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On Ramanujan's Quartic Theory of Elliptic Functions

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1. INTRODUCTION

In his famous paper [11], [12, pp. 23–39], Ramanujan offers several elegant series for $1/\pi$. He then remarks, "There are corresponding theories in which q is replaced by one or other of the functions"

$$q_r := q_r(x) := \exp\left(-\pi \csc(\pi/r)\,\frac{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{r}, \frac{r-1}{r}; 1; x\right)}\right), \tag{1.1}$$

where $r = 3$, 4, or 6 and where ${}_2F_1$ denotes the classical Gaussian hypergeometric function. In the classical theory of elliptic functions, the variable

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$q = q_2$, and Ramanujan implies that most of his series for $1/\pi$ arise not out of the classical theory but out of new alternative theories wherein q is replaced by either q_3, q_4, or q_6. Ramanujan gave no proofs of his series for $1/\pi$ or of any of his theorems in the "corresponding" or "alternative" theories. It was not until 1987 that J. M. Borwein and P. B. Borwein [6] proved Ramanujan's series formulas for $1/\pi$. In the appendix of Ramanujan's *Collected Papers* [12, p. 336], Mordell laments, "It is unfortunate that Ramanujan has not developed in detail the corresponding theories..." However, in his second notebook [13, pp. 257–262], Ramanujan records without proofs his theorems in these new theories, which were first proved in 1995 by Berndt *et al.* [5], who gave these theories the appellation, the theories of signature r ($r = 3, 4, 6$). An account of this work may also be found in Berndt's book [4, Chap. 33].

In the classical theory, the theta-functions

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \qquad \text{and} \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} \tag{1.2}$$

play key roles. In particular, Jacobi's identity [3, p. 40, Entry 25(vii)]

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2) \tag{1.3}$$

is crucially used in establishing the fundamental inversion formula [3, pp. 100–101]

$$z_2 := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x) = \varphi^2(q), \tag{1.4}$$

where $q := q_2$ is given by (1.1).

In the cubic theory, or the theory of signature 3, for $\omega = \exp(2\pi i/3)$, let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{1.5}$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \tag{1.6}$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}. \tag{1.7}$$

The functions defined in (1.5)–(1.7) are the *cubic* theta-functions, first introduced by J. M. Borwein and P. B. Borwein [7], who proved that

$$a^3(q) = b^3(q) + c^3(q). \tag{1.8}$$

Ramanujan [13, p. 258] established the fundamental inversion formula

$$z_3 := {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; x) = a(q), \tag{1.9}$$

where $q = q_3$ is given by (1.1). This theorem was first proved in print by Berndt, Bhargava, and Garvan [5], [4, p. 99], with (1.8) being a necessary ingredient in their proof.

In the theory of signature 4, or in the quartic theory, Berndt, Bhargava, and Garvan [5], [4, p. 146, Eq. (9.7)] established a type of transfer principle by which formulas in the theory of signature 4 could be derived from formulas in the classical theory. The primary purpose of this paper is to establish an analogue of both (1.4) and (1.9), and some concomitant theorems, so that results in the quartic theory can be proved without relying on corresponding theorems in the classical theory. Taking the place of $a(q)$, $b(q)$, and $c(q)$ in the cubic theory are the functions

$$A(q) := \varphi^4(q) + 16q\psi^4(q^2), \qquad B(q) := \varphi^4(q) - 16q\psi^4(q^2), \tag{1.10}$$

and

$$C(q) := 8\sqrt{q}\,\varphi^2(q)\,\psi^2(q^2) \tag{1.11}$$

(where φ and ψ are defined in (1.2)) which, by Jacobi's identity (1.3), satisfy the equality

$$A^2(q) = B^2(q) + C^2(q). \tag{1.12}$$

In Section 2, we use (1.12) to establish a quartic inversion formula (Theorem 2.3) in order to prove that

$$z_4 := {}_2F_1(\tfrac{1}{4}, \tfrac{3}{4}; 1; x) = \sqrt{A(q)}, \tag{1.13}$$

where $x = x(q)$. Clearly (1.13) is an analogue of (1.4) and (1.9). The definitions of $A(q)$, $B(q)$, and $C(q)$ were motivated by results of the Borweins [6, p. 179, Proposition 5.7(a)] and Ramanujan [4, pp. 151–152, Lemma 9.14, Theorem 9.15] pointing toward (1.13). We next prove quartic analogues of the extremely useful "principles of duplication and dimidiation" in the classical theory [3, pp. 125–126]. Using (1.12), (1.13), and the quartic analogues of duplication and dimidiation, we then easily obtain useful formulas for $B(q)$ and $C(q)$ in terms of z_4 and x.

In Section 3, by using (1.13) and the quartic versions of duplication and dimidiation, we show how to reprove many theorems of Ramanujan in the quartic theory, which were first established by Berndt, Bhargava, and Garvan [5], [4, pp. 145–149]. Define

$$f(-q) := \prod_{n=1}^{\infty} (1-q^n) := e^{-2\pi i\tau/24}\eta(\tau) \qquad (q = e^{2\pi i\tau},\ \operatorname{Im}\tau > 0), \tag{1.14}$$

where $\eta(\tau)$ is the Dedekind eta function. We first derive quartic analogues of the representations [4, p. 109, Lemma 5.1],

$$b(q) = \frac{f^3(-q)}{f(-q^3)} \qquad \text{and} \qquad c(q) = 3q^{1/3}\,\frac{f^3(-q^3)}{f(-q)},$$

in the cubic theory. Second, we establish a formula for $f(-q)$ in terms of z_4 and x. Third, we define the classical Eisenstein series $M(q)$ and $N(q)$ and derive formulas for $M(q)$ and $M(q^2)$, where, for the latter, we use the quartic version of duplication. Corresponding formulas for $N(q)$ and $N(q^2)$ can be similarly derived.

In the last section of this paper, we show how the alternative quartic theory can be utilized to derive two general formulas for Ramanujan-type series for $1/\pi$. Our results are new versions of formulas found by the Borweins [6, pp. 182–183] and considerably facilitate the determination of particular examples. The paper concludes with three such examples.

Ramanujan's theory of elliptic functions of signature 6 is not as complete as those in the cubic and quartic theories [5], [4, pp. 161–164]. In particular, we have been unable to obtain a sextic analogue of (1.4), (1.9), and (1.13).

Chan and Ong [9] have established a few results pointing toward the beginnings of a theory of signature 7. No further alternative theories have been found.

Some of the theorems in this paper were established by the third author in his doctoral dissertation [10].

2. THE QUARTIC INVERSION FORMULA AND PRINCIPLES OF DUPLICATION AND DIMIDIATION

THEOREM 2.1 [13, p. 260]. *For* $0 < x < 1$,

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \left(\frac{1-x}{1+3x}\right)^2\right) = \sqrt{1+3x}\ {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; x^2\right). \tag{2.1}$$

THEOREM 2.2. *If* $A(q)$ *and* $C(q)$ *are defined by* (1.10) *and* (1.11), *respectively, and if* $|q|<1$, *then*

$$ {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;\frac{C^2(q)}{A^2(q)}\right)=\sqrt{A(q)}. \tag{2.2}$$

A proof of Theorem 2.1 can be found in [5, p. 4214, Theorem 9.4]. Theorem 2.2 was first proved by J. M. Borwein and P. B. Borwein (see [6, p. 179, Proposition 5.7(a)] and [7, Theorem 2.6(b)]). See [5, p. 4220, Lemma 9.14] for a proof using results from Ramanujan's notebooks.

