alpha\beta)^{s/6}$ and $y := \{(1-\alpha)(1-\beta)\}^{s/6}$, when $(p+1)/3 = N/s$ and $\gcd(N,s) = 1$. Moreover, we proved that the degree of the polynomial satisfied by x and y is N. This proves the existence of modular equations of prime degrees and we conclude that when m is a multiplier of degree p,

$$\frac{dm}{d\alpha} = F_p(\alpha,\beta), \tag{3.2}$$

where F_p is a certain function in α and β. If we know the value of α_p, then the value of a_p follows by substituting $\alpha = 1-\alpha_p$ and $\beta = \alpha_p$ into (3.2) and simplifying. Our simplification is done with the help of MAPLE V.

When n is not a prime, except for modular equations of degrees 4 and 9, it is difficult to derive a modular equation of degree n. However, deriving such a modular equation is unnecessary. We illustrate our point with $n = pq$. If β has degree q over α then from a cubic modular equation of degree q and (2.7), we can write

$$m_q = \frac{K(\alpha)}{K(\beta)} = G_q(\alpha,\beta), \tag{3.3}$$

where m_q is the multiplier of degree q and G_q is a certain function of α and β. Similarly, we can deduce that if γ has degree p over β, then from a cubic modular equation of degree p, we may write

$$m_p = \frac{K(\beta)}{K(\gamma)} = G_p(\beta,\gamma), \tag{3.4}$$

where m_p is the multiplier of degree p and G_p is a certain function of β and γ. It follows that γ has degree pq over α and

$$m_{pq}(\alpha,\gamma) = \frac{K(\alpha)}{K(\gamma)} = \frac{K(\alpha)}{K(\beta)} \cdot \frac{K(\beta)}{K(\gamma)} = m_q(\alpha,\beta)\cdot m_p(\beta,\gamma).$$

Hence, differentiating with respect to α and substituting $\alpha = \alpha_{1/(pq)}$, we have

$$\begin{aligned}\frac{dm_{pq}}{d\alpha}(1-\alpha_{pq},\alpha_{pq}) = m_p(\alpha_{q/p},\alpha_{pq})\frac{dm_q}{d\alpha}(1-\alpha_{pq},\alpha_{q/p}) \\ + m_q(1-\alpha_{pq},\alpha_{q/p})\frac{d\beta}{d\alpha}\cdot\frac{dm_p}{d\beta}(\alpha_{q/p},\alpha_{pq}).\end{aligned}$$

This allows us to compute a_{pq} provided we know modular equations of degrees p and q and the singular moduli α_{pq} and $\alpha_{q/p}$.

When n is a squarefree product of more than 2 primes, say $n = p_1p_2\cdots p_l$, then the above idea can be extended with the computation of a_n reduced to that of finding modular equations of degrees p_1, p_2, $\ldots$, p_{l-1}, and p_l, and constants $\alpha_{n/(p_{i_1}^2\cdots p_{i_s}^2)}$, where $1 \le s \le l-1$ and $1 \le i_j \le l$.

4. Euler's convenient numbers, Kronecker's Limit Formula, and cubic singular moduli.

An *Euler convenient number* is a number c satisfying the following criterion:

> *Let $l > 1$ be an odd number relatively prime to c which is properly represented by $x^2 + cy^2$. If the equation $l = x^2 + cy^2$ has only one solution with $x, y \geq 0$, then l is a prime number.*

Euler was interested in these numbers because they helped him to generate large primes. The above criterion, however, is not very useful for finding these numbers.

Let d be squarefree, $K = \mathbb{Q}(\sqrt{-d})$, C_K denote the class group of K and C_K^2 be the subgroup of squares in C_K. A genus group G_K is defined as the quotient group C_K/C_K^2. Gauss observed that $G_K \simeq C_K$ if and only if d is a convenient number. (Some convenient numbers are not squarefree but Gauss' criterion is also true for class groups of orders in K.) Using this new criterion, Gauss determined 65 Euler convenient numbers [**8**], [**7**, p. 60]. We reproduce here those c's ($\neq 3$) which are squarefree and divisible by 3.

