RAMANUJAN'S CLASS INVARIANT λ_n AND A NEW CLASS OF SERIES FOR $1/\pi$

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Dedicated to Professor D. J. S. Robinson

Abstract

On page 212 of his lost notebook, Ramanujan defined a new class invariant λ_n and constructed a table of values for λ_n. The paper constructs a new class of series for $1/\pi$ associated with λ_n. The new method also yields a new proof of the Borweins' general series for $1/\pi$ belonging to Ramanujan's 'theory of q_2'.

1. *Introduction*

For any positive integer k, let

$$(a)_k := a\cdot(a+1)\dots(a+k-2)\cdot(a+k-1).$$

In his famous paper 'Modular equations and approximations to π' [**13**], S. Ramanujan recorded many elegant series for $1/\pi$. Among his list of series, he remarked that the series

$$\frac{27}{4\pi}=\sum_{k=0}^{\infty}(2+15k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(\frac{2}{27}\right)^k \tag{1.1}$$

and

$$\frac{15\sqrt{3}}{2\pi}=\sum_{k=0}^{\infty}(4+33k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(\frac{4}{125}\right)^k, \tag{1.2}$$

belong to the 'theory of q_2', where

$$q_2=\exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\alpha)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\alpha)}\right),\ 0<\alpha<1.$$

Ramanujan did not develop his theory, neither did he record his proofs of (1.1) and (1.2).

The first proofs of (1.1) and (1.2) are due to J. M. Borwein and P. B. Borwein [**7**, Chapter 5]. In order to describe their proofs, define α_n (the *cubic singular modulus*) to be the unique number satisfying the equation

$$\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\alpha_n)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\alpha_n)}=\sqrt{n}. \tag{1.3}$$

If

$$\epsilon(n)=\frac{3\sqrt{3}}{8\pi}\frac{1}{K^2}-\sqrt{n}\left(\frac{E}{K}-1\right), \tag{1.4}$$

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where

$$K = {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; \alpha_n) \tag{1.5}$$

and

$$E = {}_2F_1(-\tfrac{1}{3}, \tfrac{2}{3}; 1; \alpha_n), \tag{1.6}$$

then the Borweins show that the following theorem holds.

THEOREM 1.1.

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} (a_n + b_n k) \frac{(\frac{1}{2})_k (\frac{1}{3})_k (\frac{2}{3})_k}{(k!)^3} H_n^k, \tag{1.7}$$

where

$$a_n = \frac{8\sqrt{3}}{9}(\epsilon(n) - \sqrt{n}\alpha_n), \tag{1.8}$$

$$b_n = \frac{2\sqrt{3n}}{3}\sqrt{1 - H_n}, \tag{1.9}$$

and

$$H_n = 4\alpha_n(1 - \alpha_n). \tag{1.10}$$

For each rational number n, the Borweins' formula yields a corresponding series for $1/\pi$ if the explicit values of α_n and $\epsilon(n)$ are known. For example, the Borweins derive Ramanujan's series (1.1) and (1.2) by computing α_n and $\epsilon(n)$ for $n = 4$ and 5 respectively. They also added three new series which correspond to $n = 2$, 3 and 6. The Borweins' method of computing α_n is rather complicated. To obtain α_n, they need to solve a quartic polynomial equation which involves *Ramanujan–Weber class invariants* G_n and g_n, where

$$G_n = 2^{-1/4} e^{\pi\sqrt{n}/24} \prod_{k=1}^{\infty} (1 + e^{-\pi\sqrt{n}(2k-1)})$$

and

$$g_n = 2^{-1/4} e^{\pi\sqrt{n}/24} \prod_{k=1}^{\infty} (1 - e^{-\pi\sqrt{n}(2k-1)}).$$

When G_n or g_n gets too complicated, their method fails to yield a simple expression for α_n. The computation of $\epsilon(n)$ is even more difficult in such cases. As such, for several years, the aforementioned series have been the only known series belonging to the theory of q_2.

Around 1995, after the work of the Borweins and F. G. Garvan [**8**, **9**], B. C. Berndt, S. Bhargava and F. G. Garvan [**5**] succeeded in developing Ramanujan's theories of elliptic function to alternative bases, which includes Ramanujan's 'theory of q_2'. Recently, using identities from [**5**], H. H. Chan and W.-C. Liaw [**11**] noticed that by expressing α_n in terms of values of certain modular functions, one can evaluate α_n explicitly whenever the class groups of the imaginary quadratic fields $\mathbb{Q}(\sqrt{-3n})$ are isomorphic to $\mathbb{Z}_2^t$, $t \in \mathbb{N}$. With these new cubic singular moduli, they managed to derive many new series for $1/\pi$ by using the following alternative form of the Borweins' result.