For $0<x<1$, let

$$F(x):=\exp\left(-\pi\sqrt{2}\,\frac{{}_2F_1(\frac{1}{4},\frac{3}{4};1;1-x)}{{}_2F_1(\frac{1}{4},\frac{3}{4};1;x)}\right). \tag{2.3}$$

The following *quartic inversion formula* is the key to proving (1.13).

THEOREM 2.3. *For* $|q|<1$,

$$F\left(\frac{C^2(q)}{A^2(q)}\right)=q. \tag{2.4}$$

A series of lemmas are needed.

LEMMA 2.4. *We have*

$$\varphi^2(q)-\varphi^2(-q)=8q\psi^2(q^4) \tag{2.5}$$

and

$$\varphi^2(q)+\varphi^2(-q)=2\varphi^2(q^2). \tag{2.6}$$

These formulas are, respectively, Entries 25 (v) and (vi) of Chapter 16 in [3, p. 40].

LEMMA 2.5. *We have*

$$B(q)=\tfrac{1}{3}(4A(q^2)-A(q)) \tag{2.7}$$

and

$$C(q) = \tfrac{1}{3}(A(\sqrt{q}) - A(q)). \tag{2.8}$$

Proof. Squaring both sides of (2.5) and (2.6) and then adding the resulting identities together, we obtain

$$4(\varphi^4(q^2) + 16q^2\psi^4(q^4)) = 2(\varphi^4(q) + \varphi^4(-q)).$$

Thus,

$$4A(q^2) = 2(\varphi^4(q) + \varphi^4(-q)) = 2(2\varphi^4(q) - 16q\psi^4(q^2)) = A(q) + 3B(q), \tag{2.9}$$

where the penultimate equality is justified by (1.3). Hence the first identity (2.7) follows. Now, multiplying equalities (2.5) and (2.6) and then employing equality (1.3), we see that

$$C(q^2) = 8q\psi^4(q^2). \tag{2.10}$$

From (2.9), we also have

$$A(q^2) = \varphi^4(q) - 8q\psi^4(q^2). \tag{2.11}$$

From (1.10), (2.10), and (2.11), we immediately deduce the second identity (2.8). ∎

Equalities (2.7) and (2.8) are the quartic analogues of equalities (2.8) and (2.9) in [5].

LEMMA 2.6. *If $n = 2^m$, where m is a positive integer, then*

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{B^2(q)}{A^2(q)}\right) = \frac{\sqrt{A(q)}}{\sqrt{A(q^n)}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{B^2(q^n)}{A^2(q^n)}\right). \tag{2.12}$$

Proof. Replacing x by $(1-x)/(1+3x)$ in (2.1), we find that

$${}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - x^2\right) = \sqrt{\frac{4}{1+3x}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \left(\frac{1-x}{1+3x}\right)^2\right). \tag{2.13}$$

Let $x = B(q)/A(q)$ and use (2.7), (2.8), and (1.12) to obtain

$$\begin{aligned}
{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{B^2(q)}{A^2(q)}\right) &= \sqrt{\frac{4A(q)}{A(q)+3B(q)}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \left(\frac{A(q)-B(q)}{A(q)+3B(q)}\right)^2\right) \\
&= \sqrt{\frac{A(q)}{A(q^2)}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{C^2(q^2)}{A^2(q^2)}\right) \\
&= \sqrt{\frac{A(q)}{A(q^2)}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{B^2(q^2)}{A^2(q^2)}\right).
\end{aligned}$$

We conclude the proof by iterating this identity m times. ∎

LEMMA 2.7. *If $n = 2^m$, where m is a positive integer, then*

$$ {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{B^2(q)}{A^2(q)}\right) = \frac{\sqrt{A(q)}}{n\sqrt{A(q^n)}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{B^2(q^n)}{A^2(q^n)}\right). \tag{2.14}$$

Proof. By (1.12), (2.13) with $x = C(q)/A(q)$, (2.7), and (2.8), we deduce that

$$\begin{aligned}
{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{B^2(q)}{A^2(q)}\right) &= {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1 - \frac{C^2(q)}{A^2(q)}\right) \\
&= \sqrt{\frac{4A(q)}{A(q)+3C(q)}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \left(\frac{A(q)-C(q)}{A(q)+3C(q)}\right)^2\right) \\
&= 2\sqrt{\frac{A(q)}{A(\sqrt{q})}}\, {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{B^2(\sqrt{q})}{A^2(\sqrt{q})}\right).
\end{aligned} \tag{2.15}$$

Replacing q by q^2 in (2.15), and then iterating the resulting equality m times, we establish (2.14). ∎

LEMMA 2.8. *Let $|q| < 1$ and let $F(x)$ be as defined in (2.3). If $n = 2^m$, where m is a positive integer, then*

$$F\left(\frac{B^2(q)}{A^2(q)}\right) = F^n\left(\frac{B^2(q^n)}{A^2(q^n)}\right)$$

and

$$F^n\left(\frac{C^2(q)}{A^2(q)}\right) = F\left(\frac{C^2(q^n)}{A^2(q^n)}\right). \tag{2.16}$$

Proof. We deduce, upon division of (2.12) by (2.14), that

$$\frac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\frac{B^2(q)}{A^2(q)}\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{B^2(q)}{A^2(q)}\right)} = n\,\frac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\frac{B^2(q^n)}{A^2(q^n)}\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{B^2(q^n)}{A^2(q^n)}\right)}.$$

This proves the lemma. ∎

The cubic analogues of Lemmas 2.6, 2.7, and 2.8 are recorded, respectively, as Lemmas 2.5, 2.7, and 2.8 in [5].

We are now ready to establish the quartic inversion formula.

Proof of Theorem 2.3. We will use Ramanujan's approximation proved in Example 3 in Section 27 of Chapter 11 in Ramanujan's second notebook [2, p. 82], namely,

$$\exp\left(-\sqrt{2}\,\pi\,\frac{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; 1-x)}{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)}\right) = \frac{x}{64}\left(1+\frac{5}{8}x+\cdots\right).$$

Letting n tend to ∞ in (2.16) and applying Ramanujan's approximation above and the definitions of $C(q)$ and $A(q)$, we deduce that

$$\begin{aligned} F\left(\frac{C^2(q)}{A^2(q)}\right) &= \lim_{n\to\infty} F^{1/n}\left(\frac{C^2(q^n)}{A^2(q^n)}\right) \\ &= \lim_{n\to\infty}\left(\frac{C^2(q^n)}{64A^2(q^n)}\left(1+\frac{5}{8}\frac{C^2(q^n)}{A^2(q^n)}+\cdots\right)\right)^{1/n} \\ &= \lim_{n\to\infty}\left((q^n+\cdots)\left(1+\frac{5}{8}(64q^n+\cdots)+\cdots\right)\right)^{1/n} \\ &= q. \quad ∎ \end{aligned}$$

Using Theorems 2.2 and 2.3, we can deduce (1.13), which we state in the following theorem. We will omit the proof since it is along the same lines as that given in [3. Entry 6, p. 101].

