Ramanujan's Elliptic Functions to Alternative Bases and Approximations to π

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1 Introduction

Define $(a)_0 := 1$ and, for each positive integer m,

$$(a)_m := (a)(a+1)(a+2)\cdots(a+m-1).$$

Recall that the classical Gaussian hypergeometric series ${}_2F_1$ is defined by

$${}_2F_1(a,b;c;z) := \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m}\frac{z^m}{m!}, \quad |z|<1.$$

The classical singular modulus $\alpha_r, r>0$, is the unique real number between 0 and 1 satisfying the relation

$$\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha_r\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha_r\right)} = \sqrt{r}.$$

In [4, Chapter 5], J.M. Borwein and P.B. Borwein introduce the new singular value function (of the second kind)

$$\alpha(r) := \frac{\pi}{4K^2} - \sqrt{r}\left(\frac{E}{K}-1\right),$$

where

$$K(r) = \frac{\pi}{2}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;\alpha_r\right)$$

and

$$E(r) = \frac{\pi}{2}\,{}_2F_1\left(-\tfrac{1}{2},\tfrac{1}{2};1;\alpha_r\right),$$

and show, via the Legendre relation [4, p. 24, Theorem 1.6], that [4, p. 153, (5.1.5)]

$$\alpha\left(\frac{1}{r}\right) = \frac{\sqrt{r}-\alpha(r)}{r}. \tag{1.1}$$

Using the transformation formula (1.1), together with various identities associated with the Eisenstein series and the Jacobi theta functions, the Borweins succeeded in relating $\alpha(r)$ with Ramanujan's series for $1/\pi$, one of which is [4, p. 182] given by

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3} \left(\alpha(r) - \sqrt{r}\alpha_r + k\sqrt{r}(1-2\alpha_r)\right) \{4\alpha_r(1-\alpha_r)\}^k, \quad r > 1. \tag{1.2}$$

When $r = 3$, $\alpha(3) = \frac{\sqrt{3}-1}{2}$ and $\alpha_3 = \frac{1}{2} - \frac{\sqrt{3}}{4}$ [4, p. 172], and (1.2) takes the simple form

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3} (1+6k) \left(\frac{1}{4}\right)^k.$$

Besides the application to the derivation of rapidly convergent series for $1/\pi$, the Borweins also establish several sequences which converge rapidly to $1/\pi$ using the properties of $\alpha(r)$. For example, they obtain the following iteration:

Iteration 1.1. Let $t_0 = \alpha(r)$, $s_0 = \sqrt{\alpha_r}$,

$$s_{n+1} = \frac{1-\sqrt{1-s_n^2}}{1+\sqrt{1-s_n^2}}, \quad \text{and} \quad t_{n+1} = (1+s_{n+1})^2 t_n - 2^{n+1}\sqrt{r} s_{n+1}.$$

Then t_n^{-1} converges quadratically to π.

When $r = 1$, $\alpha(1) = \frac{1}{2}$ and $\alpha_1 = \frac{1}{2}$, Iteration 1.1 takes the following form:

Iteration 1.2. Let $t_0 = \frac{1}{2}$, $s_0 = \frac{1}{\sqrt{2}}$,

$$s_{n+1} = \frac{1-\sqrt{1-s_n^2}}{1+\sqrt{1-s_n^2}}, \quad \text{and} \quad t_{n+1} = (1+s_{n+1})^2 t_n - 2^{n+1} s_{n+1}.$$

Then t_n^{-1} converges quadratically to π.

In [9], Ramanujan introduces the theory of elliptic functions to alternative bases, where the corresponding singular modulus $\alpha_{s,r}$ is defined as the real number between 0 and 1 satisfying the relation

$$\frac{{}_2F_1\left(\frac{1}{2}-s, \frac{1}{2}+s; 1; 1-\alpha_{s,r}\right)}{{}_2F_1\left(\frac{1}{2}-s, \frac{1}{2}+s; 1; \alpha_{s,r}\right)} = \sqrt{r},$$

where $s = \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{6}$. For a complete account of these alternative theories, see [2]. In their attempts to prove all series for $1/\pi$ given in [9], the Borweins define the corresponding analogues of $\alpha(r)$, namely,

$$\alpha_s(r) := \frac{\pi}{4K_s^2}\frac{\cos(\pi s)}{1+2s} - \sqrt{r}\left(\frac{E_s}{K_s} - 1\right),$$

where

$$K_s := \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2} - s, \frac{1}{2} + s; 1; \alpha_{s,r}\right)$$

and

$$E_s := \frac{\pi}{2}\,{}_2F_1\left(-\frac{1}{2} - s, \frac{1}{2} + s; 1; \alpha_{s,r}\right).$$