THEOREM 1.2. *Let*

$$m = \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \alpha)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \beta)}, \tag{1.11}$$

where β and α satisfy the relation

$$\frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\beta)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \beta)} = n \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \alpha)}. \tag{1.12}$$

Then

$$\frac{1}{2\pi}\sqrt{\frac{3}{n}} = \sum_{k=0}^{\infty} (a_n + b_n k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3} H_n^k,$$

where

$$a_n = \frac{\alpha_n(1-\alpha_n)}{\sqrt{n}}\frac{dm}{d\alpha}(1-\alpha_n, \alpha_n),$$

$$b_n = (1-2\alpha_n),$$

and

$$H_n = 4\alpha_n(1-\alpha_n).$$

The original proof of Theorem 1.2 given by Chan and Liaw [**11**] begins with the Borweins' Theorem 1.1 and follows closely the treatment given in the classical case [**7**, Chapter 5]. One advantage of using Theorem 1.2 is that it shows clearly that the number a_n is algebraic since α_n is algebraic. This is not so apparent in the Borweins' formula since it involves two hypergeometric series in the definition of $\epsilon(n)$. A quick comparison also shows that Theorem 1.2 is simpler than that of the Borweins and that there should be a direct proof of this result.

In this paper, we provide a new proof of Theorem 1.2 using ideas which originated from Ramanujan's work [**13**]. We then derive a new class of series for $1/\pi$ corresponding to Ramanujan's class invariant λ_n defined by

$$\lambda_n = \frac{e^{(\pi/2)\sqrt{n/3}}}{3\sqrt{3}}\frac{f^6(e^{-\pi\sqrt{n/3}})}{f^6(e^{-\pi\sqrt{3n}})}, \tag{1.13}$$

where

$$f(-q) = \prod_{k=1}^{\infty}(1-q^k).$$

In terms of the Dedekind η-function,

$$\lambda_n = \frac{1}{3\sqrt{3}}\frac{\eta^6\left(\frac{1+\sqrt{-n/3}}{2}\right)}{\eta^6\left(\frac{1+\sqrt{-3n}}{2}\right)}.$$

We conclude this section with some elegant series which arise from this new class, namely,

$$\frac{4}{\pi\sqrt{3}} = \sum_{k=0}^{\infty}(5k+1)\frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{1}{2})_k}{(k!)^3}\left(-\frac{9}{16}\right)^k, \tag{1.14}$$

$$\frac{12\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(51k+7)\frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{1}{2})_k}{(k!)^3}\left(-\frac{1}{16}\right)^k, \tag{1.15}$$

$$\frac{4\sqrt{3}}{\pi\sqrt{5}} = \sum_{k=0}^{\infty}(9k+1)\frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{1}{2})_k}{(k!)^3}\left(-\frac{1}{80}\right)^k, \tag{1.16}$$

$$\frac{2^5\cdot 3\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(615k+53)\frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{1}{2})_k}{(k!)^3}\left(-\frac{1}{1024}\right)^k, \tag{1.17}$$

$$\frac{2^2\cdot 3^3}{\pi\sqrt{7}} = \sum_{k=0}^{\infty}(165k+13)\frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{1}{2})_k}{(k!)^3}\left(-\frac{1}{3024}\right)^k \tag{1.18}$$

and

$$\frac{2^2\cdot 3\cdot 5^3\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(14151k+827)\frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{1}{2})_k}{(k!)^3}\left(-\frac{1}{250000}\right)^k. \tag{1.19}$$

These series correspond to $n = 9$, 17, 25, 41, 49 and 89, respectively.

2. *Borweins' inversion formula, Clausen's formula and the series $L(q)$*

For $|q| < 1$, let

$$L(q) = 1-24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}. \tag{2.1}$$

In this section, we state a few important lemmas and derive an interesting identity associated with $L(q)$.

LEMMA 2.1 (Clausen's formula [**7**, Proposition 5.6(b), p. 178]). *For* $-1 < t < 1$,

$$({}_2F_1(\tfrac{1}{3},\tfrac{2}{3};1;t))^2 = {}_3F_2(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2};1,1;4t(1-t)).$$

LEMMA 2.2 (Borweins' inversion formula [**5**, Theorem 2.10; **10**, (4.4)]). *Let*

$$z(q) = \sum_{m,\,n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{2.2}$$

and

$$\frac{1}{\alpha(q)} = \frac{1}{27q}\left(\frac{f(-q)}{f(-q^3)}\right)^{12}+1. \tag{2.3}$$

Then

$$z(q) = {}_2F_1(\tfrac{1}{3},\tfrac{2}{3};1;\alpha(q)). \tag{2.4}$$

LEMMA 2.3 [**5**, Lemma 2.9]. *Let* $0 < q < 1$. *Then*

$$q = \exp\left\{-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\alpha(q))}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\alpha(q))}\right\}.$$

REMARK 2.4. It can be shown that $\alpha(q) = \alpha_n$ when $q = e^{-2\pi\sqrt{n/3}}$ [**11**].