THEOREM 2.9. *Let* $0<x<1$. *If* $q=q_4$ *is given by* (1.1), *then*

$$z = z_4 := {}_2F_1(\tfrac{1}{4}, \tfrac{3}{4}; 1; x) = \sqrt{A(q)}.$$

We next derive quartic versions of duplication and dimidiation. Let

$$t'^2 = 1-\left(\frac{1-t}{1+3t}\right)^2. \tag{2.17}$$

Then

$$t = \frac{1-\sqrt{1-t'^2}}{1+3\sqrt{1-t'^2}}. \tag{2.18}$$

Using (2.17) and Theorem 2.1, we find that

$$\begin{aligned} z' := z(t'^2) &= {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; t'^2\right) \\ &= {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\left(\frac{1-t}{1+3t}\right)^2\right) \\ &= \sqrt{1+3t}\; {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; t^2\right) \\ &= \sqrt{1+3t}\; z(t^2). \end{aligned} \tag{2.19}$$

Using (2.18), we see that (2.19) may be rewritten in the form

$$z^2(t^2) = \frac{z'^2}{1+3t} = \frac{1}{4}\left(1+3\sqrt{1-t'^2}\right) z'^2. \tag{2.20}$$

Next, by (2.17),

$$\begin{aligned} q' := q(t'^2) &= \exp\left(-\pi\sqrt{2}\,\frac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-t'^2\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; t'^2\right)}\right) \\ &= \exp\left(-\pi\sqrt{2}\,\frac{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \left(\frac{1-t}{1+3t}\right)^2\right)}{{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\left(\frac{1-t}{1+3t}\right)^2\right)}\right). \end{aligned} \tag{2.21}$$

Now, by the same transformation (Theorem 2.1),

$$\begin{aligned} &{}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \left(\frac{1-t}{1+3t}\right)^2\right) \\ &\quad = \frac{1}{\sqrt{1+3\,\frac{1-t}{1+3t}}}\; {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-\left(\frac{1-\frac{1-t}{1+3t}}{1+3\,\frac{1-t}{1+3t}}\right)^2\right) \\ &\quad = \frac{\sqrt{1+3t}}{2}\; {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; 1-t^2\right). \end{aligned} \tag{2.22}$$

Thus, by (2.21) and (2.22),

$$q' = q(t'^2) = \exp\left(-\pi\sqrt{2}\,\frac{\dfrac{\sqrt{1+3t}}{2}\;{}_2F_1\left(\dfrac{1}{4},\dfrac{3}{4};\,1;\,1-t^2\right)}{\sqrt{1+3t}\;{}_2F_1\left(\dfrac{1}{4},\dfrac{3}{4};\,1;\,t^2\right)}\right)$$
$$= \exp\left(-\frac{\pi}{\sqrt{2}}\,\frac{{}_2F_1\left(\dfrac{1}{4},\dfrac{3}{4};\,1;\,1-t^2\right)}{{}_2F_1\left(\dfrac{1}{4},\dfrac{3}{4};\,1;\,t^2\right)}\right)$$
$$= \sqrt{q(t^2)}. \tag{2.23}$$

Thus, we can state the following theorems.

THEOREM 2.10 (Duplication formula). *Suppose we have a relation of the form*

$$\Omega(t^2;\, q;\, z^2) = 0, \tag{2.24}$$

where

$$z = {}_2F_1(\tfrac{1}{4},\,\tfrac{3}{4};\,1;\,t^2).$$

Then

$$\Omega\left(\left\{\frac{1-\sqrt{1-t'^2}}{1+3\sqrt{1-t'^2}}\right\}^2;\; q'^2;\; \frac{1}{4}\,(1+3\sqrt{1-t'^2})\, z'^2\right) = 0.$$

Proof. The result follows immediately from (2.18), (2.23), and (2.20). ∎

THEOREM 2.11 (Dimidiation formula). *Suppose that* (2.24) *holds with* t*,* q*, and* z *replaced by* t'*,* q'*, and* z'*, respectively. Then*

$$\Omega\left(8\,\frac{t(1+t)}{(1+3t)^2};\; \sqrt{q};\; (1+3t)\, z^2\right) = 0.$$

Proof. The result is an immediate consequence of (2.17), (2.23), and (2.19). ∎

We complete this section by applying Theorems 2.10 and 2.11 in deriving representations for $B(q)$ and $C(q)$ in terms of x and z.

THEOREM 2.12. *Recall that* $B(q)$ *and* $C(q)$ *are defined by* (1.10) *and* (1.11), *respectively. Then*

$$B(q)=\sqrt{1-x}\,z^2 \tag{2.25}$$

and

$$C(q)=\sqrt{x}\,z^2. \tag{2.26}$$

Proof. By (2.7), Theorem 2.9, and Theorem 2.10,

$$B(q)=\tfrac{1}{3}(4A(q^2)-A(q))=\tfrac{1}{3}(4\cdot\tfrac{1}{4}(1+3\sqrt{1-x})\,z^2-z^2)=\sqrt{1-x}\,z^2.$$

Next, by (2.8), Theorem 2.9, and Theorem 2.11,

$$C(q)=\tfrac{1}{3}(A(\sqrt{q})-A(q))=\tfrac{1}{3}((1+3\sqrt{x})\,z^2-z^2)=\sqrt{x}\,z^2. \quad \blacksquare$$

3. THE DEDEKIND ETA FUNCTION AND EISENSTEIN SERIES

In this section, we derive some alternative representations for the quartic theta functions $A(q)$, $B(q)$, and $C(q)$ and the quartic modulus x in terms of the Dedekind eta function. We also show how the Dedekind eta function and Eisenstein series can be parametrized in terms of the quartic variables x, $q=q_4$, and $z=z_4$. Recall that the function $f(-q)=q^{-1/24}\eta(\tau)$ is defined by (1.14).

THEOREM 3.1. *We have*

$$B(q)=\left(\frac{f^2(-q)}{f(-q^2)}\right)^4, \tag{3.1}$$

$$C(q)=8\sqrt{q}\left(\frac{f^2(-q^2)}{f(-q)}\right)^4, \tag{3.2}$$

and

$$A(q)=\frac{f^8(-q)+32qf^8(-q^4)}{f^4(-q^2)}. \tag{3.3}$$

Proof. We need some representations for $\varphi(-q)$ and $\psi(q)$ in terms of $f(-q)$. These are given by [3, pp. 36–37, Entry 22(ii) and Eq. (22.4)]

$$\varphi(-q)=\frac{f^2(-q)}{f(-q^2)} \tag{3.4}$$

and

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)}. \tag{3.5}$$

Now, (3.1) follows immediately from (1.10), (1.3), and (3.4), whereas (3.2) follows from (2.10) and (3.5). By (1.10) and (2.10), we have

$$A(q) = B(q) + 4C(q^2),$$

from which we derive (3.3) in view of (3.1) and (3.2). ▮

The following curious identity is the quartic analogue of a cubic theorem of Ramanujan found in his notebooks [3, p. 346, Entry 1(iv)] and is useful in establishing an eta function representation for the quartic modulus x.