$h(-4c) := \left\lvert C_{\mathbb{Q}(\sqrt{-4c})} \right\rvert$	Euler's convenient number c
2	6, 15
4	21, 30, 33, 42, 57, 78, 93, 102, 177
8	105, 165, 210, 273, 330, 345, 357, 462
16	1365

Table 1. Convenient numbers in Gauss' table which are squarefree and divisible by 3 (except 3).

For each c in Table 1, we will deduce the corresponding values $a_{c/3}$, $b_{c/3}$, and $H_{c/3}$, which in turn yield new series for $1/\pi$.

A group homomorphism $\chi : G_K \longrightarrow \{\pm 1\}$ is known as a genus character. One can show that a genus character arises from a certain decomposition of D_K, where D_K is the discriminant of K. More precisely, if χ is a genus character, then there exist d_1 and d_2 satisfying $D_K = d_1 d_2$, $d_1 > 0$, and

$d_i \equiv 0$ or $1 \pmod 4$, such that for any prime ideal $\mathfrak{p}$ in K,

$$\chi([\mathfrak{p}]) = \begin{cases} \left(\frac{d_1}{N(\mathfrak{p})}\right), & \text{if} \quad N(\mathfrak{p}) \nmid d_1, \\ \\ \left(\frac{d_2}{N(\mathfrak{p})}\right), & \text{if} \quad N(\mathfrak{p}) \mid d_1, \end{cases} \tag{4.1}$$

where $N(\mathfrak{p})$ is the norm of the ideal $\mathfrak{p}$ and $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol. If $[\mathfrak{a}]$ is an ideal class in C_K and $\mathfrak{a} = \prod \mathfrak{p}^{\alpha_\mathfrak{p}}$, then we define

$$\chi([\mathfrak{a}]) = \prod \chi([\mathfrak{p}])^{\alpha_\mathfrak{p}}.$$

Theorem 4.1. *Let χ be a genus character arising from the decomposition $D_K = d_{1,\chi} d_{2,\chi}$. Let $h_{i,\chi}$ be the class number of the field $\mathbb{Q}(\sqrt{d_{i,\chi}})$, $w_{2,\chi}$ be the number of roots of unity in $\mathbb{Q}(\sqrt{d_{2,\chi}})$, and ϵ_χ be the fundamental unit of $\mathbb{Q}(\sqrt{d_{1,\chi}})$. Let*

$$F([\mathfrak{a}]) = \sqrt{N([1,\tau])}|\eta(\tau)|^2,$$

where $N(\cdot)$ denotes the norm of a fractional ideal, $\eta(z)$ denotes the Dedekind eta-function defined by

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}), \qquad \operatorname{Im} z > 0,$$

and

$$\tau = \frac{\tau_2}{\tau_1}, \quad \operatorname{Im} \tau > 0, \quad \textit{where} \quad \mathfrak{a} = [\tau_1, \tau_2].$$

Then

$$\epsilon_\chi^{2h_{1,\chi}h_{2,\chi}/w_{2,\chi}} = \prod_{[\mathfrak{a}] \in C_K} F([\mathfrak{a}])^{-\chi([\mathfrak{a}])}. \tag{4.2}$$

Theorem 4.1 follows from Kronecker's First Limit Formula [**11**, p. 72, Theorem 6]. In [**9**], K.G. Ramanathan applied Theorem 4.1 to compute products of the form

$$t_n = \frac{1}{5\sqrt{5}} \left(\frac{\eta(\frac{1+\sqrt{-n/5}}{2})}{\eta(\frac{1+\sqrt{-5n}}{2})} \right)^6$$

when $5n$ is a convenient number. These products are then used to deduce special values of the Rogers-Ramanujan continued fraction. In the same article, he defined [**9**, Eq. (51)]

$$\mu_n = \frac{1}{3\sqrt{3}} \left(\frac{\eta(\sqrt{-n/3})}{\eta(\sqrt{-3n})} \right)^6, \tag{4.3}$$

and remarked that μ_n can be evaluated when $3n$ is one of the convenient numbers listed in Table 1 (15 is missing from his list). Ramanathan's result can be stated as follows:

Theorem 4.2 ([**9**, Theorem 4])**.** *Let c be a convenient number listed in Table 1 and let $K = \mathbb{Q}(\sqrt{-c})$. Let $[\mathfrak{t}]$ be the ideal class containing $\mathfrak{t}$ such that $\mathfrak{t}^2 = (3)$. Then with the same notation as in Theorem 4.1*

$$\mu_{c/3} = \prod_{\chi([\mathfrak{t}])=-1} \epsilon_\chi^{3e_\chi},$$

where the exponents are given by

$$e_\chi = \frac{2wh_{1,\chi}h_{2,\chi}}{w_{2,\chi}h},$$

with h being the class number of K and w the number of roots of unity in K.