LEMMA 2.5 [**5**, (4.4); **10**, (4.7)]. *Let* $\alpha = \alpha(q)$ *and* $z = z(q)$. *Then*

$$q\frac{d\alpha}{dq} = \alpha(1-\alpha)z^2.$$

LEMMA 2.6 [**5**, Lemma 4.1; **10**, (4.8)]. *Let* $L := L(q)$. *Then*

$$L = 12\frac{q}{z}\frac{dz}{dq}+(1-4\alpha)z^2. \tag{2.5}$$

From Lemmas 2.1 and 2.2, with $t = \alpha = \alpha(q)$, we conclude that

$$z^2 = z(q)^2 = {}_3F_2(\tfrac{1}{3},\tfrac{2}{3},\tfrac{1}{2};1,1;H(q)), \tag{2.6}$$

where $H(q) = 4\alpha(1-\alpha)$. Differentiating (2.6) with respect to α, we find that

$$2z\frac{dz}{d\alpha} = \sum_{k=1}^{\infty}A_k 4k(1-2\alpha)H^{k-1}, \tag{2.7}$$

where $H := H(q)$ and

$$A_k = \frac{(\frac{1}{3})_k(\frac{2}{3})_k(\frac{1}{2})_k}{(1)_k^3}. \tag{2.8}$$

Next, by Lemmas 2.5 and 2.6, we find that

$$\frac{dz}{d\alpha} = \frac{1}{12z\alpha(1-\alpha)}(L-(1-4\alpha)z^2). \tag{2.9}$$

Substituting (2.9) into (2.7) and simplifying, we obtain our first result.

THEOREM 2.7.

$$L(q) = \sum_{k=0}^{\infty} A_k(6(1-2\alpha(q))k+1-4\alpha(q))H(q)^k.$$

From Remark 2.4, we find, after setting $q = e^{-2\pi\sqrt{n/3}}$ that

$$L(e^{-2\pi\sqrt{n/3}}) = \sum_{k=0}^{\infty}(6(1-2\alpha_n)k+1-4\alpha_n)A_k H_n^k. \tag{2.10}$$

3. *Second identity associated with* $L(e^{-2\pi\sqrt{n/3}})$ *and the proof of Theorem* 1.2

From Lemma 2.3, we deduce that for any positive integer n,

$$q^n = \exp\left\{-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\alpha(q^n))}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\alpha(q^n))}\right\}. \tag{3.1}$$

On the other hand, the definition of β in (1.12) shows that

$$q^n = \exp\left\{-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\beta)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\beta)}\right\} \tag{3.2}$$

when q is given by (3.1) with $n = 1$. Comparing (3.1) and (3.2) and using the monotonicity of ${}_2F_1(\frac{1}{3},\frac{2}{3};1;x)$ for real $0 < x < 1$, we conclude that

$$\beta = \alpha(q^n)$$

if β and $\alpha = \alpha(q)$ satisfy (1.12). Using this identification, and Lemma 2.2, we may rewrite m, which is defined in (1.11), as

$$m(q) := \frac{z(q)}{z(q^n)}. \tag{3.3}$$

Note that since $m(q)$ can be expressed in terms of $\alpha = \alpha(q)$ and $\beta := \alpha(q^n)$, we will write $m(q) = m(\alpha,\beta)$ when necessary.

Now, differentiating (3.3) with respect to q, we obtain

$$q\frac{dm(q)}{dq} = \frac{z(q)}{z(q^n)}\left(\frac{q}{z(q)}\frac{dz(q)}{dq} - \frac{q}{z(q^n)}\frac{dz(q^n)}{dq}\right),$$

which, by Lemma 2.5, can be written as

$$\alpha(1-\alpha)\frac{dm}{d\alpha} = \frac{1}{z(q)\,z(q^n)}\left(\frac{q}{z(q)}\frac{dz(q)}{dq} - \frac{q}{z(q^n)}\frac{dz(q^n)}{dq}\right), \tag{3.4}$$

with $m = m(\alpha, \beta)$. If we replace q by q^n in (2.5), we find that

$$nL(q^n) = 12\frac{q}{z(q^n)}\frac{dz(q^n)}{dq} + nz^2(q^n)(1-4\beta(q)). \tag{3.5}$$

Subtracting (2.5) from (3.5), and dividing by $z(q)\,z(q^n)$, we obtain

$$\begin{aligned}\frac{nL(q^n)-L(q)}{z(q)\,z(q^n)} &= \frac{12}{z(q)\,z(q^n)}\left(\frac{q}{z(q^n)}\frac{dz(q^n)}{dq} - \frac{q}{z(q)}\frac{dz(q)}{dq}\right)\\ &\quad + (1-4\beta)\frac{n}{m} - (1-4\alpha)\,m\\ &= (1-4\beta)\frac{n}{m} - (1-4\alpha)\,m - 12\frac{dm}{d\alpha}\alpha(1-\alpha). \end{aligned} \tag{3.6}$$

The expression $dm/d\alpha$ makes sense since β can be expressed implicitly in terms of α by (1.12).

Now, for any positive real n, the function $z(q)$ satisfies

$$z(e^{-2\pi/\sqrt{3n}}) = \sqrt{n}z(e^{-2\pi\sqrt{n/3}}). \tag{3.7}$$

This transformation formula follows from the transformation of $f(-q)$ [**1**, Entry 27(iii), Chapter 16], namely,

$$e^{-\pi/(12t)}f(-e^{-2\pi/t}) = \sqrt{t}e^{-\pi t/12}f(-e^{-2\pi t}), \;\; t \in \mathbb{R}^+, \tag{3.8}$$

and the relations

$$z(q) = 1 + 6\sum_{k=0}^{\infty}\left(\frac{q^{3k+1}}{1-q^{3k+1}} - \frac{q^{3k+2}}{1-q^{3k+2}}\right) = \frac{f^3(-q^{1/3})}{f(-q)} + 3q^{1/3}\frac{f^3(-q^3)}{f(-q)}. \tag{3.9}$$

The proof of the first equality and the second equality in (3.9) can be found in [**2**, Chapter 33, (2.6)] and [**1**, p. 346, Entry 1(v)], respectively.