THEOREM 3.2. *We have*

$$1 + 32q\,\frac{f^8(-q^4)}{f^8(-q)} = \left(1 + 64q\,\frac{f^{24}(-q^2)}{f^{24}(-q)}\right)^{1/2}. \tag{3.6}$$

Proof. Substituting (3.1) and (3.2) into (1.12), comparing the resulting expression for $A^2(q)$ with (3.3), and simplifying, we finish the proof. ▮

THEOREM 3.3. *Let* $0 < x < 1$. *If* $q = q_4$ *is given by* (1.1), *then*

$$\frac{1}{1-x} = 1 + 64q\,\frac{f^{24}(-q^2)}{f^{24}(-q)} \tag{3.7}$$

and

$$\frac{1}{x} = 1 + \frac{f^{24}(-q)}{64qf^{24}(-q^2)}. \tag{3.8}$$

Proof. To prove (3.7), solve (2.25) for $1/(1-x)$, and then use, in order, Theorem 2.9, (1.12), and Theorems 3.1 and 3.2. To prove (3.8), solve (2.26) for $1/x$ and use the same steps as above. ▮

THEOREM 3.4 [5, p. 4216, Theorem 9.9]. *We have*

$$f(-q) = q^{-1/24}\sqrt{z}\,2^{-1/4}x^{1/24}(1-x)^{1/12}. \tag{3.9}$$

Proof. By Theorem 2.12, (3.9) is equivalent to

$$f(-q) = q^{-1/24}2^{-1/4}C^{1/12}(q)\,B^{1/6}(q).$$

Thus, it suffices to prove that

$$8\sqrt{q}\, f^{12}(-q) = C(q)\, B^2(q),$$

which is obvious by Theorem 3.1. ∎

For $|q| < 1$, the classical Eisenstein series L, M, and N are defined by

$$L(q) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}, \tag{3.10}$$

$$M(q) = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k}, \tag{3.11}$$

and

$$N(q) = 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1-q^k}. \tag{3.12}$$

THEOREM 3.5. *We have*

$$M(q) = (1+3x)\, z^4.$$

Proof. We need to recall some facts about $z = {}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)$ and $L(q)$. The hypergeometric differential equation satisfied by z is given by [1, p. 1]

$$x(1-x)\frac{d^2 z}{dx^2} + (1-2x)\frac{dz}{dx} - \frac{3}{16} z = 0. \tag{3.13}$$

If

$$y := -\pi\sqrt{2}\, \frac{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; 1-x)}{{}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)}, \tag{3.14}$$

then [2, p. 88, Corollary]

$$\frac{dy}{dx} = -\frac{1}{x(1-x)\, z^2}. \tag{3.15}$$

The functions z and $L(q)$ are related by [5, Eq. (9.10)], [4, p. 149, Eq. (9.10)]

$$L(q)=12x(1-x)\,z\,\frac{dz}{dx}+(1-3x)\,z^2. \tag{3.16}$$

Last, $L(q)$ satisfies the differential equation [2, p. 330]

$$q\,\frac{dL}{dq}=\frac{L^2(q)-M(q)}{12}. \tag{3.17}$$

By the chain rule and (3.15),

$$\frac{dL}{dx}=-\frac{1}{x(1-x)\,z^2}\,\frac{dL}{dy}. \tag{3.18}$$

By (1.1) and (3.14), $q=e^{-y}$. Thus, by (3.17) and the chain rule,

$$\frac{dL}{dy}=\frac{M(q)-L^2(q)}{12}. \tag{3.19}$$

Thus, from (3.18) and (3.19),

$$-x(1-x)\,z^2\,\frac{dL}{dx}=\frac{M(q)-L^2(q)}{12}. \tag{3.20}$$

From (3.16) and (3.13),

$$\begin{aligned}\frac{dL}{dx}&=(12(1-x)-12x)\,z\,\frac{dz}{dx}+12x(1-x)\left(z\,\frac{d^2z}{dx^2}+\left(\frac{dz}{dx}\right)^2\right)\\&\quad-3z^2+(1-3x)\,2z\,\frac{dz}{dx}\\&=(14-30x)\,z\,\frac{dz}{dx}+12x(1-x)\left(z\,\frac{d^2z}{dx^2}+\left(\frac{dz}{dx}\right)^2\right)-3z^2\\&=(14-30x)\,z\,\frac{dz}{dx}+12x(1-x)\left(z\,\frac{\frac{3}{16}\,z-(1-2x)\,\frac{dz}{dx}}{x(1-x)}+\left(\frac{dz}{dx}\right)^2\right)-3z^2\\&=2(1-3x)\,z\,\frac{dz}{dx}-\frac{3}{4}\,z^2+12x(1-x)\left(\frac{dz}{dx}\right)^2.\end{aligned} \tag{3.21}$$

Thus, by (3.20), (3.16), and (3.21),

$$
\begin{aligned}
M(q) &= \left(12x(1-x)\,z\,\frac{dz}{dx} + (1-3x)\,z^2\right)^2 \\
&\quad - 12x(1-x)\,z^2\left(2(1-3x)\,z\,\frac{dz}{dx} - \frac{3}{4}z^2 + 12x(1-x)\left(\frac{dz}{dx}\right)^2\right) \\
&= (1-3x)^2\,z^4 + 9x(1-x)\,z^4 \\
&= (1+3x)\,z^4. \quad \blacksquare
\end{aligned}
$$

We now use Theorem 3.5 and the principle of duplication to find a representation for $M(q^2)$.

THEOREM 3.6. *We have*

$$M(q^2) = (1 - \tfrac{3}{4}x)\,z^4.$$

Proof. By Theorems 3.5 and 2.10,

$$
\begin{aligned}
M(q^2) &= \frac{1}{16}\left(1+3\sqrt{1-x}\right)^2\left(1+3\left(\frac{1-\sqrt{1-x}}{1+3\sqrt{1-x}}\right)^2\right)z^4 \\
&= \frac{1}{16}\left((1+3\sqrt{1-x})^2 + 3(1-\sqrt{1-x})^2\right)z^4 \\
&= \left(1-\frac{3}{4}x\right)z^4. \quad \blacksquare
\end{aligned}
$$

Similarly, we can derive formulas for $N(q)$ and $N(q^2)$. Also, one can use the principle of duplication to derive a formula for $f(-q^2)$. Alternatively, we could have reversed the procedure and used the principle of dimidiation. See [4, pp. 146–148] for a different approach to the formulas in this section.

4. SERIES FOR $1/\pi$

We conclude our paper with a new method for deriving series for $1/\pi$ associated with the theory of signature 4. We first state a general theorem of J. M. Borwein and P. B. Borwein [6, pp. 182–183, Eqs. (5.5.16), (5.5.17)]. In order to state their theorem, we need to review some definitions.

The classical singular modulus $\sqrt{\alpha_n}$ is the unique number between 0 and 1 satisfying the relation

$$\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha_n)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha_n)} = \sqrt{n}.$$

The Ramanujan–Weber class invariants G_n and g_n are defined by [4, p. 183, Eq. (1.3)]

$$G_n := 2^{-1/4} q_n^{-1/24} \frac{f(q_n)}{f(-q_n^2)} \qquad \text{and} \qquad g_n := 2^{-1/4} q_n^{-1/24} \frac{f(-q_n)}{f(-q_n^2)}, \tag{4.1}$$

where $f(-q)$ is defined by (1.14) and where $q_n = e^{-\pi\sqrt{n}}$. They can be expressed in terms of α_n by the equalities [4, p. 185, Eq. (1.6)]

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24} \qquad \text{and} \qquad g_n = \{4\alpha_n(1-\alpha_n)^{-2}\}^{-1/24}. \tag{4.2}$$

The Borweins [6, pp. 182–183, Eqs. (5.5.16), (5.5.17)] derived two classes of series for $1/\pi$, involving G_n, g_{2n}, and α_n, in the theory of signature 4, which we record in the following theorem.

