RAMANUJAN'S EXPLICIT VALUES FOR THE CLASSICAL THETA-FUNCTION

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§1. *Introduction.* Let

$$\varphi(q) = \vartheta_3(0, q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \qquad |q| < 1,$$

where φ is the notation used by Ramanujan in his notebooks [**15**], and ϑ_3 is the familiar notation of Whittaker and Watson [**20**, p. 464]. It is well known that [**1**, p. 102] (with a misprint corrected)

$$\varphi^2(q) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 \sin^2\theta}} = \frac{2}{\pi} K(k), \tag{1.1}$$

where ${}_2F_1$ denotes the ordinary or Gaussian hypergeometric function; k, $0<k<1$, is the modulus; K is the complete elliptic integral of the first kind; and

$$q = \exp(-\pi K'/K), \tag{1.2}$$

where $K' = K(k')$ and $k' = \sqrt{1-k^2}$ is the complementary modulus. Thus, an evaluation of any one of the functions φ, ${}_2F_1$, or K yields an evaluation of the other two functions. However, such evaluations may not be very explicit. For example, if $K(k)$ is known for a certain value of k, it may be difficult or impossible to explicitly determine K', and so q cannot be explicitly determined. Conversely, it may be possible to evaluate $\varphi(q)$ for a certain value of q, but it may be impossible to determine the corresponding value of k. (Recall that $k = \sqrt{1-\varphi^4(-q)/\varphi^4(q)}$ [**1**, p. 102].)

In the literature more attention has been devoted to determining ${}_2F_1$ and $K(k)$. In particular, using the Selberg–Chowla formula, I. J. Zucker [**23**] evaluated K when $K'/K = \sqrt{\lambda}$, when λ is a positive integer such that $1 \leqslant \lambda \leqslant 16$, $\lambda \neq 14$. See also papers of G. S. Joyce and Zucker [**11**], J. M. Borwein and Zucker [**7**], J. G. Huard, P. Kaplan and K. S. Williams [**10**], Kaplan and Williams [**12**], and Williams and Zhang Nan-Yue [**21**].

In his notebooks [**15**], Ramanujan recorded several values of $\varphi(q)$, some of which have never been heretofore proved in print. The purpose of this paper is to prove each of these new evaluations. We also establish some new evaluations of $\varphi(q)$ not claimed by Ramanujan. All of our proofs utilize theorems of Ramanujan, in particular, some of his modular equations and class invariants. As corollaries, we are able to obtain three new explicit determinations of ${}_2F_1$. Lastly, we explicitly determine $a(e^{-2\pi})$, where $a(q)$ is the

[MATHEMATIKA, **42** (1995), 278–294]

Borweins' cubic theta-function [**6**]; no other value of $a(q)$ had been previously determined.

In his second notebook, Ramanujan recorded the values of $\varphi(e^{-\pi})$, $\varphi(e^{-\sqrt{2}\pi})$ and $\varphi(e^{-2\pi})$ [**1**, pp. 103, 104]. In particular,

$$\varphi(e^{-\pi})=\frac{\pi^{1/4}}{\Gamma(\frac{3}{4})}. \tag{1.3}$$

These evaluations are classical and can be found in Whittaker and Watson's text [**20**, p. 525]. However, in the second notebook, Ramanujan also recorded the value of $\varphi(e^{-5\pi})$ (given in Theorem 1 below), which appears to be new. The proof that we gave in [**1**, p. 210] depends on the theta transformation formula, which does not appear to be applicable to further evaluations. Independently, and almost simultaneously, Joyce and Zucker [**11**] obtained an equivalent evaluation. Using modular equations, we shall provide a proof more natural than the two previously mentioned proofs.

At scattered places in his first notebook, Ramanujan recorded the values of $\varphi(e^{-n\pi})$ when $n=3, 9, 45$ and 7. Although the result for $n=3$ has since been proved in the literature, the remaining three values have not been previously established.

Ramanujan recorded most of his values for $\varphi(e^{-n\pi})$ in terms of $\varphi(e^{-\pi})$, but in view of (1.3), $\varphi(e^{-n\pi})$ is therefore determined explicitly.

We complete the introduction by defining a modular equation and a class invariant. Let K, K', L and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively. Suppose that, for a fixed positive integer n,

$$n\frac{K'}{K}=\frac{L'}{L}. \tag{1.4}$$

Then a relation between k and l induced by (1.4) is a *modular equation of degree n*. Following Ramanujan, set $\alpha=k^2$ and $\beta=l^2$. We say that *β has degree n over α*. The *multiplier m of degree n* is defined by

$$m=\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}=\frac{\varphi^2(q)}{\varphi^2(q^n)}, \tag{1.5}$$

by (1.1).