By setting $q = e^{-2\pi/\sqrt{3n}}$ in (3.7) we find that

$$m(e^{-2\pi/\sqrt{3n}}) = \sqrt{n}. \tag{3.10}$$

Next, by (3.8) and (2.3), we find that

$$\alpha(e^{-2\pi/\sqrt{3n}}) = 1-\alpha_n \text{ and } \beta(e^{-2\pi/\sqrt{3n}}) = \alpha(e^{-2\pi\sqrt{n/3}}) = \alpha_n. \tag{3.11}$$

Thus, using (3.10), (3.7) and (3.11) in (3.6) (with $q = e^{-2\pi/\sqrt{3n}}$), we deduce that

$$\begin{aligned}\frac{nL(e^{-2\pi\sqrt{n/3}}) - L(e^{-2\pi/\sqrt{3n}})}{\sqrt{n}z^2(e^{-2\pi\sqrt{n/3}})} &= \sqrt{n}(1-4\alpha_n) - \sqrt{n}(1-4(1-\alpha_n))\\ &\quad - 12\alpha_n(1-\alpha_n)\frac{dm}{d\alpha}(1-\alpha_n, \alpha_n)\\ &= 4\sqrt{n}(1-2\alpha_n) - 12\alpha_n(1-\alpha_n)\frac{dm}{d\alpha}(1-\alpha_n, \alpha_n),\end{aligned}$$

where we have used thwe fact that $\alpha(e^{-2\pi\sqrt{n/3}}) = \alpha_n$. Hence we obtain

$$nL(e^{-2\pi\sqrt{n/3}}) - L(e^{-2\pi/\sqrt{3n}}) = 4z_n^2\left(n(1-2\alpha_n) - 3\sqrt{n}\alpha_n(1-\alpha_n)\frac{dm}{d\alpha}(1-\alpha_n, \alpha_n)\right), \tag{3.12}$$

where

$$z_n = z(e^{-2\pi\sqrt{n/3}}). \tag{3.13}$$

Next, we establish another relation satisfied by these functions.

Now, (3.7) holds for any real number n. Thus, differentiating with respect to n, we find that

$$-\frac{\pi}{\sqrt{3}}\dot{z}(e^{-2\pi\sqrt{n/3}})+\frac{1}{2\sqrt{n}}z(e^{-2\pi\sqrt{n/3}}) = \frac{\pi}{\sqrt{3}n^{3/2}}\dot{z}(e^{-2\pi/\sqrt{3n}}), \tag{3.14}$$

where

$$\dot{z}(q) := q\frac{dz(q)}{dq}.$$

By (2.5),

$$\dot{z}(q) = z(q)\frac{L(q)-z^2(q)(1-4\alpha(q))}{12},$$

and hence

$$\dot{z}(e^{-2\pi\sqrt{n/3}}) = z_n\frac{L(e^{-2\pi\sqrt{n/3}})-z_n^2(1-4\alpha_n)}{12}, \tag{3.15}$$

where z_n and α_n are defined as in (3.13) and (1.3) respectively. Using (2.5), (3.7) and (3.11), we deduce that

$$\begin{aligned}\dot{z}(e^{-2\pi/\sqrt{3n}}) &= z(e^{-2\pi/\sqrt{3n}})\frac{L(e^{-2\pi/\sqrt{3n}})-z^2(e^{-2\pi/\sqrt{3n}})(1-4\alpha(e^{-2\pi/\sqrt{3n}}))}{12}\\ &= \sqrt{n}z_n\frac{L(e^{-2\pi/\sqrt{3n}})-nz_n^2(4\alpha_n-3)}{12}.\end{aligned} \tag{3.16}$$

Substituting (3.15) and (3.16) into (3.14) and simplifying, we conclude that

$$nL(e^{-2\pi\sqrt{n/3}})+L(e^{-2\pi/\sqrt{3n}})+2nz_n^2 = \frac{6\sqrt{3n}}{\pi}. \tag{3.17}$$

Adding (3.12) to (3.17), we arrive at a second expression for $L(e^{-2\pi\sqrt{n/3}})$, namely,

$$L(e^{-2\pi\sqrt{n/3}}) = \frac{3\sqrt{3}}{\pi\sqrt{n}}+\left(1-4\alpha_n-\frac{6}{\sqrt{n}}\alpha_n(1-\alpha_n)\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n)\right)z_n^2. \tag{3.18}$$

Now, by (2.6), we may rewrite (3.18) as

$$L(e^{-2\pi\sqrt{n/3}}) = \frac{3\sqrt{3}}{\pi\sqrt{n}}+\sum_{k=0}^{\infty}\left(1-4\alpha_n-\frac{6}{\sqrt{n}}\alpha_n(1-\alpha_n)\frac{dm}{d\alpha}(1-\alpha_n,\alpha_n)\right)A_k H_n^k, \tag{3.19}$$

where A_k and H_n are given by (2.8) and (1.10) respectively. Combining (3.19) and (2.10) we arrive at Theorem 1.2.

We conclude this section with an interesting identity which arises from our computations.