THEOREM 4.1. *Let*

$$A_k = \frac{(\frac{1}{4})_k (\frac{3}{4})_k (\frac{1}{2})_k}{(k!)^3} \tag{4.3}$$

and

$$\alpha(n) = \frac{\pi}{4K^2} - \sqrt{n}\left(\frac{E}{K} - 1\right),$$

where

$$K := \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha_n\right) \qquad \textit{and} \qquad E := \frac{\pi}{2}\, {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; \alpha_n\right).$$

Then

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} A_k \left(\left(\frac{\alpha(n)\, u_n^{-1}}{1+\alpha_n} - \frac{\sqrt{n}}{4}\, g_n^{-12}\right) + k\sqrt{n}\left(\frac{g_n^{12} - g_n^{-12}}{2}\right)\right) u_n^{2k+1} \tag{4.4}$$

and

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} (-1)^k A_k \left(\left(\frac{\alpha(n)\, v_n^{-1}}{1-2\alpha_n} + \frac{\sqrt{n}}{2}\, \alpha_n G_n^{12}\right) + k\sqrt{n}\left(\frac{G_n^{12} + G_n^{-12}}{2}\right)\right) v_n^{2k+1}, \tag{4.5}$$

where

$$u_n = \left(\frac{g_n^{12} + g_n^{-12}}{2}\right)^{-1} \quad \textit{and} \quad v_n = \left(\frac{G_n^{12} - G_n^{-12}}{2}\right)^{-1}.$$

In his papers [11], [12, pp. 23–39], Ramanujan recorded a total of 10 series [11, Eqs. (35)–(44)] for $1/\pi$ associated with the quartic theory. The series (40)–(44) are precisely (4.4) when $n=6$, 10, 18, 22, and 58, respectively. The first example of (4.4), which is the case when $n=4$, namely,

$$\frac{9}{2\pi} = \sum_{k=0}^{\infty} A_k (7k+1) \left(\frac{32}{81}\right)^k, \tag{4.6}$$

was surprisingly omitted by Ramanujan and apparently has not been recorded anywhere in the literature. However, it is not difficult to see that it follows from substituting the values [6, p. 172, Table 5.2b]

$$g_4^{12} = 2\sqrt{2} \qquad \text{and} \quad \alpha(4) = 2(\sqrt{2}-1)^2 \tag{4.7}$$

into (4.4). We shall give an alternate derivation of (4.6) from our new version of (4.4) which we prove below.

Ramanujan's series (35)–(39) in [11] are the special cases of (4.5), corresponding to $n=5$, 9, 13, 25, and 37, respectively. In this class, Ramanujan omitted the series

$$\frac{1}{\pi\sqrt{7}} = \sum_{k=0}^{\infty} A_k \left(\frac{65}{63}k + \frac{8}{63}\right)\left(-\frac{256}{3969}\right)^k. \tag{4.8}$$

This new series is the case $n=7$ of (4.5). It can be obtained by substituting the values [6, p. 172, Table 5.2a]

$$G_7^{12} = 8 \qquad \text{and} \quad \alpha(7) = \frac{\sqrt{7}-2}{2} \tag{4.9}$$

into (4.5). At the end of this section, based on our new version of (4.5), we will sketch a different proof of (4.8) without using values from the Borweins' tables.

The Borweins' series were derived from certain transformation formulas for hypergeometric series and series for $1/\pi$ in the classical base. In this section, we prove alternative forms of (4.4) and (4.5) by using the theory of signature 4 given in the previous sections. Our derivation is motivated by a new method illustrated in a recent paper by Chan, Liaw, and V. Tan [8].

We begin with Clausen's transformation [6, p. 178, Proposition 5.6(b)].

LEMMA 4.2. *For* $-1<t<1$,

$$ {}_2F_1^2(\tfrac{1}{4}, \tfrac{3}{4}; 1; t) = {}_3F_2(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{2}; 1, 1; 4t(1-t)). \tag{4.10} $$

Setting $t=x$ in (4.10), where x is given by (3.8), and using Theorem 2.9, we conclude that

$$ z^2 = {}_3F_2(\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{2}; 1, 1; H), \tag{4.11} $$

where $H=4x(1-x)$. Differentiating (4.11) with respect to $x=x(q)$, using (4.3), and substituting in (3.16), we conclude that

$$ L(q) = \sum_{k=0}^{\infty} A_k (6(1-2x(q))\, k + 1 - 3x(q))\, H^k(q). \tag{4.12} $$

We derive another formula for $L(q)$, which will be important later for us. Recall from (1.1) and (3.14) that $q=e^{-y}$. Using the chain rule and (3.15), we find that

$$ q\frac{dx}{dq} = q\frac{dx}{dy}\frac{dy}{dq} = x(1-x)\, z^2. \tag{4.13} $$

Substituting the right side of (4.13) into (3.16), we find that

$$ L(q) = 12\frac{q}{z}\frac{dz}{dq} + (1-3x(q))\, z^2(q). \tag{4.14} $$

Recall that the multiplier of degree n is defined by [3, p. 214, Eq. (24.14); p. 101, Eqs. (6.2), (6.4)]

$$ m(q) = \frac{z(q)}{z(q^n)}. \tag{4.15} $$

Note that by Theorem 2.9, $m(q)$ may be expressed in terms of $x(q)$ and $x(q^n)$; i.e., $m(q) = m(x(q), x(q^n))$. Differentiating $m(q)$ with respect to q and following exactly the steps given in [8], we deduce that

$$ \frac{nL(q^n) - L(q)}{z(q)\, z(q^n)} = (1-3x(q^n))\frac{n}{m(q)} - (1-3x(q))\, m(q) - 12x(q)(1-x(q))\frac{dm}{dx(q)}(x(q), x(q^n)). \tag{4.16} $$

To obtain an equivalent form of the Borweins' formula (4.4), we need the transformation formula [3, p. 43, Entry 27(iii)]

$$e^{-\pi/(12\tau)}f(-e^{-2\pi/\tau})=\sqrt{\tau}\,e^{-\pi\tau/12}f(-e^{-2\pi\tau}), \tag{4.17}$$

where $\tau>0$. Next set $q=e^{-2\pi/\sqrt{2n}}$ in (3.8) to deduce that

$$\frac{1}{x(e^{-2\pi/\sqrt{2n}})}=1+\frac{f^{24}(-e^{-2\pi/\sqrt{2n}})}{64e^{-2\pi/\sqrt{2n}}f^{24}(-e^{-4\pi/\sqrt{2n}})}. \tag{4.18}$$

By (4.17),

$$e^{-2\pi/\sqrt{2n}}f^{24}(-e^{-2\pi/\sqrt{2n}})=(2n)^6\,e^{-2\pi\sqrt{2n}}f^{24}(-e^{-2\pi\sqrt{2n}}) \tag{4.19}$$

and

$$e^{-2\pi\sqrt{2/n}}f^{24}(-e^{-2\pi\sqrt{2/n}})=(n/2)^6\,e^{-2\pi\sqrt{n/2}}f^{24}(-e^{-2\pi\sqrt{n/2}}), \tag{4.20}$$

which, when substituted in (4.18), gives

$$\frac{1}{x(e^{-2\pi/\sqrt{2n}})}=1+64e^{-\pi\sqrt{2n}}\,\frac{f^{24}(-e^{2\pi\sqrt{2n}})}{f^{24}(-e^{-\pi\sqrt{2n}})}=\frac{1}{1-x(e^{-\pi\sqrt{2n}})}, \tag{4.21}$$

by (3.7). Hence,

$$1-x(e^{-2\pi\sqrt{n/2}})=x(e^{-2\pi/\sqrt{2n}}). \tag{4.22}$$

We next derive a similar transformation for z. Recall from Theorem 2.9 that $z(q)=\sqrt{A(q)}$. Now go to (3.3), square it, and then use (3.6) to deduce that

$$z^4(q)=A^2(q)=\frac{f^{16}(-q)}{f^8(-q^2)}\left(1+64q\,\frac{f^{24}(-q^2)}{f^{24}(-q)}\right).$$

