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Eisenstein Series in Ramanujan's Lost Notebook

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Abstract. In his lost notebook, Ramanujan stated without proofs several beautiful identities for the three classical Eisenstein series (in Ramanujan's notation) $P(q)$, $Q(q)$, and $R(q)$. The identities are given in terms of certain quotients of Dedekind eta-functions called *Hauptmoduls*. These identities were first proved by S. Raghavan and S.S. Rangachari, but their proofs used the theory of modular forms, with which Ramanujan was likely unfamiliar. In this paper we prove all these identities by using classical methods which would have been well known to Ramanujan. In fact, all our proofs use only results from Ramanujan's notebooks.

Key words: Eisenstein series, modular equations, Ramanujan's lost notebook, theta functions, Dedekind eta function, differential equations for Eisenstein series

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1. Introduction

The Eisenstein series in the title, $P(q)$, $Q(q)$, and $R(q)$, are defined by

$$P(q) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \tag{1.1}$$

$$Q(q) := 1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}, \tag{1.2}$$

and

$$R(q) := 1 - 504\sum_{k=1}^{\infty}\frac{k^5q^k}{1-q^k}, \tag{1.3}$$

where $|q| < 1$. (The notation above is that used in Ramanujan's paper [4], [6, pp. 136–162] and in his lost notebook [7]. In his notebooks [5], Ramanujan replaced P, Q, and R by L, M, and N, respectively.) On pages 44, 50, 51, and 53 in his lost notebook [7], Ramanujan offers 12 formulas for Eisenstein series. All are connected with modular equations of either degree 5 or 7.

In a wonderful paper [3] devoted to proving identities for Eisenstein series and incomplete elliptic integrals in Ramanujan's lost notebook, S. Raghavan and S. S. Rangachari employ the theory of modular forms in establishing proofs for all of Ramanujan's identities for Eisenstein series. Most of the identities give representations for certain Eisenstein series in terms of quotients of Dedekind eta-functions, or, more precisely, Hauptmoduls. The very short proofs by Raghavan and Rangachari depend upon the finite dimensions of the spaces of the relevant modular forms, and therefore upon showing that a sufficient number of coefficients in the expansions about $q = 0$ of both sides of the proposed identities agree. Ramanujan evidently was unfamiliar with the theory of modular forms and most likely did not discover the identities by comparing coefficients.

The purpose of this paper is therefore to construct proofs in the spirit of Ramanujan's work. In fact, our proofs depend only upon theorems found in Ramanujan's notebooks [5]. Admittedly, some of our algebraic manipulations are rather laborious, and we resorted at times to *Mathematica*. It is therefore clear to us that Ramanujan's calculations, at least in some cases, were more elegant than ours. We actually have devised two approaches. In Sections 3 and 4, we use the two methods, respectively, to prove Ramanujan's quintic identities. At the end of Section 3, we prove a first order nonlinear "quintic" differential equation of Ramanujan satisfied by $P(q)$. In Section 5, we use the second approach, which is more constructive, to prove Ramanujan's septic identities. The new parametrizations for moduli of degree 7 in Section 5 appear to more useful than those given in [1, pp. 316–324]. A subset of the authors plans to utilize these parametrizations in future work. Section 7 is devoted to proving two new first order nonlinear "septic" differential equations for $P(q)$.

Page numbers placed after theorem numbers refer to their locations in the lost notebook [7].

2. Preliminary results

As usual, set

$$(a; q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n), \quad |q| < 1.$$

Define, after Ramanujan,

$$f(-q) := (q; q)_\infty =: e^{-2\pi iz/24}\eta(z), \quad q = e^{2\pi iz}, \quad \text{Im } z > 0, \tag{2.1}$$

where η denotes the Dedekind eta-function. We shall use the well-known transformation formula [1, p. 43, Entry 27(iii)]

$$\eta(-1/z) = \sqrt{z/i}\;\eta(z). \tag{2.2}$$

If $q = e^{2\pi iz}$, Im $z > 0$, $Q(q)$ and $R(q)$ obey the well-known transformation formulas [8, p. 136]

$$Q(e^{-2\pi i/z}) = z^4 Q(e^{2\pi iz}) \tag{2.3}$$

and

$$R(e^{-2\pi i/z}) = z^6 R(e^{2\pi iz}). \tag{2.4}$$

Our proofs below depend upon modular equations. As usual, set

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}$$

and

$${}_2F_1(a, b; c; x) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \quad |x| < 1.$$

Suppose that, for some positive integer n,

$$\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)} = n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}. \tag{2.5}$$

A modular equation of degree n is an equation involving α and β that is induced by (2.5). We often say that β *has degree n over* α. Also set

$$z_1 := {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \alpha\right) \quad \text{and} \quad z_n := {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \beta\right). \tag{2.6}$$

The *multiplier m* is defined by

$$m := \frac{z_1}{z_n}. \tag{2.7}$$

When

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)}\right)$$

and $z = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$, we have the "evaluations"

$$f(-q^2) = \sqrt{z}\, 2^{-1/3} \left(x(1-x)/q\right)^{1/12}, \tag{2.8}$$

$$Q(q^2) = z^4(1 - x + x^2), \tag{2.9}$$

and

$$R(q^2) = z^6(1+x)(1-x/2)(1-2x). \tag{2.10}$$

These are, respectively, Entries 12(iii), 13(i), and 13(ii) in Chapter 17 of Ramanujan's second notebook [1, pp. 124, 126].

Next, we record some relations from the theory of modular equations of degree 5. Set

$$m = 1 + 2p, \quad 0 < p < 2, \tag{2.11}$$

and

$$\rho = (m^3 - 2m^2 + 5m)^{1/2}. \tag{2.12}$$

Then [1, p. 284, Eqs. (13.4) and (13.5)]

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} = \frac{5\rho + m^2 + 5m}{4m^2}, \qquad \left(\frac{\beta^5}{\alpha}\right)^{1/8} = \frac{\rho - m - 1}{4}, \tag{2.13}$$

$$\left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = \frac{5\rho - m^2 - 5m}{4m^2}, \quad \text{and} \quad \left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} = \frac{\rho + m + 1}{4}. \tag{2.14}$$

Furthermore [1, p. 288, Entry 14(ii)]

$$4\alpha(1-\alpha) = p\left(\frac{2-p}{1+2p}\right)^5 \tag{2.15}$$

and

$$4\beta(1-\beta) = p^5\left(\frac{2-p}{1+2p}\right). \tag{2.16}$$

Also, from Entry 14(iii) in Chapter 19 of Ramanujan's second notebook [1, p. 289]

$$1 - 2\beta = (1 + p - p^2)\left(\frac{1+p^2}{1+2p}\right)^{1/2}. \tag{2.17}$$

We also need two modular equations of degree 5 from Entry 13(iv) of Chapter 18 in Ramanujan's second notebook [1, p. 281], namely,

$$m = 1 + 2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} \tag{2.18}$$

and

$$\frac{5}{m} = 1 + 2^{4/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}. \tag{2.19}$$

For Section 4, we need several modular equations of degree 7 found in Entry 19(i), (ii), (iii), and (vii) of Ramanujan's second notebook [1, pp. 314–315]. Thus, if β has degree 7 over α and m is the multiplier of degree 7,

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1, \tag{2.20}$$

$$m = -\frac{1 - 4\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}}{(\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}}, \tag{2.21}$$

$$\frac{7}{m} = \frac{1 - 4\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}}{(\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}}, \tag{2.22}$$

$$\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^7}{\alpha}\right)^{1/8} = m\left(\frac{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2}, \tag{2.23}$$

$$\left(\frac{\alpha^7}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^7}{1-\beta}\right)^{1/8} = \frac{7}{m}\left(\frac{1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}}{2}\right)^{1/2}, \tag{2.24}$$

and

$$m - \frac{7}{m} = 2((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8})(2 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}). \tag{2.25}$$

3. Quintic identities (First method)

Theorem 3.1 (p. 50). *For $Q(q)$ and $f(-q)$ defined by* (1.2) *and* (2.1), *respectively,*

$$Q(q) = \frac{f^{10}(-q)}{f^2(-q^5)} + 250q f^4(-q) f^4(-q^5) + 3125q^2 \frac{f^{10}(-q^5)}{f^2(-q)} \tag{3.1}$$

and

$$Q(q^5) = \frac{f^{10}(-q)}{f^2(-q^5)} + 10q f^4(-q) f^4(-q^5) + 5q^2 \frac{f^{10}(-q^5)}{f^2(-q)}. \tag{3.2}$$

Proof: It is slightly advantageous to first prove (3.2) with q replaced by q^2. To prove (3.2), we first write the right side of (3.2) as a function of p, where p is defined by (2.11).