It turns out that Ramanathan's μ_n is related to α_n, namely [**5**, (2.7)],

$$\frac{1}{\alpha_n} = \mu_n^2 + 1. \tag{4.4}$$

Hence, from Theorem 4.2, (4.1), and (4.4), we can determine α_n explicitly. Using the same technique as given in the proof of Theorem 4.2, one can compute $\alpha_{n/(p_{i_1}^2 \cdots p_{i_s}^2)}, 1 \le s \le l-1$ and $1 \le i_j \le l$, which will be needed in the evaluations of a_n.

We conclude this section with a list of singular moduli which will be needed in the evaluations of a_n, b_n, and H_n with $n = c/3$.

n	Cubic singular moduli
2	$\alpha_2 = \frac{1}{2} - \frac{\sqrt{2}}{4}$
5	$\alpha_5 = \frac{1}{2} - \frac{11\sqrt{5}}{50}$

Table 2. Cubic singular moduli for $h(-12n) = 2$.

n	Cubic singular moduli
7	$\alpha_7 = \frac{1}{2} - \frac{13-\sqrt{7}}{12\sqrt{3}}$
10	$\alpha_{10} = \frac{1}{2} - \frac{35\sqrt{2}+2\sqrt{5}}{108}, \quad \alpha_{5/2} = \frac{1}{2} - \frac{35\sqrt{2}-2\sqrt{5}}{108}$
11	$\alpha_{11} = \frac{1}{2} - \frac{45\sqrt{3}-5}{44\sqrt{11}}$
14	$\alpha_{14} = \frac{1}{2} - \frac{99\sqrt{6}+2\sqrt{14}}{500}, \quad \alpha_{7/2} = \frac{1}{2} - \frac{99\sqrt{6}-2\sqrt{14}}{500}$
19	$\alpha_{19} = \frac{1}{2} - \frac{301\sqrt{51}-13\sqrt{3}}{4500}$
26	$\alpha_{26} = \frac{1}{2} - \frac{6930\sqrt{2}+5\sqrt{26}}{19652}, \quad \alpha_{13/2} = \frac{1}{2} - \frac{6930\sqrt{2}-5\sqrt{26}}{19652}$
31	$\alpha_{31} = \frac{1}{2} - \frac{35113\sqrt{3}-7\sqrt{93}}{121500}$
34	$\alpha_{34} = \frac{1}{2} - \frac{17420\sqrt{17}+35\sqrt{2}}{143748}, \quad \alpha_{17/2} = \frac{1}{2} - \frac{17420\sqrt{17}-35\sqrt{2}}{143748}$
59	$\alpha_{59} = \frac{1}{2} - \frac{6367095\sqrt{177}-265\sqrt{59}}{169413308}$

Table 3. Cubic singular moduli for $h(-12n) = 4$.