COROLLARY 3.1.

$$L(e^{-2\pi/3}) = 1-24\sum_{k=1}^{\infty}\frac{ke^{-2\pi k/3}}{1-e^{-2\pi k/3}} = \frac{9}{\pi}-\frac{6\pi}{12^{1/4}(\sqrt{3}-1)\,\Gamma^4(\frac{3}{4})}, \tag{3.20}$$

where

$$\Gamma(x) = \int_0^\infty e^{-s} s^{x-1}\, ds.$$

Proof. Set $n = 1/3$ in (3.17). We find that

$$L(e^{-2\pi/3}) = \frac{18}{\pi} - 2z_{1/3}^2 - 3L(e^{-2\pi}) = \frac{18}{\pi} - 6z_3^2 - 3L(e^{-2\pi}), \tag{3.21}$$

by (3.7). Now, it is well known that [**7**, p. 163]

$$L(e^{-2\pi}) = \frac{3}{\pi}. \tag{3.22}$$

Furthermore, from [**4**, Theorem 12], we find that

$$z_3 = \sum_{m,\, n=-\infty}^{\infty} e^{-2\pi(m^2+mn+n^2)} = \frac{\sqrt{\pi}}{12^{1/8}\sqrt{\sqrt{3}-1}\,\Gamma^2(\frac{3}{4})}. \tag{3.23}$$

Substituting (3.22) and (3.23) into (3.21), we complete our proof. □

There are very few known values of $L(q)$. Previous methods used in proving those values cannot be applied to establish our new identity (3.20).

4. λ_n *and a new class of series for* $1/\pi$

The new proof of Theorem 1.2 which we present here avoids the use of the Legendre-type relation [**7**, p. 178], namely,

$$EK' + KE' - KK' = \frac{3\sqrt{3}}{16}\pi, \tag{4.1}$$

where E and K are given by (1.6) and (1.5) respectively, and

$$K' = {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; 1-\alpha_n)$$

and

$$E' = {}_2F_1(-\tfrac{1}{3}, \tfrac{2}{3}; 1; 1-\alpha_n).$$

Relation (4.1) plays an important role in the Borweins' derivation of Theorem 1.1 and the advantage of avoiding the use of this relation is that it allows us to derive new classes of series for $1/\pi$ even if analogues of (4.1) fail to exist. This is precisely the case, as we will show in this section, when we attempt to find a new class of series for $1/\pi$ using Ramanujan class invariant λ_n defined in (1.13).

Motivated by (2.2) and (2.3), we define $\mathbf{z}(q)$ and $\boldsymbol{\alpha}(q)$ by

$$\mathbf{z}(q) = z(-q) = \sum_{m,\, n=-\infty}^{\infty} (-q)^{m^2+mn+n^2} \tag{4.2}$$

and

$$\frac{1}{\boldsymbol{\alpha}(q)} = -\frac{1}{27q}\left(\frac{f(q)}{f(q^3)}\right)^{12} + 1. \tag{4.3}$$

Note that

$$\frac{1}{\boldsymbol{\alpha}(q)} = -\lambda_n^2 + 1$$

when $q = e^{-\pi\sqrt{n/3}}$. We set $\boldsymbol{\alpha}_n = \boldsymbol{\alpha}(e^{-\pi\sqrt{n/3}})$. Note that for $|q| < 1$, Lemma 2.2 holds and so, replacing q by $-q$, we deduce that

$$\mathbf{z}(q) = {}_2F_1\left(\tfrac{1}{3}, \tfrac{2}{3}; 1; \boldsymbol{\alpha}(q)\right). \tag{4.4}$$

Furthermore, analogues of Lemmas 2.5 and 2.6 hold, namely,

$$q\frac{d\boldsymbol{\alpha}}{dq} = \boldsymbol{\alpha}(1-\boldsymbol{\alpha})\,\mathbf{z}^2, \tag{4.5}$$

and

$$\mathbf{L} = 12\frac{q}{\mathbf{z}}\frac{d\mathbf{z}}{dq} + (1-4\boldsymbol{\alpha})\,\mathbf{z}^2, \tag{4.6}$$

where $\mathbf{L} := L(-q)$, $\mathbf{z} := \mathbf{z}(q)$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}(q)$. Thus, by following exactly the steps given in Section 2, we deduce the following analogue of (2.10):

$$\mathbf{L}(e^{-\pi\sqrt{n/3}}) = \sum_{k=0}^{\infty} (6(1-2\boldsymbol{\alpha}_n)\,k + 1 - 4\boldsymbol{\alpha}_n)\,A_k\,\mathbf{H}_n^k, \tag{4.7}$$

where $\mathbf{H}_n = 4\boldsymbol{\alpha}_n(1-\boldsymbol{\alpha}_n)$.

Next, define the analogue $\mathbf{m}(q)$ of $m(q)$ (see (3.3)) by

$$\mathbf{m}(q) = \frac{\mathbf{z}(q)}{\mathbf{z}(q^n)}. \tag{4.8}$$

By (4.4), one can easily express $\mathbf{m}(q)$ as

$$\mathbf{m}(q) = \mathbf{m}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \boldsymbol{\alpha})}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \boldsymbol{\beta})},$$

where $\boldsymbol{\beta} = \boldsymbol{\alpha}(q^n)$. Thus, we may write $\mathbf{m}(q) = \mathbf{m}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. At this point, in order to continue, we need two transformation formulae analogous to that of (3.7) and (3.11).