By (2.8),

$$q^4 \frac{f^{10}(-q^{10})}{f^2(-q^2)} = q^4 \frac{z_5^5 2^{-10/3}(\beta(1-\beta)/q^5)^{5/6}}{z_1 2^{-2/3}\,(\alpha(1-\alpha)/q)^{1/6}}$$
$$= 2^{-8/3}\frac{z_5^4}{m}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/6}, \tag{3.3}$$

where β has degree 5 over α, z_1 and z_5 are defined by (2.6), and m is the multiplier defined by (2.7). Using (2.13), (2.14), (2.12), and (2.11) in (3.3), we find that

$$\begin{aligned} q^4\frac{f^{10}(-q^{10})}{f^2(-q^2)} &= 2^{-8/3}\frac{z_5^4}{m}\left(\frac{\rho^2-(m+1)^2}{16}\right)^{4/3} \\ &= \frac{z_5^4(m-1)^4}{2^8 m} = \frac{z_5^4 p^4}{2^4(1+2p)}. \end{aligned} \tag{3.4}$$

Similarly, from (2.8), (2.7), (2.15), (2.16), and (2.11),

$$\begin{aligned} \frac{f^6(-q^2)}{q^2 f^6(-q^{10})} &= m^3\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} \\ &= m^3\left(\frac{2-p}{p(1+2p)}\right)^2 = \frac{(1+2p)(2-p)^2}{p^2}. \end{aligned} \tag{3.5}$$

Thus, from (3.4) and (3.5),

$$\begin{aligned} & q^4\frac{f^{10}(-q^{10})}{f^2(-q^2)}\left(\frac{f^{12}(-q^2)}{q^4 f^{12}(-q^{10})}+10\frac{f^6(-q^2)}{q^2 f^6(-q^{10})}+5\right) \\ &= \frac{z_5^4 p^4}{2^4(1+2p)}\left(\frac{(1+2p)^2(2-p)^4}{p^4}+10\frac{(1+2p)(2-p)^2}{p^2}+5\right) \\ &= \frac{z_5^4}{2^4(1+2p)}(16+32p-8p^5+4p^6) \\ &= z_5^4\left(1+\frac{p^5(-2+p)}{4(1+2p)}\right) \\ &= z_5^4(1-\beta(1-\beta)) \\ &= Q(q^{10}), \end{aligned}$$

where in the penultimate step we used (2.16), and in the last step utilized (2.9). This completes the proof of (3.2).

To prove (3.1), we first rewrite (3.2) in terms of the Dedekind eta-function, defined in (2.1). Accordingly,

$$Q(q^5) = \frac{\eta^{10}(5z)}{\eta^2(z)}\left(\left(\frac{\eta(z)}{\eta(5z)}\right)^{12}+10\left(\frac{\eta(z)}{\eta(5z)}\right)^6+5\right). \tag{3.6}$$

We now transform (3.6) by means of (2.2) and (2.3) to deduce that

$$(5z)^{-4}Q\big(e^{-2\pi i/(5z)}\big)=\frac{\eta^{10}(-1/(5z))}{(5z/i)^5}\frac{(z/i)}{\eta^2(-1/z)}$$
$$\times\left(\left(\frac{\eta(-1/z)}{\sqrt{z/i}}\frac{\sqrt{5z/i}}{\eta(-1/(5z))}\right)^{12}+10\left(\frac{\eta(-1/z)}{\sqrt{z/i}}\frac{\sqrt{5z/i}}{\eta(-1/(5z))}\right)^{6}+5\right),$$

or

$$Q\big(e^{-2\pi i/(5z)}\big)=\frac{\eta^{10}(-1/(5z))}{\eta^2(-1/z)}\left(5^5\left(\frac{\eta(-1/z)}{\eta(-1/(5z))}\right)^{12}+250\left(\frac{\eta(-1/z)}{\eta(-1/(5z))}\right)^{6}+1\right)$$
$$=5^5\frac{\eta^{10}(-1/z)}{\eta^2(-1/(5z))}+250\eta^4(-1/(5z))\eta^4(-1/z)+\frac{\eta^{10}(-1/(5z))}{\eta^2(-1/z)}.$$

If we set $q=e^{-2\pi i/(5z)}$ and use (2.1), the last equality takes the shape (3.1), and so this completes the proof of (3.1). □

Theorem 3.2 (p. 51)**.** *For* $f(-q)$ *and* $R(q)$ *defined by* (2.1) *and* (1.3), *respectively,*

$$R(q)=\left(\frac{f^{15}(-q)}{f^3(-q^5)}-500q\,f^9(-q)\,f^3(-q^5)-15625q^2\,f^3(-q)\,f^9(-q^5)\right)$$
$$\times\sqrt{1+22q\frac{f^6(-q^5)}{f^6(-q)}+125q^2\frac{f^{12}(-q^5)}{f^{12}(-q)}} \tag{3.7}$$

and

$$R(q^5)=\left(\frac{f^{15}(-q)}{f^3(-q^5)}+4q\,f^9(-q)\,f^3(-q^5)-q^2\,f^3(-q)\,f^9(-q^5)\right)$$
$$\times\sqrt{1+22q\frac{f^6(-q^5)}{f^6(-q)}+125q^2\frac{f^{12}(-q^5)}{f^{12}(-q)}}. \tag{3.8}$$

Proof: Our procedure is similar to that of the previous theorem. We establish (3.8) first, but with q replaced by q^2.

By (2.8), (2.7), (2.15), and (2.16),

$$\frac{f^{15}(-q^2)}{f^3(-q^{10})}=\frac{z_1^{15/2}}{16z_5^{3/2}}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/4}=\frac{z_1^6m^{3/2}}{64}\left(\frac{2-p}{1+2p}\right)^6. \tag{3.9}$$

Hence, from (3.9), (3.5), (2.7), and (2.11),

$$
\begin{aligned}
F(q) := {} & \frac{f^{15}(-q^2)}{f^3(-q^{10})}\left(1+4q^2\frac{f^6(-q^{10})}{f^6(-q^2)}-q^4\frac{f^{12}(-q^{10})}{f^{12}(-q^2)}\right) \\
& \times\sqrt{1+22q^2\frac{f^6(-q^{10})}{f^6(-q^2)}+125q^4\frac{f^{12}(-q^{10})}{f^{12}(-q^2)}} \\
= {} & \frac{z_1^6 m^{3/2}}{64}\left(\frac{2-p}{1+2p}\right)^6 \\
& \times\left(1+4\frac{p^2}{(1+2p)(2-p)^2}-\frac{p^4}{(1+2p)^2(2-p)^4}\right) \\
& \times\sqrt{1+22\frac{p^2}{(1+2p)(2-p)^2}+125\frac{p^4}{(1+2p)^2(2-p)^4}} \\
= {} & \frac{z_5^6}{8m^{3/2}}(4+8p-6p^2-6p^3+9p^4-5p^5+p^6) \\
& \times\sqrt{4+8p+12p^2+12p^3+9p^4+4p^5+p^6} \\
= {} & \frac{z_5^6}{8m^{3/2}}(1+p-p^2)(4+4p-6p^2+4p^3-p^4) \\
& \times\sqrt{(1+p^2)(4+8p+8p^2+4p^3+p^4)}.
\end{aligned}
\tag{3.10}
$$

Using (2.17) and (2.11), we can write (3.10) in the form

$$
\begin{aligned}
F(q) &= z_5^6(1-2\beta)\frac{4+4p-6p^2+4p^3-p^4}{8(1+2p)}\sqrt{4+8p+8p^2+4p^3+p^4} \\
&= z_5^6(1-2\beta)\frac{(4+4p-6p^2+4p^3-p^4)(2+2p+p^2)}{8(1+2p)} \\
&= z_5^6(1-2\beta)\frac{8+16p+2p^5-p^6}{8(1+2p)} \\
&= z_5^6(1-2\beta)\left(1+\frac{p^5(2-p)}{8(1+2p)}\right) \\
&= z_5^6(1-2\beta)\big(1+\tfrac{1}{2}\beta(1-\beta)\big) \\
&= z_5^6(1-2\beta)\big(1-\tfrac{1}{2}\beta\big)(1+\beta) \\
&= R(q^{10}),
\end{aligned}
\tag{3.11}
$$

where in the antipenultimate line we used (2.16), and in the last line we used (2.10). Combining (3.10) and (3.11), we deduce (3.8), but with q replaced by q^2.

The proof of (3.7) is almost exactly like the proof of (3.1), but, of course, we use (2.4) instead of (2.3). □

The next two results are algebraic combinations of the pairs of representations in Theorems 3.1 and 3.2.

Theorem 3.3 (p. 51). *Let $A = Q(q)$ and $B = Q(q^5)$. Then*

$$\sqrt{A^2+94AB+625B^2} = 12\sqrt{5}\left(\frac{f^{10}(-q)}{f^2(-q^5)} + 26qf^4(-q)\,f^4(-q^5) + 125q^2\frac{f^{10}(-q^5)}{f^2(-q)}\right). \tag{3.12}$$

Proof: Set

$$C = \frac{f^5(-q)}{f(-q^5)}, \quad D = qf^4(-q)\,f^4(-q^5), \quad \text{and} \quad E = q\frac{f^5(-q^5)}{f(-q)}. \tag{3.13}$$

Note that

$$CE = D. \tag{3.14}$$

Equalities (3.1) and (3.2) now take the shapes

$$A = C^2 + 250D + 3125E^2 \quad \text{and} \quad B = C^2 + 10D + 5E^2, \tag{3.15}$$

respectively, and the proposed equality (3.12) has the form

$$\sqrt{A^2+94AB+625B^2} = 12\sqrt{5}(C^2 + 26D + 125E^2). \tag{3.16}$$

Substitute (3.15) into (3.16), square both sides, use (3.14), and with just elementary algebra (3.16) is then verified. □

Theorem 3.4 (p. 51). *Let $A = R(q)$ and $B = R(q^5)$. Then*

$$\begin{aligned}
&\sqrt{5(A+125B)^2-(126)^2AB} \\
&\quad = 252\left(\frac{f^{10}(-q)}{f^2(-q^5)} + 62qf^4(-q)\,f^4(-q^5) + 125q^2\frac{f^{10}(-q^5)}{f^2(-q)}\right) \\
&\qquad \times \sqrt{\frac{f^{10}(-q)}{f^2(-q^5)} + 22qf^4(-q)\,f^4(-q^5) + 125q^2\frac{f^{10}(-q^5)}{f^2(-q)}}.
\end{aligned} \tag{3.17}$$

Proof: We employ the notation (3.13). Equalities (3.7) and (3.8) then may be written as, respectively,

$$A = (C^3 - 500CD - 5^6DE)\sqrt{1 + 22E^2/D + 125E^4/D^2} \tag{3.18}$$

and

$$B = (C^3 + 4CD - DE)\sqrt{1 + 22E^2/D + 125E^4/D^2}, \tag{3.19}$$

and the proposed equality (3.17) has the form

$$\sqrt{5(A+125B)^2-(126)^2AB}=252(C^2+62D+125E^2)\sqrt{C^2+22D+125E^2}. \tag{3.20}$$

Square (3.20), use (3.18), (3.19), and (3.14), and simplify to verify the truth of (3.20). □

Our next goal is to establish a differential equation satisfied by $P(q)$, defined by (1.1). We need two lemmas.