For $s = \frac{1}{6}$ and $\frac{1}{4}$, the Borweins supply the following respective analogues of Iteration 1.2 [5]:

Iteration 1.3 (Cubic, $s = \frac{1}{6}$). Let $t_0 = \frac{1}{3}$, $s_0 = \frac{\sqrt{3}-1}{2}$,

$$s_n = \frac{1-(1-s_{n-1}^3)^{1/3}}{1+2(1-s_{n-1}^3)^{1/3}}, \quad \text{and} \quad t_n = (1+2s_n)^2 t_{n-1} - 3^{n-1}((1+2s_n)^2-1).$$

Then t_n^{-1} converges cubically to π.

Iteration 1.4 (Quartic, $s = \frac{1}{4}$). Let $t_0 = \frac{1}{3}$, $s_0 = \frac{1}{3}$,

$$s_n = \frac{1-\sqrt{1-s_{n-1}^2}}{1+3\sqrt{1-s_{n-1}^2}}, \quad \text{and} \quad t_n = (1+3s_n)^2 t_{n-1} - 2^n s_n.$$

Then t_n^{-1} converges quadratically to π.

These iterations are consequences of the Borweins' cubic and quartic analogues of the classical Arithmetic-Geometric Mean, although their generalizations in the same spirit as that of Iteration 1.1 (which is a generalization of Iteration 1.2) can be given along the same line as illustrated in [4, p. 169], using $\alpha_s(r)$ for $s = \frac{1}{6}$ and $\frac{1}{4}$ respectively.

In [6], J.M. Borwein and F.G. Garvan provide an alternative approach to the derivations of Iterations 1.3 and 1.4. First, recall that the Dedekind eta-function $\eta(\tau)$ is defined by

$$\eta(\tau) := e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i \tau n}), \quad \operatorname{Im} \tau > 0.$$

For each integer $p > 1$, define the three functions

$$B_p(r) = \frac{\eta^p(\tau)}{\eta(p\tau)},$$
$$C_p(r) = \frac{\eta^p(p\tau)}{\eta(\tau)},$$

and

$$A_p(r) = q\left(\frac{24}{p^2-1}\right)\left\{\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right\},$$

where $\tau = i\sqrt{r/p}$ and $q = \exp(-2\pi\sqrt{r/p})$, and $\frac{\dot{f}(q)}{f(q)} := \frac{d\log f(q)}{dq}$. Borwein and Garvan construct an infinite family of functions $\boldsymbol{\alpha}_p$ defined by

$$\boldsymbol{\alpha}_p(r) := \frac{1}{A_p(r)}\left(\frac{1}{\pi} - q\frac{8\sqrt{r}}{(p-1)\sqrt{p}}\frac{\dot{B}}{B}\right),$$

and derive the following result:

Theorem 1.1. *Let $N, p \geq 1$ be fixed. Then*

$$\boldsymbol{\alpha}_p(N^2r) = \boldsymbol{\alpha}_p(r)m_{N,p}(r) + \sqrt{r}\epsilon_{N,p}(r),$$

where

$$\epsilon_{N,p} = \frac{p+1}{3\sqrt{p}}\left\{\frac{q\frac{\dot{B}}{B} - Nq^N\frac{\dot{B}}{B}(q^N)}{q^N\frac{\dot{C}}{C}(q^N) - q^N\frac{\dot{B}}{B}(q^N)}\right\}$$

and

$$m_{N,p} = \frac{A_p(r)}{A_p(N^2r)}.$$

Furthermore,

$$A_p = \frac{1}{p-1}(pP(q^p) - P(q)),$$

where

$$P(q) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k} = 24q\frac{\dot{\eta}}{\eta}.$$