Define, for $|q|<1$,

$$\chi(q)=(-q;q^2)_\infty,$$

where

$$(a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

The *class invariant G_n* is defined by

$$G_n=2^{-1/4}q^{-1/24}\chi(q), \tag{1.6}$$

where $q=\exp(-\pi\sqrt{n})$. Since $\chi(q)=2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24}$ [**1**, p. 124], it follows from (1.6) that

$$G_n=\{4\alpha(1-\alpha)\}^{-1/24}. \tag{1.7}$$

If β has degree n over α, it follows from (1.7) that

$$G_{n^2}=\{4\beta(1-\beta)\}^{-1/24}, \tag{1.8}$$

where now $q=\exp(-\pi)$. From [**8**, p. 214, Theorem 10.23; p. 257, Theorem 12.17], we can deduce that G_n is algebraic for each positive integer n.

§2. *Evaluations of* $\varphi(q)$.

All page numbers below refer to Ramanujan's first notebook.

THEOREM 1.

$$\frac{\varphi(e^{-5\pi})}{\varphi(e^{-\pi})}=\frac{1}{\sqrt{5\sqrt{5}-10}}.$$

THEOREM 2 (p. 284).

$$\frac{\varphi(e^{-3\pi})}{\varphi(e^{-\pi})}=\frac{1}{\sqrt[4]{6\sqrt{3}-9}}.$$

An equivalent formulation of Theorem 2 can be found in Zucker's paper [**23**].

COROLLARY 3. *If*

$$p=\frac{\sqrt{6\sqrt{3}-9}-1}{2},$$

then

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;p^3\frac{2+p}{1+2p}\right)=\frac{\sqrt{\pi}}{\sqrt{6\sqrt{3}-9}\,\Gamma^2(\frac{3}{4})}.$$

COROLLARY 4.

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{3\sqrt{3}-5}{4}\right)=\frac{\sqrt{\pi}}{(12)^{1/8}\sqrt{\sqrt{3}-1}\,\Gamma^2(\frac{3}{4})}.$$

THEOREM 5 (p. 287).

$$\frac{\varphi(e^{-9\pi})}{\varphi(e^{-\pi})}=\frac{1+\sqrt[3]{2(\sqrt{3}+1)}}{3}. \tag{2.1}$$

COROLLARY 6. *If*

$$t=\frac{(2(\sqrt{3}-1))^{1/3}-1}{(2(\sqrt{3}-1))^{1/3}+2},$$

then

$${}_2F_1\left(\frac{1}{4},\frac{1}{4};1;64t^9\frac{1-t^3}{1+8t^3}\right)=\frac{(1+(2(\sqrt{3}+1))^{1/3})^2\sqrt{\pi}}{9\Gamma^2(\frac{3}{4})}.$$

THEOREM 7 (p. 312).

$$\frac{\varphi(e^{-45\pi})}{\varphi(e^{-\pi})}=\frac{3+\sqrt{5}+(\sqrt{3}+\sqrt{5}+(60)^{1/4})\sqrt[3]{2+\sqrt{3}}}{3\sqrt{10+10\sqrt{5}}}.$$

THEOREM 8 (p. 297).

$$\frac{\varphi^2(e^{-7\pi})}{\varphi^2(e^{-\pi})}=\frac{\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}}}{14}(28)^{1/8}. \tag{2.2}$$

We give three new evaluations, not claimed by Ramanujan, below.

THEOREM 9. *Let*

$$\begin{aligned} G&=G_{169}\\ &=\frac{1}{3}\left((\sqrt{13}+2)+\left(\frac{13+3\sqrt{13}}{2}\right)^{1/3}\right.\\ &\quad\left.\times\left\{\left(\frac{11+\sqrt{13}}{2}+3\sqrt{3}\right)^{1/3}+\left(\frac{11+\sqrt{13}}{2}-3\sqrt{3}\right)^{1/3}\right\}\right) \end{aligned} \tag{2.3}$$

and

$$a=(G-G^{-1})^3+7(G-G^{-1}). \tag{2.4}$$

Then

$$\frac{\varphi(e^{-13\pi})}{\varphi(e^{-\pi})}=\left(G^{-3}\left(\frac{a+\sqrt{a^2+52}}{2}\right)\right)^{-1/2}.$$

The value for G_{169} was first stated by Ramanujan in his paper **[14]**, **[16**, p. 28], and the first published proof was given by G. N. Watson **[17]**.

THEOREM 10.

$$\frac{\varphi(e^{-27\pi})}{\varphi(e^{-3\pi})}=\frac{1}{3}\left(1+(\sqrt{3}-1)\left(\frac{\sqrt[3]{2(\sqrt{3}+1)}+1}{\sqrt[3]{2(\sqrt{3}-1)}-1}\right)^{1/3}\right).$$

THEOREM 11.

$$\frac{\varphi(e^{-63\pi})}{\varphi(e^{-7\pi})}=\frac{1}{3}\left(1+\left(\frac{\sqrt{4+\sqrt{7}}-7^{1/4}}{2}\right)^3\sqrt{\sqrt{3}+\sqrt{7}}(2+\sqrt{3})^{1/6}\right.$$
$$\left.\times\sqrt{\frac{2+\sqrt{7}+\sqrt{7+4\sqrt{7}}}{2}}\sqrt{\frac{\sqrt{3+\sqrt{7}}+(6\sqrt{7})^{1/4}}{\sqrt{3+\sqrt{7}}-(6\sqrt{7})^{1/4}}}\right).$$

Of course, by combining Theorems 2 and 10, we obtain an evaluation for $\varphi(e^{-27\pi})$, and by combining Theorems 8 and 11, we obtain an evaluation for $\varphi(e^{-63\pi})$.