Let $q=e^{-2\pi/\sqrt{2n}}$. Then, using the transformations (4.19) and (4.20), we find that

$$z^4(e^{-2\pi/\sqrt{2n}})=n^2z^4(e^{-2\pi\sqrt{n/2}}).$$

Taking the fourth root of each side and remembering that $z(q)>0$ for $q>0$, we conclude that

$$z(e^{-2\pi/\sqrt{2n}})=\sqrt{n}\,z(e^{-2\pi\sqrt{n/2}}). \tag{4.23}$$

By (4.15) and (4.23), we also find that

$$m(e^{-2\pi/\sqrt{2n}}) = \sqrt{n}. \tag{4.24}$$

If we set

$$L_n = L(e^{-2\pi\sqrt{n/2}}), \qquad L_{1/n} = L(e^{-2\pi/\sqrt{2n}}),$$

$$x_n = x(e^{-2\pi\sqrt{n/2}}), \qquad \text{and} \quad z_n = z(e^{-2\pi\sqrt{n/2}}), \tag{4.25}$$

then, by (4.16), (4.22), (4.23), and (4.24), we conclude that

$$nL_n - L_{1/n} = \sqrt{n}\, z_n^2 \left(3\sqrt{n}\,(1-2x_n) - 12x_n(1-x_n)\frac{dm}{dx}(1-x_n, x_n) \right). \tag{4.26}$$

Now, differentiating both sides of the identity (4.23) with respect to n and setting, for brevity, $q = e^{-2\pi/\sqrt{2n}}$ and $q' = e^{-2\pi\sqrt{n/2}}$, we find that

$$\frac{\pi}{\sqrt{2}\, n^{3/2}}\, q\frac{dz}{dq}(q) = -\frac{\pi}{\sqrt{2}}\, q'\frac{dz}{dq'}(q') + \frac{1}{2\sqrt{n}} z(q'). \tag{4.27}$$

Next, from (4.14), we find that

$$q'\frac{dz}{dq'}(q') = z_n \frac{L_n - z_n^2(1-3x_n)}{12} \tag{4.28}$$

and

$$q\frac{dz}{dq}(q) = z_{1/n}\frac{L_{1/n} - z_{1/n}^2(1-3x_{1/n})}{12} = \sqrt{n}\, z_n \frac{L_{1/n} - nz_n^2(3x_n - 2)}{12}, \tag{4.29}$$

by the transformation formulas (4.23) and (4.22). Substituting (4.28) and (4.29) into (4.27), we find, after simplification, that

$$nL_n + L_{1/n} + nz_n^2 = \frac{6\sqrt{2n}}{\pi}. \tag{4.30}$$

Adding (4.26) and (4.30), we conclude that

$$L_n = \frac{6}{\pi\sqrt{2n}} + z_n^2 \left(1 - 3x_n - 6\,\frac{x_n(1-x_n)}{\sqrt{n}}\,\frac{dm}{dx}(1-x_n, x_n) \right). \tag{4.31}$$

Substituting (4.12) and (4.11), with $q=e^{-2\pi\sqrt{n/2}}$, into (4.31), we deduce that

$$\frac{1}{\pi\sqrt{2n}}=\sum_{k=0}^{\infty} A_k\left((1-2x_n)\,k+\frac{x_n(1-x_n)}{\sqrt{n}}\,\frac{dm}{dx}\,(1-x_n,x_n)\right)H_n^k, \quad (4.32)$$

where

$$H_n=4x_n(1-x_n). \quad (4.33)$$

By employing (3.7), (3.8), and (4.1), we find that

$$H_n=\left(\frac{2}{g_{2n}^{12}+g_{2n}^{-12}}\right)^2. \quad (4.34)$$

Using this in (4.32), we see that (4.32) is an alternative formula to (4.4).

To derive the second general formula, first set $\mathbf{L}(q)=L(-q)$, $\mathbf{x}(q)=x(-q)$, $\mathbf{z}(q)=z(-q)$, and

$$\mathbf{m}(q)=\frac{\mathbf{z}(q)}{\mathbf{z}(q^n)}. \quad (4.35)$$

These definitions need some clarification. First, the identity of $\mathbf{L}(q)=L(-q)$ is clear from (3.10). However, throughout the theory developed in Section 2, it was necessarily assumed that $0<x<1$. In particular, see Theorem 2.9. Thus, more precisely, we define $\mathbf{x}(q)=x(-q)$ by (3.8). We next define $A(-q)$ by (1.10) and $\mathbf{z}(q)=z(-q)$ by (1.13). However, $A(-q)<0$ for $q>0$. Thus, we unambiguously define $\mathbf{z}(q)$ by taking the principal branch of $\sqrt{A(-q)}$.

Proceeding as in [8], we derive the analogue of (4.16), namely,

$$\frac{n\mathbf{L}(q^n)-\mathbf{L}(q)}{\mathbf{z}(q)\,\mathbf{z}(q^n)}=(1-3\mathbf{x}(q^n))\,\frac{n}{\mathbf{m}(q)}-(1-3\mathbf{x}(q))\,\mathbf{m}(q)$$
$$-12\mathbf{x}(q)(1-\mathbf{x}(q))\,\frac{d\mathbf{m}}{d\mathbf{x}(q)}\,(\mathbf{x}(q),\mathbf{x}(q^n)). \quad (4.36)$$

Recall the transformation formula for $f(q)$ [3, p. 43, Entry 27(iv)]. If $\tau>0$,

$$e^{-\pi/(24\tau)}f(e^{-\pi/\tau})=\sqrt{\tau}\,e^{-\pi\tau/24}f(e^{-\pi\tau}). \quad (4.37)$$

Using (4.37) with $\tau=\sqrt{n}$ and (4.19) with n replaced by $n/2$, we find from (3.8) that

$$\begin{aligned}\frac{1}{\mathbf{x}(e^{-\pi/\sqrt{n}})}&=1-\frac{f^{24}(e^{-\pi/\sqrt{n}})}{64e^{-\pi/\sqrt{n}}f^{24}(-e^{-2\pi/\sqrt{n}})}\\&=1-\frac{n^6e^{-\pi\sqrt{n}+\pi/\sqrt{n}}f^{24}(e^{-\pi\sqrt{n}})}{64e^{-\pi/\sqrt{n}}e^{-2\pi\sqrt{n}+2\pi/\sqrt{n}}f^{24}(-e^{-2\pi\sqrt{n}})}\\&=1-\frac{f^{24}(e^{-\pi\sqrt{n}})}{64e^{-\pi\sqrt{n}}f^{24}(-e^{-2\pi\sqrt{n}})}\\&=\frac{1}{\mathbf{x}(e^{-\pi\sqrt{n}})}.\end{aligned}\tag{4.38}$$

To derive the corresponding formula for $\mathbf{z}(q)$, we need the transformation formula [3, p. 43, Entry 27(ii)]

$$\varphi(-e^{-\pi\tau})=2\tau^{-1/2}e^{-\pi/(4\tau)}\psi(e^{-2\pi/\tau}),\tag{4.39}$$

where $\tau>0$. From (1.10) and (4.39),

$$\begin{aligned}A(-e^{-\pi/\sqrt{n}})&=\varphi^4(-e^{-\pi/\sqrt{n}})-16e^{-\pi/\sqrt{n}}\psi^4(e^{-2\pi/\sqrt{n}})\\&=16ne^{-\pi\sqrt{n}}\psi^4(e^{-2\pi\sqrt{n}})-e^{-\pi/\sqrt{n}}ne^{\pi/\sqrt{n}}\varphi^4(-e^{-\pi\sqrt{n}})\\&=-nA(-e^{-\pi\sqrt{n}}).\end{aligned}$$