LEMMA 4.1. *The functions* $\boldsymbol{\alpha}(q)$ *and* $\mathbf{z}(q)$ *satisfy the transformation formulae*

$$\boldsymbol{\alpha}(e^{-\pi/\sqrt{3n}}) = 1 - \boldsymbol{\alpha}(e^{-\pi\sqrt{n/3}}), \tag{4.9}$$

and

$$\mathbf{z}(e^{-\pi/\sqrt{3n}}) = -\sqrt{n}\,\mathbf{z}(e^{-\pi\sqrt{n/3}}). \tag{4.10}$$

Proof. Formula (4.9) follows immediately from (4.3) and the transformation formula for $f(q)$ [**1**, Entry 27(iv), Chapter 16], namely,

$$e^{-\pi/(24t)}f(e^{-\pi/t}) = \sqrt{t}\,e^{-\pi t/24}f(e^{-\pi t}), \quad t \in \mathbb{R}^+. \tag{4.11}$$

Formula (4.10) follows from the transformation formula [**1**, Entry 27(ii), Chapter 16]

$$\varphi(-e^{-\pi/t}) = 2e^{-\pi t/4}\sqrt{t}\,\psi(e^{-2\pi t}),$$

where

$$\varphi(q) = \sum_{k=-\infty}^{\infty} q^{k^2} \text{ and } \psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2},$$

and the identity [**3**, Entry 27, Chapter 25]

$$\mathbf{z}(q) = \varphi(-q)\,\varphi(-q^3) - 4q\psi(q^2)\,\psi(q^6). \qquad \square$$

Now, following exactly the same steps as we have given in Section 3, together with Lemma 4.1 and (2.5), we deduce the following two relations which are analogues of (3.12) and (3.17):

$$n\mathbf{L}(e^{-\pi\sqrt{n/3}})-\mathbf{L}(e^{-\pi/\sqrt{3n}}) = 4\mathbf{z}_n^2\left(n(1-2\boldsymbol{\alpha}_n)+3\sqrt{n}\boldsymbol{\alpha}_n(1-\boldsymbol{\alpha}_n)\frac{d\mathbf{m}}{d\boldsymbol{\alpha}}(1-\boldsymbol{\alpha}_n,\boldsymbol{\alpha}_n)\right), \tag{4.12}$$

$$n\mathbf{L}(e^{-\pi\sqrt{n/3}})+\mathbf{L}(e^{-\pi/\sqrt{3n}})+2n\mathbf{z}_n^2 = \frac{12\sqrt{3n}}{\pi}, \tag{4.13}$$

where $\mathbf{z}_n = \mathbf{z}(e^{-\pi\sqrt{n/3}})$. Using (4.12), (3.17) and (4.7), we conclude that the following theorem holds.

THEOREM 4.2 (new class of series for $1/\pi$).

$$\frac{1}{\pi}\sqrt{\frac{3}{n}} = \sum_{k=0}^{\infty}(\mathbf{a}_n+\mathbf{b}_n k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\mathbf{H}_n^k,$$

where

$$\mathbf{a}_n = -\frac{\boldsymbol{\alpha}_n(1-\boldsymbol{\alpha}_n)}{\sqrt{n}}\frac{d\mathbf{m}}{d\boldsymbol{\alpha}}(1-\boldsymbol{\alpha}_n,\boldsymbol{\alpha}_n),$$

$$\mathbf{b}_n = 1-2\boldsymbol{\alpha}_n,$$

and

$$\mathbf{H}_n = 4\boldsymbol{\alpha}_n(1-\boldsymbol{\alpha}_n).$$

It is unlikely that Theorem 4.2 follows from the Borweins' method, the main reason being that the analogue of the cubic singular modulus here, namely, $\boldsymbol{\alpha}_n$, is a negative number for $n > 1$. This implies that the value $1-\boldsymbol{\alpha}_n$ is greater than 1, causing the series $K'(\boldsymbol{\alpha}_n) = {}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\boldsymbol{\alpha}_n)$ to diverge and hence, the analogue of the Legendre-type identity (4.1) does not exist.

5. *Proofs of* (1.14)–(1.16)

The values $\mathbf{b}_n$ and $\mathbf{H}_n$ follow easily from Ramanujan's table of λ_n [**14**, p. 212] (see [**6**] for the proofs of these values). We reproduce these values in Table 1.

Thus, in order to derive our new series, it suffices to determine the $\mathbf{a}_n$. In [**11**], we discuss a way to compute a_n given in Section 2 via 'cubic' modular equations. A similar method can be applied to give values of $\mathbf{a}_n$, using the representation in Theorem 4.2. Unfortunately, the method given in [**11**] requires a huge amount of computation. Thus, more efficient methods are needed to derive these $\mathbf{a}_n$. We now take this opportunity to discuss some alternative methods for computing $\mathbf{a}_n$. These methods will be illustrated in the proofs of (1.14), (1.15) and (1.16).