§3. *Proofs of Ramanujan's theorems.*

Proof of Theorem 1. If β has degree 5 and m is the multiplier for degree 5, then from Chapter 19 of Ramanujan's second notebook [**1**, Entry 13(xii), pp. 281, 282]

$$m=\left(\frac{\beta}{\alpha}\right)^{1/4}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \tag{3.1}$$

and

$$\frac{5}{m}=\left(\frac{\alpha}{\beta}\right)^{1/4}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/4}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}. \tag{3.2}$$

Set $\alpha=\frac{1}{2}$, so that, by (1.2), $q=e^{-\pi}$. From (3.1) we find that

$$(2\beta)^{1/4}+(2(1-\beta))^{1/4}=m+(4\beta(1-\beta))^{1/4}, \tag{3.3}$$

and, from (3.2) and (3.3), we find that

$$\begin{aligned}\frac{5}{m}&=\frac{(2(1-\beta))^{1/4}+(2\beta)^{1/4}-1}{(4\beta(1-\beta))^{1/4}}\\ &=\frac{m+(4\beta(1-\beta))^{1/4}-1}{(4\beta(1-\beta))^{1/4}}\\ &=\frac{m+G_{25}^{-6}-1}{G_{25}^{-6}},\end{aligned} \tag{3.4}$$

by (1.8), with $n=5$. From Ramanujan's paper [**14**], [**16**, p. 26] or Weber's tables [**19**, p. 722],

$$G:=G_{25}=\frac{1+\sqrt{5}}{2}. \tag{3.5}$$

Hence, from (3.4), since $G^3=2+\sqrt{5}$,

$$G^3m-\frac{5}{G^3m}=G^3-G^{-3}=4,$$

from which we deduce that $G^3m=5$, or $m=5(\sqrt{5}-2)$. Theorem 1 now follows from (1.5).

First Proof of Theorem 2. Our first proof is similar to that for Theorem 1. From Entry 5(vii) in Chapter 19 of Ramanujan's second notebook [**1**, p. 230],

$$m^2=\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} \tag{3.6}$$

and

$$\frac{9}{m^2}=\left(\frac{\alpha}{\beta}\right)^{1/2}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/2}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}, \tag{3.7}$$

where β has degree 3. Setting $\alpha=\frac{1}{2}$ in (3.6) and (3.7) and eliminating the terms $(2\beta)^{1/2}+(2(1-\beta))^{1/2}$, we deduce that

$$m^2-\frac{9}{m^2}G^{-12}=1-G^{-12}, \tag{3.8}$$

by (1.8) with $n=3$, where, from Ramanujan's paper [**14**], [**16**, p. 24], or from Weber's tables [**19**, p. 721],

$$G:=G_9=\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{1/3}. \tag{3.9}$$

Rewriting (3.8) and employing (3.9), we arrive at

$$(G^3m)^2-\frac{9}{(G^3m)^2}=G^6-G^{-6}=2\sqrt{3}.$$

Hence, $(G^3m)^2=3\sqrt{3}$, or, by (3.9), $m^2=6\sqrt{3}-9$. Appealing to (1.5), we complete the first proof of Theorem 2.

Second Proof of Theorem 2. From the Borwein brothers' book [**5**, p. 145],

$$9\frac{\varphi^4(e^{-3n\pi})}{\varphi^4(e^{-n\pi})}=1+2\sqrt{2}\frac{G^3_{9n^2}}{G^9_{n^2}}, \tag{3.10}$$

where n is any positive integer. We provide here a proof somewhat different from that in [**5**].

From [**1**, p. 347],

$$\frac{\varphi^4(q)}{\varphi^4(q^3)}=1+8q\frac{\chi^3(q)}{\chi^9(q^3)}. \tag{3.11}$$

Recall the transformation formula for φ [**1**, p. 43]. If $a, b>0$ with $ab=\pi$, then

$$\sqrt{a}\varphi(e^{-a^2})=\sqrt{b}\varphi(e^{-b^2}). \tag{3.12}$$

Using (3.11) twice, we easily find that, for Re $(z)>0$,

$$\frac{\varphi^4(e^{-\pi z})}{\varphi^4(e^{-3\pi z})}=9\frac{\varphi^4(e^{-\pi/z})}{\varphi^4(e^{-\pi/(3z)})}. \tag{3.13}$$