Thus, recalling (1.13) and choosing the principal square root, we conclude that

$$\mathbf{z}(e^{-\pi/\sqrt{n}})=\sqrt{-n}\,\mathbf{z}(e^{-\pi\sqrt{n}}).\tag{4.40}$$

Define

$$\mathbf{L}_n=\mathbf{L}(e^{-\pi\sqrt{n}}),\qquad \mathbf{L}_{1/n}=\mathbf{L}(e^{-\pi/\sqrt{n}}),$$

$$\mathbf{x}_n=\mathbf{x}(e^{-\pi\sqrt{n}}),\qquad\text{and}\qquad \mathbf{z}_n=\mathbf{z}(e^{-\pi\sqrt{n}}).\tag{4.41}$$

Thus, by (4.36), (4.38), and (4.40),

$$n\mathbf{L}_n-\mathbf{L}_{1/n}=\left(2n(1-3\mathbf{x}_n)-12\sqrt{-n}\,\mathbf{x}_n(1-\mathbf{x}_n)\frac{d\mathbf{m}}{d\mathbf{x}}(\mathbf{x}_n,\mathbf{x}_n)\right)\mathbf{z}_n^2.\tag{4.42}$$

For brevity, we temporarily set $q = e^{-\pi/\sqrt{n}}$ and $q' = e^{-\pi\sqrt{n}}$. Thus, differentiating (4.40) with respect to n, we find that

$$\frac{\pi}{n^{3/2}}\, q\, \frac{d\mathbf{z}}{dq}\,(q) = \frac{i}{\sqrt{n}}\, \mathbf{z}_n - \pi i q'\, \frac{d\mathbf{z}}{dq'}\,(q'). \tag{4.43}$$

By (4.14),

$$q'\, \frac{d\mathbf{z}}{dq'}\,(q') = \mathbf{z}_n\, \frac{\mathbf{L}_n - \mathbf{z}_n^2(1 - 3\mathbf{x}_n)}{12}, \tag{4.44}$$

and by (4.14), (4.38), and (4.40),

$$q\, \frac{d\mathbf{z}}{dq}(q) = \mathbf{z}_{1/n}\, \frac{\mathbf{L}_{1/n} - \mathbf{z}_{1/n}^2(1 - 3\mathbf{x}_{1/n})}{12} = i\sqrt{n}\, \mathbf{z}_n\, \frac{\mathbf{L}_{1/n} + n\mathbf{z}_n^2(1 - 3\mathbf{x}_n)}{12}. \tag{4.45}$$

Substituting (4.44) and (4.45) into (4.43), we conclude, after multiplying both sides by $12n/(i\pi\mathbf{z}_n)$ and simplifying, that

$$n\mathbf{L}_n + \mathbf{L}_{1/n} = \frac{12\sqrt{n}}{\pi}. \tag{4.46}$$

Adding (4.42) and (4.46), we find that

$$2n\mathbf{L}_n = \left(2n(1 - 3\mathbf{x}_n) - 12\sqrt{-n}\, \mathbf{x}_n(1 - \mathbf{x}_n)\, \frac{d\mathbf{m}}{d\mathbf{x}}\,(\mathbf{x}_n, \mathbf{x}_n) \right) \mathbf{z}_n^2 + \frac{12\sqrt{n}}{\pi}. \tag{4.47}$$

Now, with $q = -e^{-\pi\sqrt{n}}$, substitute (4.11) and (4.12) into (4.47) to conclude that

$$\frac{1}{\pi\sqrt{n}} = \sum_{k=0}^{\infty} A_k \left((1 - 2\mathbf{x}_n)\, k - \frac{\mathbf{x}_n(1 - \mathbf{x}_n)}{\sqrt{-n}}\, \frac{d\mathbf{m}}{d\mathbf{x}}\,(\mathbf{x}_n, \mathbf{x}_n) \right) \mathbf{H}_n^k, \tag{4.48}$$

where $\mathbf{H}_n = 4\mathbf{x}_n(1 - \mathbf{x}_n)$. Using (3.7), (3.8), and (4.1), we find that

$$\frac{1}{\mathbf{x}_n} = 1 - G_n^{24} \qquad \text{and} \qquad \frac{1}{1 - \mathbf{x}_n} = 1 - \frac{1}{G_n^{24}}. \tag{4.49}$$

Thus,

$$\frac{1}{\mathbf{x}_n(1 - \mathbf{x}_n)} = 2 - G_n^{24} - \frac{1}{G_n^{24}} = -\left(G_n^{12} - \frac{1}{G_n^{12}} \right)^2,$$

and so

$$\mathbf{H}_n = -\left(\frac{2}{G_n^{12} - G_n^{-12}}\right)^2. \tag{4.50}$$

Thus, in summary, we have proved the following theorem, which should be compared with Theorem 4.1.

THEOREM 4.3. *Let A_k be defined by* (4.3). *Suppose that x_n and $\mathbf{x}_n$ are defined by* (4.25) *and* (4.41), *respectively. Define $m(q)$ and $\mathbf{m}(q)$ by* (4.15) *and* (4.35), *respectively. Last, let H_n and $\mathbf{H}_n$ be given by* (4.34) *and* (4.50), *respectively. Then*

$$\frac{1}{\pi\sqrt{2n}} = \sum_{k=0}^{\infty} A_k \left((1-2x_n)\, k + \frac{x_n(1-x_n)}{\sqrt{n}} \frac{dm}{dx}(1-x_n, x_n)\right) H_n^k \tag{4.51}$$

and

$$\frac{1}{\pi\sqrt{n}} = \sum_{k=0}^{\infty} A_k \left((1-2\mathbf{x}_n)\, k - \frac{\mathbf{x}_n(1-\mathbf{x}_n)}{\sqrt{-n}} \frac{d\mathbf{m}}{d\mathbf{x}}(\mathbf{x}_n, \mathbf{x}_n)\right) \mathbf{H}_n^k. \tag{4.52}$$

We now establish the new series for $1/\pi$ offered in (4.6). Recall, from (4.13), that

$$q\frac{dx(q)}{dq} = x(q)(1-x(q))\, z^2(q). \tag{4.53}$$

Replacing q by q^n in (4.53), we find that

$$q\frac{dx(q^n)}{dq} = nx(q^n)(1-x(q^n))\, z^2(q^n). \tag{4.54}$$

Dividing (4.54) by (4.53), we find that

$$\frac{dx(q^n)}{dx(q)} = \frac{n}{m^2(x(q), x(q^n))} \frac{x(q^n)(1-x(q^n))}{x(q)(1-x(q))}. \tag{4.55}$$

When $q = e^{-2\pi/\sqrt{2n}}$, $x(q^n) = x_n$, $x(q) = 1 - x_n$ by (4.22), and $m^2(1-x_n, x_n) = n$ by (4.24). With these substitutions in (4.55), we arrive at

$$\frac{dx(q^n)}{dx(q)}(1-x_n, x_n) = 1. \tag{4.56}$$

Examining (2.19) carefully, we see that

$$z' = z(q) = \sqrt{1 + 3\sqrt{x(q^2)}}\, z(q^2).$$

Thus, with $n = 2$,

$$m^2(x(q), x(q^2)) = 1 + 3\sqrt{x(q^2)}. \tag{4.57}$$

Differentiating (4.57) with respect to $x(q^2)$, we deduce that

$$\frac{dm}{dx(q^2)} = \frac{1}{2m}\,\frac{3}{2\sqrt{x(q^2)}}. \tag{4.58}$$

Therefore, by the chain rule, (4.56), (4.58), and the fact that $m(1 - x_n, x_n) = \sqrt{n}$,

$$\begin{aligned}\frac{dm}{dx(q)}(1 - x_2, x_2) &= \frac{dx(q^2)}{dx(q)}(1 - x_2, x_2) \cdot \frac{dm}{dx\ (q^2)}(1 - x_2, x_2)\\ &= \frac{1}{2\sqrt{2}}\,\frac{3}{2\sqrt{x_2}}.\end{aligned} \tag{4.59}$$