Lemma 3.5. *Recall that $Q(q)$ and $R(q)$ are defined by* (1.2) *and* (1.3), *respectively. Let*

$$u := q^{1/4} f(-q) f(-q^5) \quad \text{and} \quad \lambda := q\left(\frac{f(-q^5)}{f(-q)}\right)^6. \tag{3.21}$$

Then

$$Q(q) = u^4\left(\frac{1}{\lambda} + 250 + 5^5\lambda\right) \tag{3.22}$$

and

$$R(q) = u^6\left(\frac{1}{\lambda} - 500 - 5^6\lambda\right)\sqrt{\frac{1}{\lambda} + 22 + 125\lambda}. \tag{3.23}$$

Proof: Identities (3.22) and (3.23) are obtained from (3.1) and (3.7), respectively. For example, by (3.1) and (3.21),

$$\begin{aligned} Q(q) &= qf^4(-q)f^4(-q^5)\left(\frac{f^6(-q)}{qf^6(-q^5)} + 250 + 3125q\frac{f^6(-q^5)}{f^6(-q)}\right) \\ &= u^4\left(\frac{1}{\lambda} + 250 + 5^5\lambda\right). \end{aligned}$$

□

Lemma 3.6. *Recall that $f(-q)$ is defined by* (2.1). *Then*

$$\begin{aligned} &1 + 6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} - 30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}} \\ &\quad = \sqrt{\frac{f^{12}(-q) + 22qf^6(-q)f^6(-q^5) + 125q^2f^{12}(-q^5)}{f^2(-q)f^2(-q^5)}}. \end{aligned}$$

Lemma 3.6 is part of Entry 4(i) in Chapter 21 of Ramanujan's second notebook, and a proof is given in [1, p. 463]. We give here a new short proof, based on Lemma 3.5.

Proof: Using Ramanujan's differential equations [4, Eq. (30)], [6, p. 142],

$$q\,\frac{dQ}{dq} = \frac{PQ - R}{3} \quad \text{and} \quad q\,\frac{dR}{dq} = \frac{PR - Q^2}{2}, \tag{3.24}$$

we deduce that

$$Q^3 - R^2 = 3qR\,\frac{dQ}{dq} - 2qQ\,\frac{dR}{dq}. \tag{3.25}$$

From (3.22) and (3.23), we find that

$$Q^3 - R^2 = 1728\frac{u^{12}}{\lambda^2}, \tag{3.26}$$

$$\frac{dQ}{dq} = 4u^3\left(\frac{1}{\lambda} + 250 + 5^5\lambda\right)\frac{du}{dq} + u^4\left(-\frac{1}{\lambda^2} + 5^5\right)\frac{d\lambda}{dq}, \tag{3.27}$$

and

$$\begin{aligned}\frac{dR}{dq} &= 6u^5\left(\frac{1}{\lambda} - 500 - 5^6\lambda\right)\sqrt{\frac{1}{\lambda} + 22 + 125\lambda}\,\frac{du}{dq}\\ &\quad - \frac{3u^6(1 - 152\lambda + 5250\lambda^2 + 250000\lambda^3 + 1953125\lambda^4)}{2\lambda^3\sqrt{\frac{1}{\lambda} + 22 + 125\lambda}}\frac{d\lambda}{dq}.\end{aligned} \tag{3.28}$$

Using (3.22), (3.23), (3.27), and (3.28) to simplify the right hand side of (3.25), we deduce that

$$Q^3 - R^2 = 3qR\,\frac{dQ}{dq} - 2qQ\,\frac{dR}{dq} = 1728\frac{u^{10}}{\lambda^3\sqrt{\frac{1}{\lambda} + 22 + 125\lambda}}q\,\frac{d\lambda}{dq}.$$

Combining this last equation with (3.26) yields

$$q\,\frac{d\lambda}{dq} = u^2\lambda\sqrt{\frac{1}{\lambda} + 22 + 125\lambda}. \tag{3.29}$$

On the other hand, by straightforward logarithmic differentiation,

$$q\,\frac{d\lambda}{dq} = \lambda\left(1 - 30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1 - q^{5k}} + 6\sum_{k=1}^{\infty}\frac{kq^k}{1 - q^k}\right). \tag{3.30}$$

If we combine (3.29) and (3.30), we deduce Lemma 3.6. □

Theorem 3.7 (p. 44)**.** *Let $P(q)$ be defined by* (1.1)*. Then*

$$P(q)=\frac{f^5(-q)}{f(-q^5)}\left(\sqrt{1+22\lambda+125\lambda^2}-30F(\lambda)\right) \tag{3.31}$$

and

$$P(q^5)=\frac{f^5(-q)}{f(-q^5)}\left(\sqrt{1+22\lambda+125\lambda^2}-6F(\lambda)\right), \tag{3.32}$$

where λ is defined in (3.21)*, and where $F(\lambda)$ satisfies the nonlinear, first order differential equation*

$$1+\frac{25}{2}\lambda+\frac{5}{2\lambda}F^2(\lambda)=F'(\lambda)\sqrt{1+22\lambda+125\lambda^2}. \tag{3.33}$$

Proof: Assume that $F(\lambda)$ is defined by (3.31), so that (3.31) is trivially true. By (1.1) and Lemma 3.6, we have

$$\frac{5P(q^5)-P(q)}{4}=\frac{f^5(-q)}{f(-q^5)}\sqrt{1+22\lambda+125\lambda^2}, \tag{3.34}$$

with λ defined by (3.21). If we substitute (3.31) into (3.34) and solve for $P(q^5)$, we deduce (3.32). It remains to prove that $F(\lambda)$ satisfies the differential equation (3.33).

From (3.24), (3.22), (3.23), and (3.29), we find that, with the prime $'$ denoting differentiation with respect to q,

$$P(q)=12q\frac{u'}{u}-2\frac{u^2}{\sqrt{\lambda}}\sqrt{1+22\lambda+125\lambda^2}. \tag{3.35}$$

Differentiating (3.35) with the help of (3.29), we deduce that

$$q\,\frac{dP}{dq}=-u^4\frac{125\lambda^2-1}{\lambda}-4\frac{u^2}{\sqrt{\lambda}}\left(q\frac{u'}{u}\right)\sqrt{1+22\lambda+125\lambda^2}+12q\left(q\frac{u'}{u}\right)'. \tag{3.36}$$

Next, by using another differential equation of Ramanujan [4, Eq. (30)], [6, p. 142],

$$q\,\frac{dP}{dq}=\frac{P^2-Q}{12}, \tag{3.37}$$

(3.22), (3.23), (3.35), and (3.36), we conclude that

$$12q\left(q\frac{u'}{u}\right)'-12\left(q\frac{u'}{u}\right)^2=-\frac{3}{4}\frac{u^4}{\lambda}\left(1+125\lambda^2+18\lambda\right). \tag{3.38}$$

We now identify Ramanujan's function $F(\lambda)$. By comparing (3.31) and (3.35), we conclude that

$$F(\lambda)=-\frac{2}{5}q\frac{u'}{u}\frac{\sqrt{\lambda}}{u^2}+\frac{1}{10}\sqrt{1+22\lambda+125\lambda^2}. \tag{3.39}$$

Rewriting (3.39) in the form,

$$\frac{F(\lambda)}{\sqrt{\lambda}}-\frac{1}{10\sqrt{\lambda}}\sqrt{1+22\lambda+125\lambda^2}=-\frac{2}{5}q\frac{u'}{u}\frac{1}{u^2}, \tag{3.40}$$

and differentiating with respect to q, we deduce that

$$\begin{aligned}&-\frac{1}{2}\frac{u^2}{\sqrt{\lambda}}\sqrt{\frac{1+22\lambda+125\lambda^2}{\lambda}}F(\lambda)+\frac{1}{\sqrt{\lambda}}q\frac{dF(\lambda)}{dq}-\frac{u^2}{20}\frac{125\lambda^2-1}{\lambda}\\&\quad=-\frac{2}{5}\left\{q\left(q\frac{u'}{u}\right)'\frac{1}{u^2}-\frac{2}{u^2}\left(q\frac{u'}{u}\right)^2\right\}.\end{aligned} \tag{3.41}$$

Using (3.38) and (3.40), we may rewrite the right hand side of (3.41) and deduce that

$$\begin{aligned}&-\frac{1}{2}\frac{u^2}{\sqrt{\lambda}}\sqrt{\frac{1+22\lambda+125\lambda^2}{\lambda}}F(\lambda)+\frac{1}{\sqrt{\lambda}}q\frac{dF(\lambda)}{dq}-\frac{u^2}{20}\frac{125\lambda^2-1}{\lambda}\\&\quad=u^2\frac{1+18\lambda+125\lambda^2}{40\lambda}+\frac{5}{2}u^2\frac{F^2(\lambda)}{\lambda}+u^2\frac{1+22\lambda+125\lambda^2}{40\lambda}\\&\qquad-u^2\frac{1}{2\lambda}F(\lambda)\sqrt{1+22\lambda+125\lambda^2}.\end{aligned} \tag{3.42}$$

Simplifying (3.42) with the use of (3.29), we deduce Ramanujan's differential equation (3.33). □

4. Quintic identities (Second method)

The alternative method to proving Theorems 3.1, 3.2, and 3.7 that we present in this section is more constructive than that in Section 3, but, although no less elementary, is perhaps slightly more removed from procedures that Ramanujan might have employed. On the other hand, the method here is more amenable to proving further theorems of this sort, especially if one does not know their formulations beforehand.