Recall also the transformation formula for χ [**1**, p. 43]. If $a, b>0$ with $ab=\pi^2$, then

$$e^{a/24}\chi(e^{-a})=e^{b/24}\chi(e^{-b}). \tag{3.14}$$

Hence, from (3.14), we deduce that

$$\begin{aligned} e^{-\pi z}\frac{\chi^3(e^{-\pi z})}{\chi^9(e^{-3\pi z})} &= \frac{e^{3\pi z/24}\chi^3(e^{-\pi z})}{e^{27\pi z/24}\chi^9(e^{-3\pi z})} \\ &= \frac{(e^{\pi/(24z)}\chi(e^{-\pi/z}))^3}{(e^{\pi/(24(3z))}\chi(e^{-\pi/(3z)}))^9} \\ &= \frac{\chi^3(e^{-\pi/z})}{\chi^9(e^{-\pi/(3z)})}. \end{aligned} \tag{3.15}$$

Utilizing (3.13) and (3.15) in (3.11), we deduce that

$$9\frac{\varphi^4(q_1^3)}{\varphi^4(q_1)}=1+8\frac{\chi^3(q_1^3)}{\chi^9(q_1)}, \tag{3.16}$$

where $q_1=e^{-\pi/(3z)}$. If we now set $q_1=e^{-\pi n}$ in (3.16), we deduce (3.10).

Setting $n=1$ in (3.10), employing (3.9), and noting that $G_1=1$, we find that

$$9\frac{\varphi^4(e^{-3\pi})}{\varphi^4(e^{-\pi})}=1+2\sqrt{2}\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)=3+2\sqrt{3}.$$

The desired formula now follows by elementary algebra.

Third Proof of Theorem 2. From Entry 6a of Chapter 19 in Ramanujan's second notebook [**1**, p. 238],

$$\frac{\varphi^2(q)}{\varphi^2(q^3)}=\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;p\left(\frac{2+p}{1+2p}\right)^3\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;p^3\left(\frac{2+p}{1+2p}\right)\right)}=1+2p, \tag{3.17}$$

where $0<p<1$. Set $\alpha=\frac{1}{2}$, so that, by (1.5) and (3.17),

$$p\left(\frac{2+p}{1+2p}\right)^3=\frac{1}{2}.$$

Hence,

$$p^4+2p^3+6p^2+5p-\tfrac{1}{2}=0.$$

To solve this quartic equation, we use Ferrari's method, as found, for example, in Hall and Knight's text [**9**, pp. 483, 484]. Thus, adding $(ap+b)^2$ to both sides above, writing the left side as $(p^2+p+k)^2$, and equating coefficients of like powers of p, we are led to the equations

$$\begin{aligned} 1+2k&=6+a^2,\\ k&=\tfrac{5}{2}+ab,\\ k^2&=-\tfrac{1}{2}+b^2. \end{aligned}$$

Hence,

$$(k-\tfrac{5}{2})^2=a^2b^2=(2k-5)(k^2+\tfrac{1}{2}).$$

Obviously, $k=\frac{5}{2}$ is the real root. It follows that $a=0$ and $b=\frac{3}{2}\sqrt{3}$. Hence,

$$(p^2+p+\tfrac{5}{2})^2=\tfrac{27}{4}.$$

Since $p>0$,

$$p=\frac{-1+\sqrt{6\sqrt{3}-9}}{2}. \tag{3.18}$$

Using (3.18) in (3.17), we complete the proof.

Proof of Corollary 3. From (1.1) and (1.3),

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}\right)=\frac{\sqrt{\pi}}{\Gamma^2(\frac{3}{4})}. \tag{3.19}$$

Hence, the desired result follows from (3.17) and (3.18).

Proof of Corollary 4. By a transformation of Ramanujan from his second notebook that was proved by Berndt, Bhargava and Garvan [**3**, Theorem 5.6], if

$$\alpha=p^3\frac{2+p}{1+2p} \quad \text{and} \quad \beta=\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}, \tag{3.20}$$

where $0\leqslant p<1$, then

$$(1+p+p^2)\,{}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\alpha)=\sqrt{1+2p}\,{}_2F_1(\tfrac{1}{3},\tfrac{2}{3};1;\beta). \tag{3.21}$$

If p is given by (3.18), then, by (3.20) and a straightforward calculation, we find that

$$\beta=\frac{3\sqrt{3}-5}{4}. \tag{3.22}$$

Another elementary calculation shows that

$$\frac{1+p+p^2}{\sqrt{6\sqrt{3}-9}\sqrt{1+2p}}=\frac{1}{(12)^{1/8}\sqrt{\sqrt{3}-1}}. \tag{3.23}$$

Using Corollary 3, (3.22) and (3.23) in (3.21), we readily complete the proof.

First proof of Theorem 5. If β and γ have degrees 3 and 9, respectively, over α, then from Entries 3(vi),(x) of Chapter 20 in Ramanujan's second notebook [**1**, p. 352]

$$(\alpha(1-\gamma))^{1/8}+(\gamma(1-\alpha))^{1/8}=2^{1/3}(\beta(1-\beta))^{1/24} \tag{3.24}$$

and

$$\left(\frac{\gamma}{\alpha}\right)^{1/8}+\left(\frac{1-\gamma}{1-\alpha}\right)^{1/8}-\left(\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}\right)^{1/8}=\sqrt{mm'}, \tag{3.25}$$

respectively, where m is the multiplier connecting α and β and m' is the multiplier relating β and γ. Setting $\alpha=\frac{1}{2}$ in (3.24) and (3.25), we find that, respectively,

$$(2(1-\gamma))^{1/8}+(2\gamma)^{1/8}=\sqrt{2}G_9^{-1} \tag{3.26}$$

and

$$(2\gamma)^{1/8}+(2(1-\gamma))^{1/8}-G_{81}^{-3}=\frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})}. \tag{3.27}$$