The function $\epsilon_{N,p}$ is very complicated but when $N = p$, it simplifies to give

$$\epsilon_{p,p}(r) = \frac{\sqrt{p}}{3}(1 - m_{p,p}(r)),$$

so that

$$\boldsymbol{\alpha}_p(p^2r) = \boldsymbol{\alpha}_p(r)m_{p,p}(r) + \frac{\sqrt{rp}}{3}(1 - m_{p,p}(r)). \tag{1.3}$$

Using (1.3) and various modular equations, Borwein and Garvan discovered many new results analogous to Iterations 1.1–1.4. Their results for $p = 3$ and 2 coincide respectively, with the Borweins' Iterations 1.3 and 1.4.

In this paper, we construct a new class of functions $\mathcal{K}_{\Psi_s}(n)$ and establish two properties of these functions. The properties of these functions are then used to establish the following iterations:

Iteration 1.5.[1] Let $k_0 = 0$ and $s_0 = \frac{1}{\sqrt{2}}$. Set

$$s_n = \frac{1 - \sqrt{1 - s_{n-1}^2}}{1 + \sqrt{1 - s_{n-1}^2}}$$

and

$$k_n = (1 + s_n)^2 k_{n-1} + 2^n(1 - s_n)s_n.$$

Then k_n^{-1} tends to π quadratically.

Iteration 1.6. Let $k_0 = 0$ and $s_0 = \frac{1}{\sqrt[4]{2}}$. Set

$$s_n = \frac{1 - \sqrt[4]{1 - s_{n-1}^4}}{1 + \sqrt[4]{1 + s_{n-1}^4}}$$

and

$$k_n = (1 + s_n)^4 k_{n-1} + 4^n s_n \frac{1 - s_n^4}{1 + s_n}.$$

Then k_n^{-1} tends to π quartically.

Iteration 1.7. Let $k_0 = 0$ and $s_0 = \frac{1}{\sqrt{2}}$. Set

$$s_n = \frac{1 - \sqrt{1 - s_{n-1}^2}}{1 + 3\sqrt{1 - s_{n-1}^2}}$$

[1]The author wishes to thank J.M. Borwein for suggesting the use of the sequence $\{s_n\}$ which simplifies considerably his original iteration.

and

$$k_n = (1+3s_n)k_{n-1} + 3 \cdot 2^{n-1}\sqrt{2}s_n \frac{1-s_n^2}{1+3s_n}.$$

Then k_n^{-1} tends to π quadratically.

Iteration 1.8. Let $k_0 = 0$ and $s_0 = \frac{1}{\sqrt[3]{2}}$. Set

$$s_n = \frac{1-\sqrt[3]{1-s_{n-1}^3}}{1+2\sqrt[3]{1-s_{n-1}^3}}$$

and

$$k_n = (1+2s_n)^2 k_{n-1} + 8 \cdot 3^{n-2}\sqrt{3}s_n \frac{1-s_n^3}{1+2s_n}.$$

Then k_n^{-1} tends to π cubically.

2 The Definition of $\mathcal{K}_{\Psi_s}(n)$ and its Properties

Suppose $\Psi_s(q)$ is a function satisfying the relation

$$\sqrt{n}\Psi_s\left(e^{-2\pi\sqrt{n/s}}\right) = \Psi_s\left(e^{-2\pi/\sqrt{sn}}\right). \tag{2.1}$$

Let

$$\mathcal{K}_{\Psi_s}(n) = \frac{1}{\pi\Psi_s^2\left(e^{-2\pi\sqrt{n/s}}\right)} - 4\sqrt{\frac{n}{s}}\frac{\widetilde{\Psi}_s\left(e^{-2\pi\sqrt{n/s}}\right)}{\Psi_s^3\left(e^{-2\pi\sqrt{n/s}}\right)}, \tag{2.2}$$

where $\widetilde{\Psi}_s(q) = q\frac{d\Psi_s(q)}{dq}$.