By (4.7), $g_4^{12} = 2\sqrt{2}$, and so by (4.34), $H_2 = \frac{32}{81}$. It follows from (4.33) that $x_2 = \frac{1}{9}$. Using (4.56) and this value of x_2 in (4.59), we conclude that

$$\frac{x_2(1 - x_2)}{\sqrt{2}}\,\frac{dm}{dx}(1 - x_2, x_2) = \frac{1}{9}.$$

Together with the values $1 - 2x_2 = \frac{7}{9}$ and $H_2 = \frac{32}{81}$, we immediately deduce (4.6) from (4.51).

In our derivation of (4.6), we calculated directly the value

$$x_n(1 - x_n)\,\frac{dm}{dx}(1 - x_n, x_n).$$

We now illustrate another method of deriving such series for $1/\pi$ in our second example, (4.8).

By (4.42) and (4.52), we see that to calculate

$$\mathbf{x}_n(1 - \mathbf{x}_n)\,\frac{d\mathbf{m}}{d\mathbf{x}}(\mathbf{x}_n, \mathbf{x}_n)$$

it suffices to calculate

$$\frac{n\mathbf{L}_n - \mathbf{L}_{1/n}}{\sqrt{-n}\,\mathbf{z}_n^2}. \tag{4.60}$$

To calculate (4.60) when $n=7$, we employ the identity

$$\frac{7L(q^7)-L(q)}{\sqrt{A(q)\,A(q^7)}} = 6\left(\sqrt{\frac{B(q)\,B(q^7)}{A(q)\,A(q^7)}} + 2\sqrt{\frac{C(q)\,C(q^7)}{A(q)\,A(q^7)}} + 2\sqrt{2}\left\{\frac{B(q)\,B(q^7)\,C(q)\,C(q^7)}{A^2(q)\,A^2(q^7)}\right\}^{1/4}\right), \tag{4.61}$$

which we discovered with the help of Garvan's *etamake* package on Maple V.

By (3.8), observe that

$$\frac{1}{\mathbf{x}(q)} = \left(1 - \frac{f^{24}(q)}{64q\,f^{24}(-q^2)}\right). \tag{4.62}$$

Setting $n=7$ in (4.49) and employing (4.9), we conclude that

$$\mathbf{x}_7 = -\tfrac{1}{63}. \tag{4.63}$$

It follows from (2.26) and the equality $\mathbf{z}_n^2 = A(-q_n)$ that

$$\frac{C^2(-q_7)}{A^2(-q_7)} = -\frac{1}{63}. \tag{4.64}$$

Recall from (1.11) that

$$C(-q_n) = 8i\sqrt{q_n}\,\varphi^2(-q_n)\,\psi^2(q_n^2).$$

Using the transformation formula (4.39), we deduce that

$$C(-q_7)\,C(-q_{1/7}) = -7C^2(-q_7). \tag{4.65}$$

Recall also from (4.40) that

$$A(-q_{1/7}) = -7A(-q_7). \tag{4.66}$$

Thus, by (4.65) and (4.66),

$$\sqrt{\frac{C(-q_7)\,C(-q_{1/7})}{A(-q_7)\,A(-q_{1/7})}} = \sqrt{\frac{C^2(-q_7)}{A^2(-q_7)}} = -\frac{i}{\sqrt{63}}. \tag{4.67}$$

By (1.12) and (4.64),

$$\frac{B^2(-q_7)}{A^2(-q_7)}=\frac{64}{63}. \tag{4.68}$$

Also, by (1.10), (2.25), and the equality $\mathbf{z}_n^2=A(-q_n)$, we can see that

$$B(-q_7)\,B(-q_{1/7})=7B^2(-q_7). \tag{4.69}$$

Thus, from (4.66), (4.69), and (4.68),

$$\sqrt{\frac{B(-q_7)\,B(-q_{1/7})}{A(-q_7)\,A(-q_{1/7})}}=\sqrt{\frac{B^2(-q_7)}{-A^2(-q_7)}}=-i\,\frac{8}{\sqrt{63}}. \tag{4.70}$$

It follows from (4.70) and (4.67) that

$$\begin{aligned}&\left\{\frac{B(-q_{1/7})\,B(-q_7)\,C(-q_{1/7})\,C(-q_7)}{A^2(-q_{1/7})\,A^2(-q_7)}\right\}^{1/4}\\&\quad=\left\{\frac{B^2(-q_7)}{-A^2(-q_7)}\,\frac{C^2(-q_7)}{A^2(-q_7)}\right\}^{1/4}=i\,\frac{2\sqrt{2}}{\sqrt{63}}.\end{aligned} \tag{4.71}$$

Substituting (4.67), (4.70), and (4.71) into (4.61), we conclude that

$$\frac{7\mathbf{L}_7-\mathbf{L}_{1/7}}{\sqrt{-7}\,\mathbf{z}_7^2}=-\frac{4i}{\sqrt{7}}. \tag{4.72}$$

Now using (4.42), (4.52), and (4.72), we may complete the proof of (4.8).

We conclude this section with one additional series for $1/\pi$. By using (2.25), (2.26), (3.7), (3.8), (4.1), and (4.21), we find that

$$\begin{aligned}\frac{C^2(e^{-\pi\sqrt{2/7}})}{A^2(e^{-\pi\sqrt{2/7}})}&=\frac{B^2(e^{-\pi\sqrt{14}})}{A^2(e^{-\pi\sqrt{14}})}\\&=\left(1+\frac{1}{g_{14}^{24}}\right)^{-1}=\frac{1}{2}-\frac{1}{7}\sqrt{-50+44\sqrt{2}}\end{aligned} \tag{4.73}$$

and

$$\begin{aligned}\frac{C^2(e^{-\pi\sqrt{14}})}{A^2(e^{-\pi\sqrt{14}})}&=\frac{B^2(e^{-\pi\sqrt{2/7}})}{A^2(e^{-\pi\sqrt{2/7}})}\\&=(1+g_{14}^{24})^{-1}=\frac{1}{2}+\frac{1}{7}\sqrt{-50+44\sqrt{2}}\,.\end{aligned} \tag{4.74}$$

We have also used the value of g_{14}, given by Weber [14, p. 721] in the form

$$g_{14}^2 = \sqrt{\frac{3+2\sqrt{2}}{4}} + \sqrt{\frac{-1+2\sqrt{2}}{4}}.$$

Substituting these values into (4.61) and using (4.51), we conclude that

$$\frac{1}{\pi\sqrt{14}\sqrt{-50+44\sqrt{2}}} = \sum_{k=0}^{\infty} A_k \left(\frac{2}{7}k + \frac{3}{196}(3-\sqrt{2})\right)\left(\frac{249-176\sqrt{2}}{49}\right)^k. \tag{4.75}$$

Deriving (4.75) by using the Borweins' method would require the value of $\alpha(14)$, which is more complicated than the radicals used here.

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