TABLE 1.

n	λ_n
9	3
17	$4+\sqrt{17}$
25	$(2+\sqrt{5})^2$
41	$32+5\sqrt{41}$
49	$55+12\sqrt{21}$
89	$500+53\sqrt{89}$

Proof of (1.14). From [**1**, p. 460, Entry 3(i)] and (3.9) with q replaced by $-q$, we have

$$3\mathbf{L}(q^3)-\mathbf{L}(q) = 2\mathbf{z}(q)^2. \tag{5.1}$$

By setting $n = 1$ in (4.10), we find that

$$\mathbf{z}(e^{-\pi/\sqrt{3}}) = 0. \tag{5.2}$$

Substituting $q = e^{-\pi/\sqrt{3}}$ in (5.1) and using (5.2), we conclude that

$$3\mathbf{L}(e^{-3\pi/\sqrt{3}}) = \mathbf{L}(e^{-\pi/\sqrt{3}}). \tag{5.3}$$

Next, set $n = 1$ in (4.13) to deduce that

$$\mathbf{L}(e^{-\pi/\sqrt{3}}) = \frac{6\sqrt{3}}{\pi}-\mathbf{z}_1^2. \tag{5.4}$$

Therefore, by (5.2) and (5.3). we deduce that

$$\mathbf{L}(e^{-\pi\sqrt{3}}) = \frac{2\sqrt{3}}{\pi}.$$

This is of course another new value for $\mathbf{L}(q)$. Finally, by (4.7), we conclude our result. □

Proof of (1.15). Now, set

$$t_k(q) = \frac{k\mathbf{L}(q^k)-\mathbf{L}(q)}{\mathbf{z}(q)\,\mathbf{z}(q^k)}$$

and for $k \equiv 2 \pmod 3$, let

$$x_k(q) = (\boldsymbol{\alpha}(q)\,\boldsymbol{\alpha}(q^k))^{1/3}.$$

It can be shown that when $q = e^{-\pi/\sqrt{3n}}$,

$$x_k(q) = (\boldsymbol{\alpha}_n(1-\boldsymbol{\alpha}_n))^{1/3} = (\mathbf{H}_n/4)^{1/3}. \tag{5.5}$$

Then from the theory of modular forms, there exists a relation between $t_k(q)$ and $x_k(q)$. When $k = 17$, we find that

$$\begin{aligned} &S^6-2^{18}\cdot 3^{12}\cdot T-2^{15}\cdot 3^{10}\cdot 37\cdot TS-2^{12}3^8\cdot 7\cdot 73\cdot T\cdot S^2\\ &-2^{10}\cdot 3^9\cdot 61\cdot T\cdot S^3-2^6 3^9\cdot 41\cdot T\cdot S^4-2^5\cdot 3^3\cdot 947\cdot TS^5\\ &+2^{18}\cdot 3^{12}\cdot 139\cdot T^2+2^{16}\cdot 3^{10}\cdot 1117\cdot T^2S+2^{12}\cdot 3^{12}\cdot 7\cdot T^2S^3\\ &+2^{12}\cdot 3^{11}\cdot 11\cdot 47\cdot T^2S^2+2^7\cdot 3^5\cdot 7951\cdot T^2\cdot S^4-2^{18}\cdot 3^{12}\cdot 5\cdot 71T^3\\ &-2^{16}\cdot 3^{13}\cdot 67\cdot T^3S-2^{12}\cdot 3^{13}\cdot 7^2\cdot T^3S^2-2^{12}\cdot 3^6\cdot 1699\cdot T^3S^3\\ &+2^{18}\cdot 3^{17}\cdot 11\cdot T^4+2^{16}\cdot 3^{15}\cdot 41\cdot T^4S+2^{12}\cdot 3^9\cdot 12107\cdot T^4S^2\\ &-2^{22}\cdot 3^{17}\cdot T^5-2^{17}\cdot 3^{11}\cdot 1187\cdot T^5S+2^{20}\cdot 3^{12}\cdot 457\cdot T^6 = 0 \end{aligned} \tag{5.6}$$

where $S = t_{17}(q)-16$ and $T = x_{17}(q)/9$. Now, since $\lambda_{17} = 4+\sqrt{17}$, by (5.5),

$$x_{17}(q) = (\boldsymbol{\alpha}_{17}(1-\boldsymbol{\alpha}_{17}))^{1/3} = -1/4.$$

Therefore, when $q = e^{-\pi/\sqrt{3\cdot 17}}$, $T = -1/36$ and substituting this value of T into (5.6), we conclude that $S = -26$. From (4.12) and (4.10), we conclude that

$$\mathbf{a}_{17} = -\frac{1}{3}\left(\frac{S+16}{4\sqrt{17}} - (1-2\boldsymbol{\alpha}_{17})\right) = \tfrac{9}{68}\sqrt{17}.$$

Together with the values $\mathbf{b}_{17} = \sqrt{17}/4$ and $\mathbf{H}_{17} = -1/16$, we complete the proof of (1.15). □

The method illustrated in the above proof works for $n = 41$ and 89. However, modular equations of degrees 41 and 89 (of polynomial degrees 14 and 30) would need to be generated respectively in order to compute the $\mathbf{a}_n$.