By logarithmically differentiating (2.1) with respect to n, we deduce that

$$-\frac{\pi}{\sqrt{s}}\widetilde{\Psi}_s\left(e^{-2\pi\sqrt{n/s}}\right) + \frac{1}{2\sqrt{n}}\Psi_s\left(e^{-2\pi\sqrt{n/s}}\right) = \frac{\pi}{n\sqrt{ns}}\widetilde{\Psi}_s\left(e^{-2\pi/\sqrt{sn}}\right),$$

which, upon simplifying, yields

$$\begin{aligned} -2\sqrt{\frac{n}{s}}\frac{\widetilde{\Psi}_s\left(e^{-2\pi\sqrt{n/s}}\right)}{\Psi_s\left(e^{-2\pi\sqrt{n/s}}\right)} + \frac{1}{\pi} &= \frac{2}{\sqrt{s}n}\frac{\widetilde{\Psi}_s\left(e^{-2\pi/\sqrt{ns}}\right)}{\Psi_s\left(e^{-2\pi\sqrt{n/s}}\right)} \\ &= \frac{2}{\sqrt{sn}}\frac{\widetilde{\Psi}_s\left(e^{-2\pi/\sqrt{ns}}\right)}{\Psi_s\left(e^{-2\pi/\sqrt{ns}}\right)}, \end{aligned}$$

where the last equality follows from (2.1). This implies that

$$\frac{2}{\sqrt{ns}}\frac{\widetilde{\Psi}_s\left(e^{-2\pi/\sqrt{sn}}\right)}{\Psi_s\left(e^{-2\pi/\sqrt{sn}}\right)}+2\sqrt{\frac{n}{s}}\frac{\widetilde{\Psi}_s\left(e^{-2\pi\sqrt{n/s}}\right)}{\Psi_s\left(e^{-2\pi\sqrt{n/s}}\right)}=\frac{1}{\pi}. \tag{2.3}$$

Next, by (2.3) and (2.1),

$$\begin{aligned}
\mathcal{K}_{\Psi_s}\left(\frac{1}{r}\right) &= \frac{1}{\pi\Psi_s^2\left(e^{-2\pi/\sqrt{sr}}\right)}-\frac{4}{\sqrt{rs}}\frac{\widetilde{\Psi}_s\left(e^{-2\pi/\sqrt{sr}}\right)}{\Psi_s^3\left(e^{-2\pi/\sqrt{sr}}\right)} \\
&= \frac{1}{\Psi_s^2\left(e^{-2\pi/\sqrt{sr}}\right)}\left(\frac{1}{\pi}-\frac{4}{\sqrt{sr}}\frac{\widetilde{\Psi_s}\left(e^{-2\pi/\sqrt{sr}}\right)}{\Psi_s\left(e^{-2\pi/\sqrt{sr}}\right)}\right) \\
&= \frac{1}{\Psi_s^2\left(e^{-2\pi/\sqrt{sr}}\right)}\left(\frac{1}{\pi}-\left\{\frac{2}{\pi}-4\sqrt{\frac{r}{s}}\frac{\widetilde{\Psi_s}\left(e^{-2\pi\sqrt{r/s}}\right)}{\Psi_s\left(e^{-2\pi\sqrt{r/s}}\right)}\right\}\right) \\
&= -\frac{1}{r}\left(\frac{\pi}{\Psi_s^2\left(e^{-2\pi\sqrt{r/s}}\right)}-4\sqrt{\frac{r}{s}}\frac{\widetilde{\Psi_s}\left(e^{-2\pi\sqrt{r/s}}\right)}{\Psi_s^3\left(e^{-2\pi\sqrt{r/s}}\right)}\right) \\
&= -\frac{1}{r}\mathcal{K}_{\Psi_s}(r).
\end{aligned}$$

This gives our first identity, namely,

$$r\mathcal{K}_{\Psi_s}\left(\frac{1}{r}\right)+\mathcal{K}_{\Psi_s}(r)=0. \tag{2.4}$$