We begin by introducing some simplifying notation and making some useful preliminary calculations. Set

$$p_1:=\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}, \tag{4.1}$$

$$p_2:=\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}, \tag{4.2}$$

and

$$C := \frac{\sqrt{z_5^5}}{16\sqrt{z_1}}. \tag{4.3}$$

Observe that, by (2.18) and (2.19), respectively,

$$p_1 = \frac{m-1}{2^{4/3}} \tag{4.4}$$

and

$$p_2 = \frac{5-m}{2^{4/3}m}. \tag{4.5}$$

It follows that

$$\alpha(1-\alpha) = p_1 p_2^5 = \frac{m-1}{2^{4/3}}\left(\frac{5-m}{2^{4/3}m}\right)^5 = -\frac{(m-1)(m-5)^5}{16^2 m^5} \tag{4.6}$$

and

$$\beta(1-\beta) = p_1^5 p_2 = \left(\frac{m-1}{2^{4/3}}\right)^5 \frac{5-m}{2^{4/3}m} = -\frac{(m-1)^5(m-5)}{16^2 m}. \tag{4.7}$$

We also note that, by (4.3),

$$\frac{z_1^4}{16^2 m^5} = \frac{z_1^4}{16^2 (z_1/z_5)^5} = \frac{z_5^5}{16^2 z_1} = C^2 \tag{4.8}$$

and

$$\frac{z_5^4}{16^2 m} = \frac{z_5^4}{16^2 (z_1/z_5)} = \frac{z_5^5}{16^2 z_1} = C^2. \tag{4.9}$$

Since, by (4.3),

$$C^3 = \frac{\sqrt{z_5^{15}}}{16^3\sqrt{z_1^3}} = \frac{z_5^6}{16^3}\frac{1}{\sqrt{(z_1/z_3)^3}} = \frac{z_5^6}{16^3 m\sqrt{m}},$$

we find that

$$\frac{z_1^6}{16^3 m^6} = \frac{z_1^6}{16^3\left(z_1^6/z_5^6\right)} = \frac{z_5^6}{16^3} = C^3 m\sqrt{m}. \tag{4.10}$$

We shall use (4.8)–(4.10) in our alternative proofs of Theorems 3.1 and 3.2.

In view of (3.1), it is natural to introduce abbreviated notation for certain quotients of eta-functions. Our goal is to represent these quotients as polynomials in the multiplier m. First, by (2.8), (4.3), (4.2), and (4.5),

$$\begin{aligned} r_1 := \frac{f^5(-q^2)}{f(-q^{10})} &= \frac{\sqrt{z_1^5}2^{-5/3}(\alpha(1-\alpha)/q)^{5/12}}{\sqrt{z_5}2^{-1/3}(\beta(1-\beta)/q^5)^{1/12}} \\ &= \frac{\sqrt{z_5^5 z_1^3}}{2^{4/3}\sqrt{z_1}z_5^3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12} \\ &= \frac{16C}{2^{4/3}}m^3 p_2^2 = \frac{16C}{2^{4/3}}m^3\left(\frac{5-m}{2^{4/3}m}\right)^2 \\ &= Cm(m-5)^2, \end{aligned} \tag{4.11}$$

and, by (2.8), (4.3), (4.1), and (4.4),

$$\begin{aligned} r_2 := q^2\frac{f^5(-q^{10})}{f(-q^2)} &= q^2\frac{\sqrt{z_5^5}2^{-5/3}(\beta(1-\beta)/q^5)^{5/12}}{\sqrt{z_1}2^{-1/3}(\alpha(1-\alpha)/q)^{1/12}} \\ &= \frac{\sqrt{z_5^5}}{2^{4/3}\sqrt{z_1}}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12} \\ &= \frac{16C}{2^{4/3}}p_1^2 = \frac{16C}{2^{4/3}}\left(\frac{m-1}{2^{4/3}}\right)^2 \\ &= C(m-1)^2. \end{aligned} \tag{4.12}$$

Hence, by (4.11) and (4.12),

$$r_1r_2 = q^2 f^4(-q^2)f^4(-q^{10}) = C^2m(m-5)^2(m-1)^2. \tag{4.13}$$

The following lemma will be very useful.

Lemma 4.1. *Let*

$$g(m) := C^2\left(\sum_{k=0}^{6} c_k m^k\right).$$

If furthermore, we set, for some numbers x_1, x_2, *and* x_3,

$$g(m) = x_1r_1^2 + x_2r_1r_2 + x_3r_2^2,$$

then

$$x_1 = c_6, \quad x_2 = c_5 + 20c_6, \quad \textit{and} \quad x_3 = c_0.$$

Proof: Since, by (4.11)–(4.13),

$$\begin{aligned} x_1 r_1^2 + x_2 r_1 r_2 + x_3 r_2^2 = C^2(&x_3 + m(25x_2 - 4x_3) + m^2(625x_1 - 60x_2 + 6x_3) \\ &+ m^3(-500x_1 + 46x_2 - 4x_3) + m^4(150x_1 - 12x_2 + x_3) \\ &+ m^5(-20x_1 + x_2) + m^6 x_1), \end{aligned}$$

by matching the coefficients of m^k, $k = 0, \ldots, 6$, we find that

$$\begin{aligned} c_0 &= x_3, \\ c_1 &= 25x_2 - 4x_3, \\ c_2 &= 625x_1 - 60x_2 + 6x_3, \\ c_3 &= -500x_1 + 46x_2 - 4x_3, \\ c_4 &= 150x_1 - 12x_2 + x_3, \\ c_5 &= -20x_1 + x_2, \\ c_6 &= x_1. \end{aligned}$$

Therefore, if the system above is not overdetermined, then $g(m)$ can be expressed as a linear combination of $r_1^2, r_1 r_2,$ and r_2^2. By solving the linear system of equations,

$$\begin{aligned} c_0 &= x_3, \\ c_5 &= -20x_1 + x_2, \\ c_6 &= x_1, \end{aligned}$$

for $x_1, x_2,$ and x_3, and noting that $c_1, c_2, c_3,$ and c_4 are then uniquely determined, we complete the proof. □

We are now ready for our second proof of Theorem 3.1.

Proof of Theorem 3.1: By (2.9), (4.6), and (4.8),

$$\begin{aligned} Q(q^2) &= z_1^4(1 - \alpha(1 - \alpha)) \\ &= z_1^4\left(1 + \frac{(m-1)(m-5)^5}{16^2 m^5}\right) \\ &= \frac{z_1^4}{16^2 m^5}(16^2 m^5 + (m-1)(m-5)^5) \\ &= C^2(m^6 + 230m^5 + \cdots + 5^5) \\ &= r_1^2 + 2 \cdot 5^3 r_1 r_2 + 5^5 r_2^2, \end{aligned}$$

upon the use of Lemma 4.1. Replacing q^2 by q, we complete the proof of (3.1).

By (2.9), (4.7), and (4.9),

$$
\begin{aligned}
Q(q^{10}) &= z_5^4(1-\beta(1-\beta)) \\
&= z_5^4\left(1+\frac{(m-1)^5(m-5)}{16^2 m}\right) \\
&= \frac{z_5^4}{16^2 m}(16^2 m + (m-1)^5(m-5)) \\
&= C^2(m^6 - 10m^5 + \cdots + 5) \\
&= r_1^2 + 10r_1r_2 + 5r_2^2,
\end{aligned}
$$

by an application of Lemma 4.1. Replacing q^2 by q, we complete the proof of (3.2). □

For the proof of Theorem 3.2, it will be convenient to define

$$
\begin{aligned}
D &:= m^2 - 2m + 5, \\
E &:= m^2 + 2m + 5, \\
F &:= m^2 + 20m - 25,
\end{aligned}
$$

and

$$G := m^2 - 4m - 1.$$

Solving (4.6) and (4.7) and using the notation above, we deduce that

$$\alpha = \frac{1}{2} + \frac{\sqrt{D/m}\,F}{16m^2} \tag{4.14}$$

and

$$\beta = \frac{1}{2} + \frac{\sqrt{D/m}\,G}{16}. \tag{4.15}$$

(See also [1, p. 289, Eq. (14.2); p. 290, Eq. (14.4)].)

Using the notation above and Lemma 4.1, we may readily deduce the following lemma.