n	Cubic singular moduli
35	$\alpha_{35} = \frac{1}{2} + \frac{549}{50}\sqrt{5} + \frac{63}{10}\sqrt{15} - \frac{83}{20}\sqrt{35} - \frac{243}{100}\sqrt{105}$
	$\alpha_{7/5} = \frac{1}{2} + \frac{549}{50}\sqrt{5} - \frac{63}{10}\sqrt{15} + \frac{83}{20}\sqrt{35} - \frac{243}{100}\sqrt{105}$
55	$\alpha_{55} = \frac{1}{2} + \frac{92995}{88434}\sqrt{3} - \frac{8259}{9826}\sqrt{5} - \frac{13755}{19652}\sqrt{11} + \frac{25879}{176868}\sqrt{165}$
	$\alpha_{11/5} = \frac{1}{2} - \frac{92995}{88434}\sqrt{3} + \frac{8259}{9826}\sqrt{5} - \frac{13755}{19652}\sqrt{11} + \frac{25879}{176868}\sqrt{165}$
70	$\alpha_{70} = \frac{1}{2} + \frac{43971}{608350}\sqrt{5} - \frac{10023}{121670}\sqrt{14} - \frac{38621}{1095030}\sqrt{21} - \frac{383383}{10950300}\sqrt{30}$
	$\alpha_{35/2} = \frac{1}{2} - \frac{43971}{608350}\sqrt{5} - \frac{10023}{121670}\sqrt{14} + \frac{38621}{1095030}\sqrt{21} - \frac{383383}{10950300}\sqrt{30}$
	$\alpha_{7/10} = \frac{1}{2} + \frac{43971}{608350}\sqrt{5} - \frac{10023}{121670}\sqrt{14} + \frac{38621}{1095030}\sqrt{21} + \frac{383383}{10950300}\sqrt{30}$
91	$\alpha_{91} = \frac{1}{2} - \frac{53880905}{23706108}\sqrt{3} + \frac{10182535}{11853054}\sqrt{21} - \frac{8256325}{11853054}\sqrt{39} + \frac{5523815}{23706108}\sqrt{273}$
	$\alpha_{13/7} = \frac{1}{2} + \frac{53880905}{23706108}\sqrt{3} + \frac{10182535}{11853054}\sqrt{21} - \frac{8256325}{11853054}\sqrt{39} - \frac{5523815}{23706108}\sqrt{273}$
110	$\alpha_{110} = \frac{1}{2} - \frac{1204702947}{25128011089}\sqrt{5} - \frac{7922677455}{50256022178}\sqrt{6}$ $- \frac{995983605}{50256022178}\sqrt{33} + \frac{1027373417}{100512044356}\sqrt{110}$
	$\alpha_{55/2} = \frac{1}{2} + \frac{1204702947}{25128011089}\sqrt{5} - \frac{7922677455}{50256022178}\sqrt{6}$ $- \frac{995983605}{50256022178}\sqrt{33} - \frac{1027373417}{100512044356}\sqrt{110}$
	$\alpha_{11/10} = \frac{1}{2} + \frac{1204702947}{25128011089}\sqrt{5} - \frac{7922677455}{50256022178}\sqrt{6}$ $+ \frac{995983605}{50256022178}\sqrt{33} + \frac{1027373417}{100512044356}\sqrt{110}$
115	$\alpha_{115} = \frac{1}{2} - \frac{132769793}{1177056540}\sqrt{3} + \frac{14283759}{163480075}\sqrt{5} + \frac{1293747}{32696015}\sqrt{23} - \frac{218554427}{5885282700}\sqrt{345}$
	$\alpha_{23/5} = \frac{1}{2} + \frac{132769793}{1177056540}\sqrt{3} + \frac{14283759}{163480075}\sqrt{5} - \frac{1293747}{32696015}\sqrt{23} - \frac{218554427}{5885282700}\sqrt{345}$
119	$\alpha_{119} = \frac{1}{2} + \frac{50276655}{165246443}\sqrt{17} - \frac{162387225}{660985772}\sqrt{51} + \frac{19683565}{165246443}\sqrt{119} - \frac{45457245}{660985772}\sqrt{357}$
	$\alpha_{17/7} = \frac{1}{2} - \frac{50276655}{165246443}\sqrt{17} - \frac{162387225}{660985772}\sqrt{51} + \frac{19683565}{165246443}\sqrt{119} + \frac{45457245}{660985772}\sqrt{357}$
154	$\alpha_{154} = \frac{1}{2} - \frac{1255233}{5404300}\sqrt{2} + \frac{3988079}{121596750}\sqrt{21} - \frac{14120327}{668782125}\sqrt{66} - \frac{254553}{14861825}\sqrt{77}$
	$\alpha_{77/2} = \frac{1}{2} - \frac{1255233}{5404300}\sqrt{2} - \frac{3988079}{121596750}\sqrt{21} - \frac{14120327}{668782125}\sqrt{66} + \frac{254553}{14861825}\sqrt{77}$
	$\alpha_{11/14} = \frac{1}{2} - \frac{1255233}{5404300}\sqrt{2} + \frac{3988079}{121596750}\sqrt{21} + \frac{14120327}{668782125}\sqrt{66} + \frac{254553}{14861825}\sqrt{77}$