It is also possible to obtain the value of $t_{17}(e^{-\pi/\sqrt{3\cdot 17}})$ using the identity

$$Z = 16 + 36X + 32Y - 36\sqrt{XY} - 12\sqrt{X(5Y - 6\sqrt{XY} + 5X + 4)}, \tag{5.7}$$

where $Z = t_{17}(q)$, $X = x_{17}(q)$ and

$$Y = y_{17}(q) := \{(1-\boldsymbol{\alpha}(q))(1-\boldsymbol{\alpha}(q^{17}))\}^{1/3}.$$

Note that when $q = e^{-\pi/\sqrt{3\cdot 17}}$, $X = -1/4 = Y$ and substituting these values into (5.7), we conclude that $Z = -10$. Analogues of (5.7) exist when $n = 41$ and 89, but at present, we have not determined these identities.

We now illustrate a method used to compute $\mathbf{a}_n$ when n is a perfect square.

Proof of (1.16). It can be shown that

$$\frac{5L(q^5) - L(q)}{z(q)\,z(q^5)} = 4(1 + 2(\alpha(q)\,\alpha(q^5))^{1/3}). \tag{5.8}$$

For an elementary proof of this, see [**12**]. Now, when $q = -e^{-\pi/\sqrt{3}}$, $\mathbf{z}(-e^{\pi/\sqrt{3}}) = 0$, by (5.2). Hence,

$$5\mathbf{L}(e^{-5\pi/\sqrt{3}}) - \mathbf{L}(e^{-\pi/\sqrt{3}}) = 8\frac{c(-e^{-\pi/\sqrt{3}})}{c(-e^{-5\pi/\sqrt{3}})}\cdot \boldsymbol{\alpha}_{25}^{2/3}\cdot \mathbf{z}^2(e^{-5\pi/\sqrt{3}}), \tag{5.9}$$

where $c(q)$ satisfies the relation

$$\frac{c^3(-q)}{\mathbf{z}^3(q)} = \boldsymbol{\alpha}(q).$$

The function $c(q)$ has a product representation given by [**5**, (5.5)]

$$c(q) = 3q^{1/3}\frac{f^3(-q^3)}{f(-q)}. \tag{5.10}$$

Now, using (5.10) and (4.11), we find that

$$\frac{c(-e^{-\pi/\sqrt{3}})}{c(-e^{-5\pi/\sqrt{3}})} = \frac{1}{\sqrt{5}}e^{\pi/\sqrt{3}}\frac{f^2(e^{-\pi\sqrt{3}})}{f^2(e^{-5\pi\sqrt{3}})}\cdot e^{\sqrt{3}\pi/5}\frac{f(e^{-\pi\sqrt{3}/5})}{f(e^{-5\pi\sqrt{3}})}. \tag{5.11}$$

Now, it is known that if $S(q)$ is the Rogers–Ramanujan continued fraction

$$S(q) = \frac{q^{1/5}}{1} - \frac{q}{1} + \frac{q^2}{1} - \frac{q^3}{1} + \dots, \quad |q| < 1,$$

then

$$\frac{1}{S(q)}+1-S(q)=q^{-1/5}\frac{f(q^{1/5})}{f(q^5)},\tag{5.12}$$

and

$$\frac{1}{S^5(q)}+11-S^5(q)=\frac{f^6(q)}{qf^6(q^5)}.\tag{5.13}$$

$S(e^{-\pi\sqrt{3}})$ was evaluated by first establishing [**10**] that

$$\frac{e^{\sqrt{3}\pi/5}f(e^{-\pi\sqrt{3}/5})}{f(e^{-5\pi\sqrt{3}})}=\frac{5+\sqrt{5}}{2}.\tag{5.14}$$

Hence we find that

$$S(e^{-\pi\sqrt{3}})=\frac{-3-\sqrt{5}+\sqrt{30+6\sqrt{5}}}{4}.\tag{5.15}$$

Substituting (5.15) into (5.13), we immediately deduce that

$$e^{\pi/\sqrt{3}}\frac{f^2(e^{-\pi\sqrt{3}})}{f^2(e^{-5\pi\sqrt{3}})}=(50\sqrt{5}+125)^{1/3}.\tag{5.16}$$

Substituting (5.14) and (5.16) into (5.11) and using the value $\boldsymbol{\alpha}_{25}=\frac{1}{2}-\frac{9}{40}\sqrt{5}$, we deduce that

$$\begin{aligned}5\mathbf{L}(e^{-5\pi/\sqrt{3}})-\mathbf{L}(e^{-\pi/\sqrt{3}})&=\frac{8}{\sqrt{5}}(50\sqrt{5}+125)^{1/3}\frac{5+\sqrt{5}}{2}\left(\frac{1}{2}-\frac{9}{40}\sqrt{5}\right)^{2/3}\cdot\mathbf{z}^2(e^{-5\pi/\sqrt{3}})\\&=\sum_{k=0}^{\infty}(-5+3\sqrt{5})\,A_k\,(-\tfrac{1}{80})^k,\end{aligned}\tag{5.17}$$

by (4.4) and Lemma 2.1. Using (4.7) and (5.4), we complete our proof of (1.16). □

The proof of (1.18) is similar to that of (1.16).

We have shown here a few ways to obtain the value of $\mathbf{a}_n$ rigorously. However, these methods are difficult to apply for large integer n even if the corresponding $\mathbf{a}_n$ is relatively simple, and therefore more efficient algorithms are needed.

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