From Ramanujan's paper [**14**], [**16**, p. 27],

$$G_{81}=\left(\frac{\sqrt[3]{2(\sqrt{3}+1)}+1}{\sqrt[3]{2(\sqrt{3}-1)}-1}\right)^{1/3}, \tag{3.28}$$

which was first proved in print by Watson [**18**]. Thus, from (3.26), (3.27), (3.9) and (3.28), we conclude that

$$\begin{aligned}\frac{\varphi(e^{-\pi})}{\varphi(e^{-9\pi})}&=\sqrt{2}\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{-1/3}-\frac{\sqrt[3]{2(\sqrt{3}-1)}-1}{\sqrt[3]{2(\sqrt{3}+1)}+1}\\&=\frac{2}{2^{1/3}(\sqrt{3}+1)^{1/3}}-\frac{2-(2(\sqrt{3}+1))^{1/3}}{(2(\sqrt{3}+1))^{1/3}(\sqrt[3]{2(\sqrt{3}+1)}+1)}\\&=\frac{3}{\sqrt[3]{2(\sqrt{3}+1)}+1},\end{aligned} \tag{3.29}$$

and so the proof is complete.

Another proof can be constructed by combining (3.25) and the "reciprocal" modular equation [**1**, p. 352, Entry 3(xi)]

$$\left(\frac{\alpha}{\gamma}\right)^{1/8}+\left(\frac{1-\alpha}{1-\gamma}\right)^{1/8}-\left(\frac{\alpha(1-\alpha)}{\gamma(1-\gamma)}\right)^{1/8}=\frac{3}{\sqrt{mm'}},$$

when $\alpha=\frac{1}{2}$. However, the resulting radicals are more difficult to simplify.

Second proof of Theorem 5. From the Borweins' book [**5**, p. 145], for any positive integer n,

$$3\frac{\varphi(e^{-9n\pi})}{\varphi(e^{-n\pi})}=1+\sqrt{2}\frac{G_{9n^2}}{G_{n^2}^3}. \tag{3.30}$$

The proof of (3.30) is very similar to that for (3.10). We begin with Ramanujan's identity [**1**, p. 345, Entry 1(ii)]

$$\frac{\varphi(q^{1/3})}{\varphi(q^3)}=1+2q^{1/3}\frac{\chi(q)}{\chi^3(q^3)} \tag{3.31}$$

and set $q=e^{-\pi z}$. After applying the transformation formulas (3.12) and (3.14) to (3.31), we obtain the identity

$$3\frac{\varphi(q_1^3)}{\varphi(q_1^{1/3})}=1+2\frac{\chi(q_1)}{\chi^3(q_1^{1/3})}, \tag{3.32}$$

where $q_1=e^{-\pi/z}$. If we now set $q_1=e^{-3n\pi}$ in (3.32), we easily deduce (3.30).

Setting $n=1$ in (3.30), we deduce that

$$3\frac{\varphi(e^{-9\pi})}{\varphi(e^{-\pi})}=1+\sqrt{2}G_9=1+(2(1+\sqrt{3}))^{1/3},$$

by (3.9). This completes the second proof.

Third proof of Theorem 5. From [**1**, p. 354, eqs. (3.10), (3.11)],

$$\frac{\varphi^4(q)}{\varphi^4(q^9)}=m^2m'^2=\frac{(1+2t)^4}{1+8t^3}(1+8t^3)=(1+2t)^4. \tag{3.33}$$

Setting $q=e^{-\pi}$ and comparing (3.33) with (2.1), we see that it remains to show that

$$1+2t=\frac{3}{1+(2(\sqrt{3}+1))^{1/3}}. \tag{3.34}$$

But from (3.33) and Theorem 2, we know that

$$m^2=\frac{(1+2t)^4}{1+8t^3}=6\sqrt{3}-9.$$

After some elementary algebra, we find that

$$t^4+\left(\frac{3}{2}+\frac{1}{2}(1-\sqrt{3})^3\right)t^3+\frac{3}{2}t^2+\frac{1}{2}t+\frac{(1-\sqrt{3})^3}{16}=0.$$

It is easily checked that $t=-\frac{1}{2}$ is a root, and so upon dividing by $t+\frac{1}{2}$, we deduce that

$$t^3+(6-3\sqrt{3})t^2-\tfrac{3}{2}(1-\sqrt{3})t+\tfrac{1}{8}(1-\sqrt{3})^3=0.$$

This equation has two complex roots and a real root given by

$$t=\frac{(2(\sqrt{3}-1))^{1/3}-1}{(2(\sqrt{3}+1))^{1/3}+2}, \tag{3.35}$$

a fact easily verified via *Mathematica.* By elementary algebra, similar to that used in (3.29), it may be verified that (3.34) and (3.35) are equivalent, and so the third proof is complete.