Table 4. Cubic singular moduli for $h(-12n) = 8$.

n	Cubic singular moduli
455	$\alpha_{455} = \frac{1}{2} - \frac{5260259275017205046267 7}{2487179487425541756119 50}\sqrt{5} - \frac{5668214189343349857381}{4974358974851083512239 0}\sqrt{15}$
	$- \frac{5696579790027594607146 1}{4974358974851083512239 0}\sqrt{35} + \frac{5384626339246783716 78}{2487179487425541756119 5}\sqrt{65}$
	$- \frac{6825036373044166275577}{9948717949702167024478}\sqrt{105} + \frac{2514650992719613832076 3}{4974358974851083512239 00}\sqrt{195}$
	$+ \frac{109593923135795012632}{4974358974851083512239}\sqrt{455} + \frac{1196360473602901817979}{9948717949702167024478 0}\sqrt{1365}$
$\alpha_{91/5} = \frac{1}{2}$	$- \frac{5260259275017205046267 7}{2487179487425541756119 50}\sqrt{5} + \frac{5668214189343349857381}{4974358974851083512239 0}\sqrt{15}$
	$+ \frac{5696579790027594607146 1}{4974358974851083512239 0}\sqrt{35} - \frac{5384626339246783716 78}{2487179487425541756119 5}\sqrt{65}$
	$- \frac{6825036373044166275577}{9948717949702167024478}\sqrt{105} + \frac{2514650992719613832076 3}{4974358974851083512239 00}\sqrt{195}$
	$+ \frac{109593923135795012632}{4974358974851083512239}\sqrt{455} - \frac{1196360473602901817979}{9948717949702167024478 0}\sqrt{1365}$
$\alpha_{65/7} = \frac{1}{2}$	$+ \frac{5260259275017205046267 7}{2487179487425541756119 50}\sqrt{5} - \frac{5668214189343349857381}{4974358974851083512239 0}\sqrt{15}$
	$+ \frac{5696579790027594607146 1}{4974358974851083512239 0}\sqrt{35} + \frac{5384626339246783716 78}{2487179487425541756119 5}\sqrt{65}$
	$- \frac{6825036373044166275577}{9948717949702167024478}\sqrt{105} - \frac{2514650992719613832076 3}{4974358974851083512239 00}\sqrt{195}$
	$+ \frac{109593923135795012632}{4974358974851083512239}\sqrt{455} - \frac{1196360473602901817979}{9948717949702167024478 0}\sqrt{1365}$
$\alpha_{35/13} = \frac{1}{2}$	$+ \frac{5260259275017205046267 7}{2487179487425541756119 50}\sqrt{5} + \frac{5668214189343349857381}{4974358974851083512239 0}\sqrt{15}$
	$- \frac{5696579790027594607146 1}{4974358974851083512239 0}\sqrt{35} - \frac{5384626339246783716 78}{2487179487425541756119 5}\sqrt{65}$
	$- \frac{6825036373044166275577}{9948717949702167024478}\sqrt{105} - \frac{2514650992719613832076 3}{4974358974851083512239 00}\sqrt{195}$
	$+ \frac{109593923135795012632}{4974358974851083512239}\sqrt{455} + \frac{1196360473602901817979}{9948717949702167024478 0}\sqrt{1365}$

Table 5. Cubic singular moduli for $h(-12n) = 16$.

5. Values of a_n, b_n, and H_n.

The values of H_n follow immediately from the values of α_n by (1.7). From (1.6), it appears that we need to denest the expression $\sqrt{1-H_n}$ in order to determine b_n. The next simple lemma shows that this is not necessary.