Lemma 4.2. *For D, E, F, and G defined above, for C defined by* (4.3)*, and for r_1 and r_2, defined in* (4.11) *and* (4.12)*, respectively, we have*

$$C^2DE^2 = r_1^2 + 22r_1r_2 + 5^3r_2^2,$$

$$C^2F(m^4 - 540m^3 + 1350m^2 - 14\cdot 5^3 m + 5^4) = r_1^2 - 4\cdot 5^3 r_1r_2 - 5^6 r_2^2,$$

and

$$C^2G(m^4 - 12m^3 + 54m^2 - 108m + 1) = r_1^2 + 4r_1r_2 - r_2^2.$$

Proof of Theorem 3.2: By (2.10), (4.14), (4.10), and Lemma 4.2,

$$\begin{aligned}
R(q^2) &= z_1^6(1+\alpha)(1-\alpha/2)(1-2\alpha)\\
&= z_1^6\frac{\sqrt{D/m}F}{16^3m^6}(\sqrt{D/m}F-24m^2)(\sqrt{D/m}F+24m^2)\\
&= \frac{z_1^6}{16^3m^6}\sqrt{D/m}F((D/m)F^2-24^2m^4)\\
&= (C^3m\sqrt{m})\sqrt{D/m}F(E(m^4-540m^3+1350m^2-14\cdot 5^3m+5^4)/m)\\
&= \sqrt{C^2DE^2}(C^2F(m^4-540m^3+1350m^2-14\cdot 5^3m+5^4))\\
&= \sqrt{r_1^2+22r_1r_2+5^3r_2^2}\big(r_1^2-4\cdot 5^3r_1r_2-5^6r_2^2\big)\\
&= \sqrt{\big(r_1^2+22r_1r_2+5^3r_2^2\big)\big/r_1^2}\cdot r_1\big(r_1^2-4\cdot 5^3r_1r_2-5^6r_2^2\big).
\end{aligned}$$

Replacing q^2 by q, we complete the proof of (3.7).

By (2.10), (4.15), (4.10), and Lemma 4.2,

$$\begin{aligned}
R(q^{10}) &= z_5^6(1+\beta)(1-\beta/2)(1-2\beta)\\
&= z_5^6\frac{\sqrt{D/m}G}{16^3}(\sqrt{D/m}G-24)(\sqrt{D/m}G+24)\\
&= \frac{z_5^6}{16^3}\sqrt{D/m}G((D/m)G^2-24^2)\\
&= (C^3m\sqrt{m})\sqrt{D/m}G(E(m^4-12m^3+54m^2-108m+1)/m)\\
&= \sqrt{C^2DE^2}(C^2G(m^4-12m^3+54m^2-108m+1))\\
&= \sqrt{r_1^2+22r_1r_2+5^3r_2^2}\big(r_1^2+4r_1r_2-r_2^2\big)\\
&= \sqrt{\big(r_1^2+22r_1r_2+5^3r_2^2\big)\big/r_1^2}\cdot r_1\big(r_1^2+4r_1r_2-r_2^2\big).
\end{aligned}$$

Replacing q^2 by q, we complete the proof of (3.8). □

We now give an alternate proof of Theorem 3.7. Recall that λ is defined in (3.21). For convenience, define

$$H := \sqrt{1+22\lambda+5^3\lambda^2} \tag{4.16}$$

and

$$J := \frac{f^5(-q)}{f(-q^5)}. \tag{4.17}$$

Then Eq. (3.1) can be written in the form

$$Q(q) = J^2(1 + 2 \cdot 5^3\lambda + 5^5\lambda^2), \tag{4.18}$$

and (3.34) takes the shape

$$5P(q^5) = P(q) + 4HJ. \tag{4.19}$$

Furthermore, (3.29) may be written as

$$\frac{d\lambda}{dq} = \frac{\lambda HJ}{q}. \tag{4.20}$$

By logarithmic differentiation, we deduce that

$$\begin{aligned}
\frac{1}{J}\frac{dJ}{dq} &= 5\sum_{k=1}^{\infty}\frac{(-k)q^{k-1}}{1-q^k} - \sum_{k=1}^{\infty}\frac{(-5k)q^{5k-1}}{1-q^{5k}} \\
&= \frac{1}{24q}(5P(q) - 5P(q^5)) \\
&= \frac{1}{24q}(5P(q) - (P(q) + 4HJ)) \\
&= \frac{1}{6q}(P(q) - HJ),
\end{aligned}$$

or

$$P(q) = JH + \frac{6q}{J}\frac{dJ}{dq} = J\left(H + \frac{6q}{J^2}\frac{dJ}{dq}\right). \tag{4.21}$$

Now define

$$F := -\frac{q}{5J^2}\frac{dJ}{dq}.$$

Then, by (4.21),

$$P(q) = J(H - 30F) \tag{4.22}$$

and

$$\frac{dJ}{dq} = -\frac{5J^2F}{q}. \tag{4.23}$$

Differentiating (4.16) with respect to λ, we find that

$$H'(\lambda) = \frac{22 + 2 \cdot 5^3\lambda}{2\sqrt{1 + 22\lambda + 5^3\lambda^2}} = \frac{11 + 5^3\lambda}{H}.$$

Using this, (4.22), (4.23), and (4.20), we deduce that

$$\begin{aligned} q\frac{dP}{dq} &= q\frac{d}{dq}(J(H-30F)) \\ &= q\frac{dJ}{dq}(H-30F) + qJ\frac{d\lambda}{dq}\frac{d}{d\lambda}(H-30F) \\ &= q\left(-\frac{5J^2F}{q}\right)(H-30F) + qJ\left(\frac{\lambda HJ}{q}\right)(H'(\lambda)-30F'(\lambda)) \\ &= J^2(-5FH+150F^2+11\lambda+5^3\lambda^2-30\lambda F'(\lambda)H). \end{aligned} \tag{4.24}$$

On the other hand, by (4.22) and (4.18),

$$\begin{aligned} &\frac{1}{12}(P^2(q)-Q(q)) \\ &\quad= \frac{1}{12}(J^2(H-30F)^2 - J^2(1+2\cdot 5^3\lambda+5^5\lambda^2)) \\ &\quad= \frac{J^2}{12}\Big(\big(\sqrt{1+22\lambda+5^3\lambda^2}\big)^2 - 60FH + 30^2F^2 - (1+250\lambda+5^5\lambda^2)\Big) \\ &\quad= J^2\big(-5FH+75F^2-19\lambda-2\cdot 5^3\lambda^2\big). \end{aligned} \tag{4.25}$$

Equating (4.24) and (4.25) by (3.37), we arrive at

$$F'(\lambda)H = 1 + \frac{25}{2}\lambda + \frac{5}{2\lambda}F^2,$$

which is (3.33).

By (4.19) and (4.22), we deduce that

$$P(q^5) = \frac{1}{5}(P(q)+4HJ) = \frac{J}{5}(H-30F+4H) = J(H-6F),$$

which completes the proof of (3.32).

5. Septic identities

Theorem 5.1. *For* $|q| < 1$,

$$\begin{aligned} Q(q) = &\left(\frac{f^7(-q)}{f(-q^7)} + 5\cdot 7^2 q f^3(-q) f^3(-q^7) + 7^4 q^2 \frac{f^7(-q^7)}{f(-q)}\right) \\ &\times\left(\frac{f^7(-q)}{f(-q^7)} + 13q f^3(-q) f^3(-q^7) + 49q^2 \frac{f^7(-q^7)}{f(-q)}\right)^{1/3} \end{aligned} \tag{5.1}$$

and

$$Q(q^7) = \left(\frac{f^7(-q)}{f(-q^7)} + 5qf^3(-q)\,f^3(-q^7) + q^2\frac{f^7(-q^7)}{f(-q)}\right)$$
$$\times\left(\frac{f^7(-q)}{f(-q^7)} + 13qf^3(-q)\,f^3(-q^7) + 49q^2\frac{f^7(-q^7)}{f(-q)}\right)^{1/3}. \tag{5.2}$$

We shall prove these identities with q replaced by q^2.

For convenience, define

$$\begin{aligned} C &:= \frac{\sqrt{z_1 z_7}}{4}, \\ p_1 &:= 4\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}, \end{aligned} \tag{5.3}$$

and

$$p_2 := 4\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}.$$

By (2.8), (2.7), and the definitions above,

$$\begin{aligned} r_1 := \frac{f^7(-q^2)}{f(-q^{14})} &= \frac{\sqrt{z_1^7}2^{-7/3}(\alpha(1-\alpha)/q)^{7/12}}{\sqrt{z_7}2^{-1/3}\big(\beta(1-\beta)/q^7\big)^{1/12}} \\ &= \frac{\sqrt{z_1^3 z_7^3}z_1^2}{4z_7^2}\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/12} = \frac{\sqrt{z_1^3 z_7^3}}{4}m^2\left(\frac{p_2}{4}\right)^2 \\ &= C^3 m^2 p_2^2. \end{aligned} \tag{5.4}$$

Furthermore,

$$p_1 p_2 = 16(\alpha\beta(1-\alpha)(1-\beta))^{1/4}, \tag{5.5}$$

$$r_2 := q^2 f^3(-q^2)\,f^3(-q^{14}) = C^3 p_1 p_2, \tag{5.6}$$

and

$$r_3 := q^4\frac{f^7(-q^{14})}{f(-q^2)} = C^3\frac{p_1^2}{m^2}. \tag{5.7}$$

Thus,

$$r_1 + 5\cdot 7^2 r_2 + 7^4 r_3 = C^3\left(m^2 p_2^2 + 5\cdot 7^2 p_1 p_2 + \frac{7^4 p_1^2}{m^2}\right) \tag{5.8}$$

and

$$r_1 + 13r_2 + 49r_3 = C^3\left(m^2 p_2^2 + 13p_1p_2 + \frac{49p_1^2}{m^2}\right). \tag{5.9}$$