Proof of Corollary 6. From Entry 41(iii) and (41.3) in Chapter 25 of [**2**, pp. 193, 194],

$$\begin{aligned}{}_2F_1\left(\frac{1}{4},\frac{1}{4};1;64t^9\frac{1-t^3}{1+8t^3}\right)&=\frac{1}{(1+2t)^2}\,{}_2F_1\left(\frac{1}{4},\frac{1}{4};1;4\alpha(1-\alpha)\right)\\&=\frac{1}{(1+2t)^2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right).\end{aligned}$$

Set $\alpha=\frac{1}{2}$ and use (3.19), (3.34) and (3.35) to complete the proof.

Proof of Theorem 7. Setting $n=5$ in (3.30), we find that

$$3\,\frac{\varphi(e^{-45\pi})}{\varphi(e^{-5\pi})}=1+\sqrt{2}\,\frac{G_{225}}{G_{25}^3}. \tag{3.36}$$

Now from [**14**], [**16**, p. 28],

$$G_{225}=\left(\frac{1+\sqrt{5}}{4}\right)(2+\sqrt{3})^{1/3}\left(\sqrt{4+\sqrt{15}}+(15)^{1/4}\right), \tag{3.37}$$

which was first proved in print by Watson [**17**]. Hence, by (3.36), (3.37), (3.5) and Theorem 1,

$$\begin{aligned}\frac{\varphi(e^{-45\pi})}{\varphi(e^{-\pi})}&=\frac{\varphi(e^{-45\pi})}{\varphi(e^{-5\pi})}\,\frac{\varphi(e^{-5\pi})}{\varphi(e^{-\pi})}\\&=\frac{1}{3\sqrt{5\sqrt{5}-10}}\left(\sqrt{2}\,\frac{G_{225}}{G_{25}^3}+1\right)\\&=\frac{1}{3\sqrt{5\sqrt{5}-10}}\left(\sqrt{2}\left(\frac{1+\sqrt{5}}{4}\right)\left(\frac{\sqrt{5}-1}{2}\right)^3\right.\\&\qquad\left.\times(2+\sqrt{3})^{1/3}\left(\sqrt{4+\sqrt{15}}+(15)^{1/4}\right)+1\right)\\&=\frac{1}{3(3+\sqrt{5})\sqrt{5\sqrt{5}-10}}\\&\qquad\times\left(\sqrt{2}(2+\sqrt{3})^{1/3}\left(\sqrt{4+\sqrt{15}}+(15)^{1/4}\right)+3+\sqrt{5}\right).\end{aligned} \tag{3.38}$$

Now, if $a^2-qb^2=d^2$, then [13, p. 53]

$$\sqrt{a+b\sqrt{q}}=\sqrt{\frac{a+d}{2}}+(\operatorname{sgn} b)\sqrt{\frac{a-d}{2}}, \tag{3.39}$$

where we have corrected a misprint in [13]. Thus, by (3.39),

$$\sqrt{4+\sqrt{15}}=\sqrt{\tfrac{5}{2}}+\sqrt{\tfrac{3}{2}}.$$

Using this in (3.38), we find that

$$\frac{\varphi(e^{-45\pi})}{\varphi(e^{-\pi})}=\frac{1}{3\sqrt{10\sqrt{5}+10}}((2+\sqrt{3})^{1/3}(\sqrt{5}+\sqrt{3}+(60)^{1/4})+3+\sqrt{5}),$$

and so the proof of Theorem 7 is complete.

Proof of Theorem 8. From Entry 19(v) of Chapter 19 in Ramanujan's second notebook [1, p. 314],

$$m^2=\left(\frac{\beta}{\alpha}\right)^{1/2}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/2}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2}-8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3} \tag{3.40}$$

and

$$\frac{49}{m^2}=\left(\frac{\alpha}{\beta}\right)^{1/2}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/2}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}-8\left(\frac{\alpha(1-a)}{\beta(1-\beta)}\right)^{1/3}, \tag{3.41}$$

where β has degree 7 over α. From Ramanujan's paper [14], [16, p. 26] or Weber's tables [19, p. 723],

$$G:=G_{49}=\frac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}. \tag{3.42}$$

Putting $\alpha=\frac{1}{2}$ in (3.40) and (3.41) and combining them, we find that, by (3.42),

$$\begin{aligned}(G^3m)^2-\frac{49}{(G^3m)^2}&=(G^6-G^{-6})+8(G^2-G^{-2})\\&=(G^2-G^{-2})^3+11(G^2-G^{-2})\\&=(G^2-G^{-2})\{(G^2-G^{-2})^2+11\}\\&=\sqrt{4\sqrt{7}+7}\{4\sqrt{7}+18\}.\end{aligned} \tag{3.43}$$

Solving (3.43), we find that

$$\frac{\varphi^4(e^{-\pi})}{\varphi^4(e^{-7\pi})}=m^2=G^{-6}((2\sqrt{7}+9)\sqrt{4\sqrt{7}+7}+\sqrt{1820+688\sqrt{7}}). \tag{3.44}$$