Lemma 5.1. *Let μ_n be defined as in* (4.3). *Then*

$$b_n = \frac{2\sqrt{3n}}{3}\frac{\mu_n^2-1}{\mu_n^2+1}.$$

Proof. From (4.4), we deduce that

$$\frac{1}{1-\alpha_n} = \frac{1}{\mu_n^2} + 1. \tag{5.1}$$

Hence, by (1.7), (4.4), and (5.1), we conclude that

$$\frac{4}{H_n} = \mu_n^2 + \frac{1}{\mu_n^2} + 2.$$

Hence,

$$\begin{aligned}\sqrt{1-H_n} &= \sqrt{1-\frac{4}{\mu_n^2+\mu_n^{-2}+2}} \\ &= \frac{\mu_n-\mu_n^{-1}}{\mu_n+\mu_n^{-1}}.\end{aligned} \tag{5.2}$$

Substituting (5.2) into (1.6) completes our proof of the lemma. □

Finally, to compute a_n, we use the method outlined in Section 3, together with the singular moduli given in Section 4. Our final results are shown in the following tables, grouped once again according to class numbers.

n	a_n	b_n	H_n
2	$\frac{1}{3\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\frac{1}{2}$
5	$\frac{8}{15\sqrt{3}}$	$\frac{22}{5\sqrt{3}}$	$\frac{4}{125}$

Table 6. a_n, b_n, and H_n for $h(-12n) = 2$.

n	a_n	b_n	H_n
7	$-\frac{10}{27}+\frac{7}{27}\sqrt{7}$	$\frac{13}{9}\sqrt{7}-\frac{7}{9}$	$-\frac{17}{27}+\frac{13}{54}\sqrt{7}$
10	$\frac{25}{243}\sqrt{15}-\frac{8}{243}\sqrt{6}$	$\frac{70}{81}\sqrt{15}+\frac{10}{81}\sqrt{6}$	$\frac{223}{1458}-\frac{35}{729}\sqrt{10}$
11	$\frac{6}{11}-\frac{13}{99}\sqrt{3}$	$\frac{45}{11}-\frac{5}{33}\sqrt{3}$	$-\frac{194}{1331}+\frac{225}{2662}\sqrt{3}$
14	$\frac{21}{125}\sqrt{7}-\frac{82}{1125}\sqrt{3}$	$\frac{198}{125}\sqrt{7}+\frac{28}{375}\sqrt{3}$	$\frac{1819}{31250}-\frac{198}{15625}\sqrt{21}$
19	$\frac{1654}{3375}-\frac{133}{3375}\sqrt{19}$	$\frac{5719}{1125}-\frac{13}{1125}\sqrt{19}$	$-\frac{8522}{421875}+\frac{3913}{843750}\sqrt{19}$
26	$\frac{1118}{14739}\sqrt{39}-\frac{3967}{44217}\sqrt{3}$	$\frac{4620}{4913}\sqrt{39}+\frac{130}{14739}\sqrt{3}$	$\frac{249913}{48275138}-\frac{34650}{24137569}\sqrt{13}$
31	$-\frac{14662}{91125}+\frac{7843}{91125}\sqrt{31}$	$-\frac{217}{30375}+\frac{35113}{30375}\sqrt{31}$	$-\frac{684197}{307546875}+\frac{245791}{615093750}\sqrt{31}$
34	$-\frac{7157}{323433}\sqrt{51}+\frac{62896}{323433}\sqrt{6}$	$\frac{70}{107811}\sqrt{51}+\frac{296140}{107811}\sqrt{6}$	$\frac{3555313}{2582935938}-\frac{304850}{1291467969}\sqrt{34}$
59	$\frac{342786}{717853}-\frac{593849}{6460677}\sqrt{3}$	$\frac{6367095}{717853}-\frac{265}{2153559}\sqrt{3}$	$-\frac{1461224894}{30403462846931}$ $+\frac{1687280175}{60806925693862}\sqrt{3}$

Table 7. a_n, b_n, and H_n for $h(-12n)=4$.