Now define

$$T := (\alpha\beta)^{1/8} - (1-\alpha)^{1/8}(1-\beta)^{1/8}. \tag{5.10}$$

Then, by (2.21) and (2.22),

$$-T = \frac{1-p_1}{m} \tag{5.11}$$

and

$$7T = (1-p_2)m. \tag{5.12}$$

Eliminating T in (5.11) and (5.12), we deduce that

$$mp_2 + \frac{7p_1}{m} = m + \frac{7}{m}. \tag{5.13}$$

By (2.20) and (5.10),

$$(\alpha\beta)^{1/8} = \frac{1+T}{2} \tag{5.14}$$

and

$$(1-\alpha)^{1/8}(1-\beta)^{1/8} = \frac{1-T}{2}. \tag{5.15}$$

Thus, by (5.5),

$$p_1p_2 = 16((\alpha\beta)^{1/8}(1-\alpha)^{1/8}(1-\beta)^{1/8})^2 = (1-T^2)^2. \tag{5.16}$$

By (2.25), (5.10), (5.14), and (5.15), we deduce that

$$m - \frac{7}{m} = 2T\left(2 + \left(\frac{1+T}{2}\right)^2 + \left(\frac{1-T}{2}\right)^2\right).$$

Rewriting this, we have the following lemma.

Lemma 5.2. *For the multiplier m, and T defined in* (5.10),

$$m - \frac{7}{m} = 5T + T^3. \tag{5.17}$$

Applying Lemma 5.2 repeatedly, one can derive the following expressions.

Lemma 5.3.

$$\begin{aligned}
m^2 &= 7 + m(5T + T^3),\\
m^3 &= (35T + 7T^3) + m(7 + 25T^2 + 10T^4 + T^6),\\
m^4 &= (49 + 175T^2 + 70T^4 + 7T^6)\\
&\quad + m(70T + 139T^3 + 75T^5 + 15T^7 + T^9),\\
\frac{1}{m} &= -\frac{1}{7}(T^3 + 5T) + \frac{1}{7}m,\\
\frac{1}{m^2} &= \frac{1}{49}(7 + 25T^2 + 10T^4 + T^6) + \frac{1}{49}m(-5T - T^3).
\end{aligned}$$

Lemma 5.4.

$$m^2 p_2^2 + 13 p_1 p_2 + \frac{49 p_1^2}{m^2} = (3 + T^2)^3. \tag{5.18}$$

Proof: By (5.13), (5.17), and (5.16),

$$\begin{aligned}
m^2 p_2^2 + 13 p_1 p_2 + \frac{49 p_1^2}{m^2} &= \left(m p_2 + \frac{7 p_1}{m}\right)^2 - p_1 p_2\\
&= \left(m + \frac{7}{m}\right)^2 - p_1 p_2\\
&= \left(m - \frac{7}{m}\right)^2 + 28 - p_1 p_2\\
&= (T^3 + 5T)^2 + 28 - (1 - T^2)^2\\
&= (3 + T^2)^3,
\end{aligned}$$

which completes the proof. □

By Lemma 5.4 and (5.9), we find that

$$(r_1 + 13r_2 + 49r_3)^{1/3} = C\left(m^2 p_2^2 + 13 p_1 p_2 + \frac{49 p_1^2}{m^2}\right)^{1/3} = C(3 + T^2). \tag{5.19}$$

By (5.12),

$$m p_2 = m - 7T. \tag{5.20}$$

By Lemma 5.4, (5.20), and (5.16),

$$\frac{49 p_1^2}{m^2} = (3 + T^2)^3 - (m - 7T)^2 - 13(1 - T^2)^2. \tag{5.21}$$

With the help of Lemma 5.3, we can now express each of $r_1, r_2,$ and r_3 in the form

$$f_1(T) + mf_2(T).$$

Lemma 5.5.

$$\begin{aligned} r_1 &= C^3((49T^2 + 7) + m(T^3 - 9T)), \\ r_2 &= C^3((T^2 - 1)^2), \\ r_3 &= \frac{C^3}{49}((7 + 4T^2 - 4T^4 + T^6) + m(9T - T^3)). \end{aligned}$$

Proof: Use (5.4) and (5.20) to deduce the formula for r_1; use (5.6) and (5.16) to prove the formula for r_2; and use (5.7) and (5.21) for the formula for r_3. □

Lemma 5.6. *For the multiplier m, and T defined in* (5.10),

$$\alpha = \frac{1}{16m}(1 + T)(21 + 8m - 21T + 7T^2 - 7T^3) \tag{5.22}$$

and

$$\beta = \frac{1}{16}(1 + T)(8 - 3m + 3mT - mT^2 + mT^3). \tag{5.23}$$

Proof: By (5.14) and (5.15), we deduce that

$$\begin{aligned} \left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} &= \frac{1-\beta}{(1-\alpha)^{1/8}(1-\beta)^{1/8}} = \frac{1-\beta}{(1-T)/2}, \\ \left(\frac{\beta^7}{\alpha}\right)^{1/8} &= \frac{\beta}{(\alpha\beta)^{1/8}} = \frac{\beta}{(1+T)/2}, \\ \left(\frac{\alpha^7}{\beta}\right)^{1/8} &= \frac{\alpha}{(\alpha\beta)^{1/8}} = \frac{\alpha}{(1+T)/2}, \end{aligned}$$

and

$$\left(\frac{(1-\alpha)^7}{1-\beta}\right)^{1/8} = \frac{1-\alpha}{(1-\alpha)^{1/8}(1-\beta)^{1/8}} = \frac{1-\alpha}{(1-T)/2}.$$

Using these identities, (5.14), and (5.15) in (2.24) and (2.23), and then solving the linear equations for α and β, we obtain (5.22) and (5.23). □

Lemma 5.7. *Let*

$$g(T) := C^3\left(\sum_{k=0}^{3} c_{2k}T^{2k} + m\sum_{k=0}^{1} d_{2k+1}T^{2k+1}\right).$$

If

$$g(T) = x_1 r_1 + x_2 r_2 + x_3 r_3,$$

for some real numbers $x_1, x_2,$ *and* x_3*, then*

$$x_1 = d_3 + c_6, \quad x_2 = c_4 + 4c_6, \quad \textit{and} \quad x_3 = 49c_6.$$

Proof: Since

$$\begin{aligned}(x_1 r_1 + x_2 r_2 + x_3 r_3)/C^3 = &\left(7x_1 + x_2 + \frac{1}{7}x_3\right) + \left(49x_1 - 2x_2 + \frac{4}{49}x_3\right)T^2 \\ &+ \left(x_2 - \frac{4}{49}x_3\right)T^4 + \frac{1}{49}x_3 T^6 \\ &+ m\left\{\left(-9x_1 + \frac{9}{49}x_3\right)T + \left(x_1 - \frac{1}{49}x_3\right)T^3\right\},\end{aligned}$$

by Lemma 5.5, we deduce the following equalities.

$$\begin{aligned}c_4 &= x_2 - \frac{4}{49}x_3, \\ c_6 &= \frac{1}{49}x_3, \\ d_3 &= x_1 - \frac{1}{49}x_3.\end{aligned}$$

Thus, by solving the linear system above for x_1, x_2, and x_3, we complete the proof. □

We are now ready to prove Theorem 5.1.

Proof of (5.1): By (2.9), (5.3), (5.22), and Lemma 5.3,

$$\begin{aligned}Q(q^2) &= z_1^4(1 - \alpha + \alpha^2) \\ &= \left(\frac{\sqrt{z_1 z_7}}{4}\right)^4 (4^4 m^2)(1 - \alpha + \alpha^2) \\ &= C^4(3 + T^2)(147 + 64m^2 + 112mT - 245T^2 - 112mT^3 \\ &\quad + 49T^4 + 49T^6) \\ &= C(3 + T^2) \cdot C^3(147 + 64(7 + m(5T + T^3)) + 112mT \\ &\quad - 245T^2 - 112mT^3 + 49T^4 + 49T^6) \\ &= C(3 + T^2) \cdot C^3(595 - 245T^2 + 49T^4 + 49T^6 \\ &\quad + m(432T - 48T^3)).\end{aligned}$$

Thus, applying Lemma 5.7, we find that $x_1 = 1$, $x_2 = 5 \cdot 7^2$, and $x_3 = 7^4$.