Comparing (3.44) and (2.2) and using (3.42), we see that we must prove that

$$(28)^{1/4}(\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}})^2((2\sqrt{7}+9)\sqrt{4\sqrt{7}+7}+\sqrt{1820+688\sqrt{7}})=(14)^2\left(\frac{7^{1/4}+\sqrt{4+\sqrt{7}}}{2}\right)^6. \tag{3.45}$$

Let A denote the left side of (3.45). Now

$$(\sqrt{13+\sqrt{7}}+\sqrt{7+3\sqrt{7}})^2=20+4\sqrt{7}+2\sqrt{2}\sqrt{56+23\sqrt{7}}$$

and

$$\sqrt{4\sqrt{7}+7}=7^{1/4}\sqrt{4+\sqrt{7}}.$$

Using these calculations in (3.45) and then multiplying out, we deduce that

$$\begin{aligned}A=&4\sqrt{2}(133+59\sqrt{7})\sqrt{4+\sqrt{7}}+4(14+9\sqrt{7})\sqrt{4+\sqrt{7}}\sqrt{56+23\sqrt{7}}\\&+2\sqrt{2}\,.\,7^{1/4}(20+4\sqrt{7})\sqrt{455+172\sqrt{7}}\\&+8\,.\,7^{1/4}\sqrt{53172+20097\sqrt{7}}.\end{aligned}\tag{3.46}$$

Next, observe that

$$56+23\sqrt{7}=\sqrt{7}(4+\sqrt{7})^2$$

and

$$53172+20097\sqrt{7}=63(844+319\sqrt{7})=63(4+\sqrt{7})(127+48\sqrt{7}).$$

Thus,

$$\begin{aligned}A=&4\sqrt{2}(133+59\sqrt{7})\sqrt{4+\sqrt{7}}+4\,.\,7^{1/4}(14+9\sqrt{7})(4+\sqrt{7})^{3/2}\\&+2\sqrt{2}\,.\,7^{1/4}(20+4\sqrt{7})\sqrt{455+172\sqrt{7}}\\&+24\,.\,7^{3/4}\sqrt{4+\sqrt{7}}\sqrt{127+48\sqrt{7}}.\end{aligned}\tag{3.47}$$

Now $16-7\,.\,1^2=9$ and $127^2-7\,.\,48^2=1$. Thus, two applications of (3.39) yield

$$\sqrt{4+\sqrt{7}}=\sqrt{\frac{4+3}{2}}+\sqrt{\frac{4-3}{2}}=\sqrt{\frac{7}{2}}+\sqrt{\frac{1}{2}}\tag{3.48}$$

and

$$\sqrt{127+48\sqrt{7}}=\sqrt{\frac{127+1}{2}}+\sqrt{\frac{127-1}{2}}=8+3\sqrt{7}.\tag{3.49}$$

To denest $\sqrt{455+172\sqrt{7}}$ we need the following denesting theorem **[22]**. If $b^2q^2-qa^2=d^2$, then

$$\sqrt{a+b\sqrt{q}}=\frac{1}{4q(bq+d)}\sqrt[4]{(4q(bq+d)^2)^3}+\frac{a}{2(bq+d)}\sqrt[4]{4q(bq+d)^2}.\tag{3.50}$$

(Unfortunately, three mistakes in the proof led to three errors in the final formula in **[22**, p. 203]. The same incorrect formula is repeated in **[13**, p. 53].)

Observe that $172^2\,.\,7^2-7\,.\,455^2=21^2$. Thus, applying (3.50), we find that, after much simplification

$$\sqrt{455+172\sqrt{7}}=\frac{5\,.\,7^{3/4}}{\sqrt{2}}+\frac{91}{7^{3/4}\sqrt{2}}.\tag{3.51}$$

Thus, using (3.48), (3.49) and (3.51), we deduce from (3.47) that

$$A=4\sqrt{2}(133+59\sqrt{7})\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{1}{2}}\right)+4\,.\,7^{1/4}(14+9\sqrt{7})(4+\sqrt{7})\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{1}{2}}\right)$$
$$+2\,.\,7^{1/4}(20+4\sqrt{7})(5\,.\,7^{3/4}+91\,.\,7^{-3/4})$$
$$+24\,.\,7^{3/4}\left(\sqrt{\frac{7}{2}}+\sqrt{\frac{1}{2}}\right)(8+3\sqrt{7}). \tag{3.52}$$

Also, by (3.48), if B denotes the right side of (3.45), then

$$B=(14)^2\left(\frac{7^{1/4}+\sqrt{\frac{7}{2}}+\sqrt{\frac{1}{2}}}{2}\right)^6. \tag{3.53}$$

With both sides of (3.45) now denested in (3.52) and (3.53), respectively, it is now a simple matter to use computer algebra to show that $A=B$.

Clearly, Ramanujan had found a more transparent proof of Theorem 8.