n	a_n, b_n, and H_n
35	$a_{35} = \frac{558}{5} - \frac{364}{15}\sqrt{21} - 42\sqrt{7} + \frac{577}{9}\sqrt{3}$ $b_{35} = \frac{1701}{5} + \frac{581}{3}\sqrt{3} - 126\sqrt{7} - \frac{366}{5}\sqrt{21}$ $H_{35} = -\frac{1210352}{125} - \frac{279531}{50}\sqrt{3} + \frac{91494}{25}\sqrt{7} + \frac{264132}{125}\sqrt{21}$
55	$a_{55} = -\frac{1411054}{132651} + \frac{14375}{4913}\sqrt{15} + \frac{26884}{14739}\sqrt{33} - \frac{194150}{132651}\sqrt{55}$ $b_{55} = -\frac{1423345}{44217} + \frac{50435}{4913}\sqrt{15} + \frac{27530}{4913}\sqrt{33} - \frac{185990}{44217}\sqrt{55}$ $H_{55} = -\frac{40461639767}{651714363} + \frac{2329268305}{144825414}\sqrt{15} + \frac{782606510}{72412707}\sqrt{33} - \frac{5473886320}{651714363}\sqrt{55}$
70	$a_{70} = \frac{57239}{1642545}\sqrt{7} + \frac{217912}{1642545}\sqrt{10} + \frac{18154}{182505}\sqrt{15} - \frac{5432}{60835}\sqrt{42}$ $b_{70} = \frac{766766}{547515}\sqrt{7} + \frac{540694}{547515}\sqrt{10} + \frac{93548}{60835}\sqrt{15} - \frac{29314}{60835}\sqrt{42}$ $H_{70} = \frac{263701974157}{999242250750} - \frac{3413048639}{55513458375}\sqrt{6} + \frac{8992317139}{499621125375}\sqrt{70} - \frac{1429629212}{55513458375}\sqrt{105}$
91	$a_{91} = -\frac{513055226}{17779581} + \frac{197125250}{17779581}\sqrt{7} - \frac{140862644}{17779581}\sqrt{13} + \frac{52944437}{17779581}\sqrt{91}$ $b_{91} = -\frac{502667165}{5926527} + \frac{214664450}{5926527}\sqrt{7} - \frac{142555490}{5926527}\sqrt{13} + \frac{53880905}{5926527}\sqrt{91}$ $H_{91} = -\frac{3020198045742832}{11707907427243} + \frac{1141527555432550}{11707907427243}\sqrt{7} - \frac{838583339971300}{11707907427243}\sqrt{13}$ $+\frac{633909424388075}{23415814854486}\sqrt{91}$
110	$a_{110} = -\frac{51466456301}{226152099801}\sqrt{3} - \frac{1605347400}{25128011089}\sqrt{10} + \frac{2302296150}{25128011089}\sqrt{55}$ $+\frac{2180745776}{75384033267}\sqrt{66}$ $b_{110} = -\frac{113011075870}{75384033267}\sqrt{3} + \frac{21911639310}{25128011089}\sqrt{10} + \frac{31690709820}{25128011089}\sqrt{55}$ $+\frac{8031352980}{25128011089}\sqrt{66}$ $H_{110} = \frac{328032510163806603637}{1262833882577813931842} - \frac{34968343286005152660}{631416941288906965921}\sqrt{22}$ $-\frac{26922173637682728405}{631416941288906965921}\sqrt{30} + \frac{11479638881035691730}{631416941288906965921}\sqrt{165}$

Table 8. a_n, b_n, and H_n for $h(-12n) = 8$.

n	a_n, b_n, and H_n
115	$a_{115} = \frac{2453452114}{882792405} - \frac{58294124}{98088045}\sqrt{15} - \frac{7317496}{32696015}\sqrt{69} + \frac{139937129}{882792405}\sqrt{115}$ $b_{115} = \frac{5026751821}{294264135} - \frac{39674908}{32696015}\sqrt{15} - \frac{19045012}{32696015}\sqrt{69} + \frac{132769793}{294264135}\sqrt{115}$ $H_{115} = -\frac{195193666847694106}{144318968578830375} + \frac{5599864542570116}{16035440953203375}\sqrt{15}$ $+\frac{2653754247048632}{16035440953203375}\sqrt{69} - \frac{37000600425371947}{288637937157660750}\sqrt{115}$
119	$a_{119} = \frac{103789302}{9720379} - \frac{547343732}{87483411}\sqrt{3} + \frac{43409625}{9720379}\sqrt{7} - \frac{72149336}{29161137}\sqrt{21}$ $b_{119} = \frac{318200715}{9720379} - \frac{551139820}{29161137}\sqrt{3} + \frac{162387225}{9720379}\sqrt{7} - \frac{67035540}{9720379}\sqrt{21}$ $H_{119} = -\frac{49978710596750603}{1606258054361897} + \frac{28855221888962700}{1606258054361897}\sqrt{3}$ $-\frac{37979008521886575}{3212516108723794}\sqrt{7} + \frac{10963595445145200}{1606258054361897}\sqrt{21}$
154	$a_{154} = \frac{965168}{4053225}\sqrt{6} - \frac{3910004}{182395125}\sqrt{7} - \frac{28870936}{182395125}\sqrt{22} + \frac{142457}{4053225}\sqrt{231}$ $b_{154} = \frac{2375828}{1351075}\sqrt{6} + \frac{112962616}{60798375}\sqrt{7} - \frac{55833106}{60798375}\sqrt{22} + \frac{836822}{1351075}\sqrt{231}$ $H_{154} = \frac{7319532242037247}{27107244286031250} - \frac{14157410807176}{301191603178125}\sqrt{33} + \frac{8770226416943}{301191603178125}\sqrt{42}$ $-\frac{206104571568818}{13553622143015625}\sqrt{154}$