Since, by Lemma 5.5,

$$r_1 + 5 \cdot 7^2 r_2 + 7^4 r_3 = C^3(595 - 245T^2 + 49T^4 + 49T^6 + m(432T - 48T^3)),$$

we deduce that

$$\begin{aligned} Q(q^2) &= C(3 + T^2) \cdot (r_1 + 5 \cdot 7^2 r_2 + 7^4 r_3) \\ &= (r_1 + 13r_2 + 49r_3)^{1/3}(r_1 + 5 \cdot 7^2 r_2 + 7^4 r_3), \end{aligned}$$

by (5.19). Thus, replacing q^2 by q, we complete the proof of (5.1). □

Proof of (5.2): By (2.9), (5.3), (5.23), and Lemma 5.3,

$$\begin{aligned} Q(q^{14}) &= z_7^4(1 - \beta + \beta^2) \\ &= \left(\frac{\sqrt{z_1 z_7}}{4}\right)^4 \frac{4^4}{m^2}(1 - \beta + \beta^2) \\ &= C^4(3 + T^2)\left(\frac{64}{m^2} + \frac{16}{m}(-T + T^3) + 3 - 5T^2 + T^4 + T^6\right) \\ &= C^4(3 + T^2)\Bigg(64\bigg(\frac{1}{49}(7 + 25T^2 + 10T^4 + T^6) \\ &\quad + \frac{1}{49}m(-5T - T^3)\bigg) + 16\left(-\frac{1}{7}(T^3 + 5T) + \frac{1}{7}m\right)(-T + T^3) \\ &\quad + 3 - 5T^2 + T^4 + T^6\Bigg) \\ &= C(3 + T^2) \cdot \frac{C^3}{49}(595 + 1915T^2 + 241T^4 + T^6 \\ &\quad + m(-432T + 48T^3)). \end{aligned}$$

Thus, by Lemma 5.7, $x_1 = 1$, $x_2 = 5$, and $x_3 = 1$.

Since, by Lemma 5.5,

$$r_1 + 5r_2 + r_3 = \frac{C^3}{49}(595 + 1915T^2 + 241T^4 + T^6 + m(-432T + 48T^3)),$$

we deduce that

$$\begin{aligned} Q(q^{14}) &= C(3 + T^2) \cdot (r_1 + 5r_2 + r_3) \\ &= (r_1 + 13r_2 + 49r_3)^{1/3}(r_1 + 5r_2 + r_3), \end{aligned}$$

by (5.19).

Thus, upon replacing q^2 by q, we complete the proof of (5.2). □

Theorem 5.8. *For $|q| < 1$,*

$$R(q) = \left(\frac{f^7(-q)}{f(-q^7)} - 7^2(5+2\sqrt{7})qf^3(-q)f^3(-q^7) - 7^3(21+8\sqrt{7})q^2\frac{f^7(-q^7)}{f(-q)}\right) \times\left(\frac{f^7(-q)}{f(-q^7)} - 7^2(5-2\sqrt{7})qf^3(-q)f^3(-q^7) - 7^3(21-8\sqrt{7})q^2\frac{f^7(-q^7)}{f(-q)}\right) \tag{5.24}$$

and

$$R(q^7) = \left(\frac{f^7(-q)}{f(-q^7)} + (7+2\sqrt{7})qf^3(-q)f^3(-q^7) + (21+8\sqrt{7})q^2\frac{f^7(-q^7)}{f(-q)}\right) \times\left(\frac{f^7(-q)}{f(-q^7)} + (7-2\sqrt{7})qf^3(-q)f^3(-q^7) + (21-8\sqrt{7})q^2\frac{f^7(-q^7)}{f(-q)}\right). \tag{5.25}$$

We shall prove the identities with q replaced by q^2.

By straightforward calculations, we deduce the following lemma.

Lemma 5.9. *If*

$$C^6\left(\sum_{k=0}^{6} c_{2k}T^{2k} + m\sum_{k=0}^{4} d_{2k+1}T^{2k+1}\right) = (r_1 + x_2r_2 + x_3r_3)(r_1 + y_2r_2 + y_3r_3),$$

for some real numbers x_2, x_3, y_2, and y_3, then

$$\begin{aligned} c_8 &= x_3 + y_3 + x_2y_2 - \frac{6}{7^2}(x_2y_3 + x_3y_2) + \frac{24}{7^4}x_3y_3, \\ c_{10} &= \frac{1}{7^2}(x_2y_3 + x_3y_2) - \frac{8}{7^4}x_3y_3, \\ c_{12} &= \frac{1}{7^4}x_3y_3, \\ d_1 &= -126 - 9(x_2 + y_2) + \frac{9}{7^2}(x_2y_3 + x_3y_2) + \frac{18}{7^3}x_3y_3. \end{aligned}$$

Proof of (5.24): By (2.10), (5.3), (5.22), and Lemma 5.3,

$$\begin{aligned} \frac{1}{C^6}R(q^2) &= \frac{z_1^6}{C^6}(1+\alpha)(1-\alpha/2)(1-2\alpha) \\ &= \left(\frac{\sqrt{z_1z_7}}{4C}\right)^6 m^3 4^6(1+\alpha)(1-\alpha/2)(1-2\alpha) \\ &= -75411 - 95130T^2 - 1841T^4 + 3780T^6 - 1029T^8 - 2058T^{10} \\ &\quad - 343T^{12} + m(-82152T - 77344T^3 - 16816T^5 - 160T^7 + 344T^9). \end{aligned}$$

If we use Lemma 5.9 to find real solutions (x_2, x_3, y_2, y_3) satisfying $x_2 \leq y_2$, we find that

$$x_2 = -7^2(5+2\sqrt{7}), \qquad x_3 = -7^3(21+8\sqrt{7}),$$
$$y_2 = -7^2(5-2\sqrt{7}), \qquad y_3 = -7^3(21-8\sqrt{7}).$$

By Lemmas 5.3 and 5.5,

$$\begin{aligned}
&\frac{1}{C^6}(r_1 - 7^2(5+2\sqrt{7})r_2 - 7^3(21+8\sqrt{7})r_3)(r_1 - 7^2(5-2\sqrt{7})r_2 - 7^3(21-8\sqrt{7})r_3)\\
&\quad = -75411 - 95130T^2 - 1841T^4 + 3780T^6 - 1029T^8 - 2058T^{10} - 343T^{12}\\
&\qquad + m(-82152T - 77344T^3 - 16816T^5 - 160T^7 + 344T^9)\\
&\quad = \frac{1}{C^6}R(q^2).
\end{aligned}$$

Thus we complete the proof of (5.24) after replacing q^2 by q. □

Proof of (5.25): By (2.10), (5.3), (5.23), and Lemma 5.3,

$$\begin{aligned}
\frac{1}{C^6}R(q^{14}) &= \frac{z_7^6}{C^6}(1+\beta)\left(1-\frac{\beta}{2}\right)(1-2\beta)\\
&= \left(\frac{\sqrt{z_1 z_7}}{4C}\right)^6 \frac{4^6}{m^3}(1+\beta)\left(1-\frac{\beta}{2}\right)(1-2\beta)\\
&= 75411 + 505890T^2 + 470713T^4 + 157644T^6 + 18645T^8 + 498T^{10}\\
&\quad - T^{12} + m(-82152T - 77344T^3 - 16816T^5 - 160T^7 + 344T^9).
\end{aligned}$$

If we use Lemma 5.9 to find real solutions (x_2, x_3, y_2, y_3) satisfying $x_2 \geq y_2$, we find that

$$x_2 = 7+2\sqrt{7}, \qquad x_3 = 21+8\sqrt{7},$$
$$y_2 = 7-2\sqrt{7}, \qquad y_3 = 21-8\sqrt{7}.$$

By Lemmas 5.3 and 5.5,

$$\begin{aligned}
&\frac{1}{C^6}(r_1 + (7+2\sqrt{7})r_2 + (21+8\sqrt{7})r_3)(r_1 + (7-2\sqrt{7})r_2 + (21-8\sqrt{7})r_3)\\
&\quad = 75411 + 505890T^2 + 470713T^4 + 157644T^6 + 18645T^8 + 498T^{10}\\
&\qquad - T^{12} + m(-82152T - 77344T^3 - 16816T^5 - 160T^7 + 344T^9)\\
&\quad = \frac{1}{C^6}R(q^{14}).
\end{aligned}$$

Thus we complete the proof of (5.25) after replacing q^2 by q. □

6. Septic differential equations

In this section, we derive two new septic differential equations for $P(q)$, defined in (1.1). Both involve variations of the same variable, but one is connected with the beautiful identities in Theorem 5.1, while the other is connected with an emerging alternative septic theory of elliptic functions, initially begun in a recent paper by Chan and Ong [2].

Theorem 6.1. *For* $|q| < 1$,

$$P(q) = \left(\frac{f^7(-q)}{f(-q^7)}\right)^{2/3}((1+13\lambda+49\lambda^2)^{2/3} - 28F(\lambda)) \tag{6.1}$$

and

$$P(q^7) = \left(\frac{f^7(-q)}{f(-q^7)}\right)^{2/3}((1+13\lambda+49\lambda^2)^{2/3} - 4F(\lambda)), \tag{6.2}$$

where

$$\lambda = q\,\frac{f^4(-q^7)}{f^4(-q)},$$

and where $F(\lambda)$ *satisfies the nonlinear, first order differential equation*

$$1 + \frac{28}{3}\lambda + \frac{7F^2(\lambda)}{3\lambda\sqrt[3]{1+13\lambda+49\lambda^2}} = F'(\lambda)\sqrt[3]{1+13\lambda+49\lambda^2}. \tag{6.3}$$

For convenience, define

$$H := \sqrt[3]{1+13\lambda+49\lambda^2}$$

and

$$J := \frac{f^7(-q)}{f(-q^7)}.$$

Then the identity (5.1) in Theorem 5.1 can be written in the abbreviated form

$$Q(q) = J^{4/3}(1 + 5\cdot 7^2\lambda + 7^4\lambda^2)H. \tag{6.4}$$

Lemma 6.2. *For* $|q| < 1$,

$$7P(q^7) = P(q) + 6H^2J^{2/3}.$$

This lemma is Entry 5(i) in Chapter 21 of Ramanujan's second notebook [1, p. 467].