Proof of Theorem 9. From Entries 8(iii),(iv) in Chapter 20 of Ramanujan's second notebook [**1**, p. 376],

$$m=\left(\frac{\beta}{\alpha}\right)^{1/4}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/4}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}-4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6} \tag{3.54}$$

and

$$\frac{13}{m}=\left(\frac{\alpha}{\beta}\right)^{1/4}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/4}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}-4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}, \tag{3.55}$$

where β has degree 13. Let $\alpha=\frac{1}{2}$. Eliminating $(2\beta)^{1/4}+(2(1-\beta))^{1/4}$ from (3.54) and (3.55), we find that

$$m-\frac{13}{m}G^{-6}=1-G^{-6}+4(G^{-2}-G^{-4}),$$

where G is given by (2.3). Hence,

$$G^3m-\frac{13}{G^3m}=G^3-G^{-3}+4(G-G^{-1})$$
$$=(G-G^{-1})^3+7(G-G^{-1})=a,$$

by (2.4). Solving for G^3m, we deduce that

$$G^3m=\frac{a+\sqrt{a^2+52}}{2},$$

which, by (1.5), yields the required result.

Proofs of Theorems 10 *and* 11. Let $n=3$ in (3.30) and use (3.9) and (3.28) to deduce Theorem 10. To prove Theorem 11, let $n=7$ in (3.30) and use (3.42) and the value

$$G_{441}=\sqrt{\frac{\sqrt{3}+\sqrt{7}}{2}}\,(2+\sqrt{3})^{1/6}\sqrt{\frac{2+\sqrt{7}+\sqrt{7+4\sqrt{7}}}{2}}\sqrt{\frac{\sqrt{3+\sqrt{7}}+(6\sqrt{7})^{1/4}}{\sqrt{3+\sqrt{7}}-(6\sqrt{7})^{1/4}}}.$$

The value of G_{441} can be found in Ramanujan's paper **[14]**, [**16**, p. 29], and the first published proof is by the present two authors and L.-C. Zhang **[4]**.

§4. *A value for the Borweins' cubic theta-function.* Let **[6]**

$$a(q)=\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \qquad |q|<1.$$

Clearly, $a(q)$ is an analogue of $\varphi^2(q)$.

THEOREM 12.

$$\frac{a(e^{-2\pi})}{\varphi^2(e^{-\pi})}=\frac{1}{(12)^{1/8}\sqrt{\sqrt{3}-1}}.$$

Proof. From [**3**, Theorem 2.12], if

$$q=\exp\left(-\frac{2\pi}{\sqrt{3}}\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-x)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;x)}\right), \qquad |x|<1, \tag{4.1}$$

then

$$a(q)={}_2F_1(\tfrac{1}{3},\tfrac{2}{3};1;x). \tag{4.2}$$

We also need Ramanujan's cubic transformation [**3**, Corollary 2.4],

$${}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\left(\frac{1-x}{1+2x}\right)^3\right)=(1+2x)\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x^3\right). \tag{4.3}$$

Now let $x=(\sqrt{3}-1)/2$. Then a simple calculation shows that $(1-x)/(1+2x)=(\sqrt{3}-1)/2$. Using these values in (4.3), we find that

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\left(\frac{\sqrt{3}-1}{2}\right)^3\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\left(\frac{\sqrt{3}-1}{2}\right)^3\right)}=\sqrt{3}. \tag{4.4}$$

Substituting (4.4) into (4.1), we see that $q=e^{-2\pi}$. Using this value of q in (4.2) and noting that $x^3=(3\sqrt{3}-5)/4$, we then find that

$$a(e^{-2\pi})={}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\frac{3\sqrt{3}-5}{4}\right)=\frac{\varphi^2(e^{-\pi})}{(12)^{1/8}\sqrt{\sqrt{3}-1}},$$

by Corollary 4 and (1.3). This completes the proof.

Another short proof of Theorem 12 can be effected by employing a formula of Ramanujan for $a(q^2)$ [1, p. 460] and Theorem 2.

By using (4.1), Theorem 2, and Theorem 5, we can also easily determine $a(e^{-6\pi})$, but its value is not very elegant.

§5. *Concluding remarks.* Of course, other values of $\varphi(e^{-n\pi})$ can be computed from (3.10) and (3.30), provided that the requisite invariants are known.

The following beautiful modular equation of Ramanujan could also be used to compute $\varphi(e^{-n\pi})$ for certain n [2, p. 235, Entry 67]. If

$$P=\frac{\varphi(q)}{\varphi(q^5)} \quad \text{and} \quad Q=\frac{\varphi(q^3)}{\varphi(q^{15})},$$

then

$$PQ+\frac{5}{PQ}=\left(\frac{Q}{P}\right)^2+3\frac{Q}{P}+3\frac{P}{Q}-\left(\frac{P}{Q}\right)^2.$$

Thus, if the values of three of the four theta-functions were known, the value of the fourth could be determined. Generally, however, one would need to find the appropriate root of a quartic polynomial, and this inevitably would lead to unsightly radicals.

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33E05: *SPECIAL FUNCTIONS; Other special functions; Elliptic functions and integrals.*

Received on the 5th of January, 1995.