Table 8 (continuation). a_n, b_n, and H_n for $h(-12n) = 8$.

$$
\begin{aligned}
a_{455} = &-\frac{35958686812804845816546}{4974358974851083512239} - \frac{19963924183950996708800 8}{4476923077365975161015 1}\sqrt{3} \\
&-\frac{91866311009633295364887}{24871794874255417561195}\sqrt{7} + \frac{12114289251501127493868}{4974358974851083512239}\sqrt{13} \\
&-\frac{2117048987345310400144 0}{1492307692455325053671 7}\sqrt{21} + \frac{19045288924435485549578}{1492307692455325053671 7}\sqrt{39} \\
&+\frac{35014000860193357422 42}{4974358974851083512239}\sqrt{91} + \frac{36482832707135043514012}{74615384622766252683585}\sqrt{273} \\
b_{455} = &-\frac{10886880309786406543608 9}{4974358974851083512239} - \frac{19946094010714692299024 0}{1492307692455325053671 7}\sqrt{3} \\
&-\frac{32690462905354979816991 9}{24871794874255417561195}\sqrt{7} + \frac{47775254611309163928990}{4974358974851083512239}\sqrt{13} \\
&-\frac{9333352321361091775752}{4974358974851083512239}\sqrt{21} + \frac{26584123954621081666818}{4974358974851083512239}\sqrt{39} \\
&+\frac{11336428378686699714762}{4974358974851083512239}\sqrt{91} + \frac{35068395166781366975118}{24871794874255417561195}\sqrt{273} \\
H_{455} = &-\frac{25593277575291678024530931497850444197001585383}{12372123605340761245091680055188536031396560500} \\
&-\frac{72643185984960781658348798523261066603020759 7}{6186061802670380622545840027594268015698280 25}\sqrt{3} \\
&-\frac{38393472765343588398997432583733106676992097 91}{49488494421363044980366720220754144125586242 00}\sqrt{7} \\
&+\frac{35164937451660133810043433731787245847033340 2}{6186061802670380622545840027594268015698280 25}\sqrt{13} \\
&-\frac{56196149792473665089405401522944398076147222}{12372123605340761245091680055188536031396560 5}\sqrt{21} \\
&+\frac{10324026218960132531682158187187806415515362 94}{30930309013351903112729200137971340078491401 25}\sqrt{39} \\
&+\frac{53580377184022523304261118862331927596113329}{24744247210681522490183360110377072062793121 0}\sqrt{91} \\
&+\frac{15252583716403938992503650442096696119092916 3}{12372123605340761245091680055188536031396560 50}\sqrt{273}
\end{aligned}
$$

Table 9. a_n, b_n, and H_n for $h(-12n) = 16$.

Concluding remarks.

The common feature of all our series computed here is that they involve only simple quadratic numbers. The series corresponding to $n = 455$ gives us approximately 33 additional digits per term and it is the fastest convergent series belonging to *the theory of* q_2 known so far. It might also be the fastest convergent series for $1/\pi$ which involves only real quadratic numbers. One should compare this with the spectacular series discovered by the Borweins [**4**] which gives "25 digits per term" using only real quadratics. The fastest convergent series known so far is that given by the Borweins [**4**] which gives roughly 50 additional digits per term.

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