Proof of Theorem 6.1: Since

$$\begin{aligned}\log\lambda &= \log q + 4\log f(-q^7) - 4\log f(-q)\\ &= \log q + 4\sum_{k=1}^{\infty}\log(1-q^{7k}) - 4\sum_{k=1}^{\infty}\log(1-q^k),\end{aligned}$$

differentiating both sides with respect to q, we deduce that

$$\begin{aligned}\frac{1}{\lambda}\frac{d\lambda}{dq} &= \frac{1}{q} + 4\sum_{k=1}^{\infty}\frac{(-7k)q^{7k-1}}{1-q^{7k}} - 4\sum_{k=1}^{\infty}\frac{(-k)q^{k-1}}{1-q^k}\\ &= \frac{1}{q}\left(1 - 28\sum_{k=1}^{\infty}\frac{kq^{7k}}{1-q^{7k}} + 4\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}\right)\\ &= H^2J^{2/3}/q. \qquad (6.5)\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}\frac{1}{J}\frac{dJ}{dq} &= 7\sum_{k=1}^{\infty}\frac{(-k)q^{k-1}}{1-q^k} - \sum_{k=1}^{\infty}\frac{(-7k)q^{7k-1}}{1-q^{7k}}\\ &= \frac{1}{24q}(7P(q) - 7P(q^7))\\ &= \frac{1}{24q}(7P(q) - (P(q) + 6H^2J^{2/3}))\\ &= \frac{1}{4q}(P(q) - H^2J^{2/3}).\end{aligned}$$

Thus,

$$P(q) = J^{2/3}H^2 + \frac{4q}{J}\frac{dJ}{dq} = J^{2/3}\left(H^2 + \frac{4q}{J^{5/3}}\frac{dJ}{dq}\right).$$

Now define

$$F := -\frac{q}{7J^{5/3}}\frac{dJ}{dq}.$$

Then

$$P(q) = J^{2/3}\left(H^2 - 28F\right) \qquad (6.6)$$

and

$$\frac{dJ}{dq} = -\frac{7J^{5/3}F}{q}. \qquad (6.7)$$

By the fact that

$$(H^2)' = 2HH' = \frac{2(13 + 2 \cdot 7^2 \lambda)}{3H},$$

and (6.6), (6.7), and (6.5), we deduce that

$$\begin{aligned}
q\,\frac{d}{dq}(P(q)) &= q\,\frac{d}{dq}(J^{2/3}(H^2 - 28F)) \\
&= q\left(\frac{2}{3}J^{-1/3}\right)\frac{dJ}{dq}(H^2 - 28F) + qJ^{2/3}\,\frac{d\lambda}{dq}\,\frac{d}{d\lambda}(H^2 - 28F) \\
&= q\left(\frac{2}{3}J^{-1/3}\right)\left(-\frac{7J^{5/3}F}{q}\right)(H^2 - 28F) \\
&\quad + qJ^{2/3}\left(\frac{\lambda H^2 J^{2/3}}{q}\right)((H^2)' - 28F') \\
&= \frac{2J^{4/3}}{3}(-7FH^2 + 4 \cdot 7^2 F^2 + \lambda H(13 + 2 \cdot 7^2\lambda - 42F'H)). \qquad (6.8)
\end{aligned}$$

On the other hand, by (6.6) and (6.4),

$$\begin{aligned}
\frac{1}{12}(P^2(q) - Q(q)) &= \frac{1}{12}(J^{4/3}(H^2 - 28F)^2 - J^{4/3}(1 + 5 \cdot 7^2\lambda + 7^4\lambda^2)H) \\
&= \frac{J^{4/3}}{12}((1 + 13\lambda + 49\lambda^2)H - 56FH^2 + 28^2F^2 \\
&\quad - (1 + 5 \cdot 7^2\lambda + 7^4\lambda^2)H) \\
&= \frac{2J^{4/3}}{3}(-7FH^2 + 2 \cdot 7^2F^2 - \lambda H(29 + 6 \cdot 7^2\lambda)). \qquad (6.9)
\end{aligned}$$

Equating (6.8) and (6.9) and using (3.37), we obtain

$$F'(\lambda)H = 1 + \frac{28}{3}\lambda + \frac{7}{3\lambda H}F^2,$$

which is (6.3), and by (6.6), we complete the proof of (6.1).

By Lemma 6.2 and (6.6), we deduce that

$$P(q^7) = \frac{1}{7}(P(q) + 6H^2J^{2/3}) = \frac{J^{2/3}}{7}(H^2 - 28F + 6H^2) = J^{2/3}(H^2 - 4F),$$

which completes the proof of (6.2). □

Theorem 6.3. *Recall that $P(q)$ is defined in* (1.1). *Let*

$$z = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}$$

and define x by

$$\frac{1-x}{x} = \frac{1}{7q}\left(\frac{f(-q)}{f(-q^7)}\right)^4. \tag{6.10}$$

Then

$$P(q) = z^2(1 + 12F_1(x)) \quad \textit{and} \quad P(q^7) = z^2\big(1 + \tfrac{12}{7}F_1(x)\big),$$

where $F_1(x)$ satisfies the differential equation

$$\frac{dF_1(x)}{dx}x(1-x) + F_1^2(x) + \frac{2}{3}F_1(x)\frac{x^2+13x}{7-x+x^2} + \frac{7}{9}\frac{x(3+x)}{7-x+x^2} = 0. \tag{6.11}$$

Proof: Throughout the proof, the prime $'$ denotes differentiation with respect to q.

In [2, Eq. (2.27)] it was shown that

$$P(q) = z^2\frac{7-53x-3x^2}{7-x+x^2} + 12q\frac{z'}{z}. \tag{6.12}$$

Using the same method that was used in [2], we can also show that

$$P(q^7) = \frac{z^2}{7}\frac{3x^2-59x+49}{7-x+x^2} + \frac{12}{7}q\frac{z'}{z}. \tag{6.13}$$

If we write $P(q) = H + 7J$ and $P(q^7) = H + J$, we find from (6.12) and (6.13) that

$$H = z^2 \quad \text{and} \quad J = -\frac{4}{7}z^2\frac{x^2+13x}{7-x+x^2} + \frac{12}{7}q\frac{z'}{z}.$$

Hence, we may let

$$P(q) = z^2(1 + 12F_1(x)) \quad \text{and} \quad P(q^7) = z^2\big(1 + \tfrac{12}{7}F_1(x)\big),$$

where

$$F_1(x) = -\frac{1}{3}\frac{x^2+13x}{7-x+x^2} + \frac{q}{z^2}\frac{z'}{z}. \tag{6.14}$$

Now,

$$\begin{aligned} q\left(\frac{q}{z^2}\frac{z'}{z}\right)' &= \frac{q}{z^2}\left(q\frac{z'}{z}\right)' - \frac{2}{z^2}\left(q\frac{z'}{z}\right)^2 \\ &= \frac{1}{z^2}\left\{q\left(q\frac{z'}{z}\right)' - \left(q\frac{z'}{z}\right)^2 - \left(q\frac{z'}{z}\right)^2\right\}. \end{aligned} \tag{6.15}$$

Since [2, Eq. (2.30)]

$$q\left(q\frac{z'}{z}\right)' - \left(q\frac{z'}{z}\right)^2 = -2z^4x(1-x)\frac{2x^2-2x-7}{(7-x+x^2)^2},$$

and, by (6.14),

$$q\frac{z'}{z} = \frac{z^2}{3}\frac{x^2+13x}{7-x+x^2} + z^2F_1(x),$$

we may rewrite (6.15) as

$$q\left(F_1(x) + \frac{1}{3}\frac{x^2+13x}{7-x+x^2}\right)' = \frac{1}{z^2}\left\{-2z^4x(1-x)\frac{2x^2-2x-7}{(7-x+x^2)^2} - \frac{z^4}{9}\left(\frac{x^2+13x}{7-x+x^2}\right)^2 - z^4F_1^2(x) - \frac{2}{3}z^4F_1(x)\frac{x^2+13x}{7-x+x^2}\right\}. \tag{6.16}$$

Simplifying (6.16) with the aid of the differentiation formula [2, Thm. 2.4]

$$\frac{dx}{dq} = \frac{z^2}{q}x(1-x),$$

we obtain Theorem 6.3. □

The differential equation of Theorem 6.1 was discovered by Raghavan and Rangachari [3] and can be deduced from (6.11) by setting

$$F_1(x) = -\frac{7}{3}F(\lambda)\left(\sqrt[3]{1+13\lambda+49\lambda^2}\right)^{-2} \tag{6.17}$$

where λ is given in Theorem 6.1. From (6.17),

$$\begin{aligned}\frac{dF_1}{dx} &= \frac{dF_1}{d\lambda}\frac{d\lambda}{dx}\\ &= -\frac{1}{3}\left(\frac{dF(\lambda)}{d\lambda}\frac{1}{(1+13\lambda+49\lambda^2)^{2/3}} - \frac{2}{3}F(\lambda)\frac{13+98\lambda}{(1+13\lambda+49\lambda^2)^{5/3}}\right)\frac{d\lambda}{dx},\end{aligned} \tag{6.18}$$

since

$$\frac{d\lambda}{dx} = \frac{1}{7}\frac{1}{(1-x)^2},$$

by (6.10). Substituting (6.18) and (6.17) into (6.11), we easily deduce the differential equation given in Theorem 6.1.

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