Next, set

$$m_{N,\Psi_s}(q)\Psi_s\left(q^N\right)=\Psi_s(q). \tag{2.5}$$

Setting $q=e^{-2\pi\sqrt{r/s}}$ and differentiating (2.5) with respect to r and simplifying, we find that

$$\begin{aligned}
&-\frac{\pi}{\sqrt{rs}}\widetilde{m}_{N,\Psi_s}\left(e^{-2\pi\sqrt{r/s}}\right)\Psi_s\left(e^{-2\pi\sqrt{N^2r/s}}\right) \\
&\qquad -\frac{\pi N}{\sqrt{rs}}m_{N,\Psi_s}\left(e^{-2\pi\sqrt{r/s}}\right)\widetilde{\Psi_s}\left(e^{-2\pi\sqrt{N^2r/s}}\right) \\
&\qquad\qquad = -\frac{\pi}{\sqrt{rs}}\widetilde{\Psi_s}\left(e^{-2\pi\sqrt{r/s}}\right),
\end{aligned} \tag{2.6}$$

where

$$\widetilde{m}_{N,\Psi_s}(q) = q\frac{dm_{N,\Psi_s}(q)}{dq}.$$

Simplifying (2.6), we deduce that

$$\widetilde{m}_{N,\Psi_s}\left(e^{-2\pi\sqrt{r/s}}\right) + Nm_{N,\Psi_s}\left(e^{-2\pi\sqrt{r/s}}\right)\frac{\widetilde{\Psi_s}}{\Psi_s}\left(e^{-2\pi\sqrt{N^2r/s}}\right) = \frac{\widetilde{\Psi_s}\left(e^{-2\pi\sqrt{r/s}}\right)}{\Psi_s\left(e^{-2\pi N\sqrt{r/s}}\right)},$$

or

$$\frac{\widetilde{m}_{N,\Psi_s}}{m_{N,\Psi_s}}\left(e^{-2\pi\sqrt{r/s}}\right) + N\frac{\widetilde{\Psi_s}}{\Psi_s}\left(e^{-2\pi\sqrt{N^2r/s}}\right) = \frac{\widetilde{\Psi_s}}{\Psi_s}\left(e^{-2\pi\sqrt{r/s}}\right), \tag{2.7}$$

by (2.5). Rewriting (2.7) in terms of $\mathcal{K}_{\Psi_s}$ and using (2.2), we deduce that

$$\begin{aligned}\frac{\widetilde{m}_{N,\Psi_s}}{m_{N,\Psi_s}}&\left(e^{-2\pi\sqrt{r/s}}\right)\\ &+ \frac{\Psi_s^2\left(e^{-2\pi\sqrt{N^2r/s}}\right)}{4}\sqrt{\frac{s}{r}}\left\{\frac{1}{\pi\Psi_s^2\left(e^{-2\pi\sqrt{N^2r/s}}\right)} - \mathcal{K}_{\Psi_s}(N^2r)\right\}\\ &= \frac{\Psi_s^2\left(e^{-2\pi\sqrt{r/s}}\right)}{4}\sqrt{\frac{s}{r}}\left\{\frac{1}{\pi\Psi_s^2\left(e^{-2\pi\sqrt{r/s}}\right)} - \mathcal{K}_{\Psi_s}(r)\right\}.\end{aligned} \tag{2.8}$$

Simplifying (2.8), we find that

$$\begin{aligned}\mathcal{K}_{\Psi_s}(N^2r) &= \frac{4}{\Psi_s^2\left(e^{-2\pi\sqrt{N^2r/s}}\right)}\sqrt{\frac{r}{s}}\frac{\widetilde{m}_{N,\Psi_s}}{m_{N,\Psi_s}}\left(e^{-2\pi\sqrt{r/s}}\right)\\ &\quad + \mathcal{K}_{\Psi_s}(r)m_{N,\Psi_s}^2\left(e^{-2\pi\sqrt{r/s}}\right)\\ &= m_{N,\Psi_s}^2(q)\left(\frac{4}{\Psi_s^2\left(e^{-2\pi\sqrt{r/s}}\right)}\sqrt{\frac{r}{s}}\frac{\widetilde{m}_{N,\Psi_s}}{m_{N,\Psi_s}}\left(e^{-2\pi\sqrt{r/s}}\right) + \mathcal{K}_{\Psi_s}(r)\right).\end{aligned} \tag{2.9}$$

We are now ready to prove Iterations 1.5–1.8. Our main idea is to express $\widetilde{m}_{N,\Psi_s}(q)$ in terms of $\Psi_s^2(q)$ and $\alpha_{\Psi_s}(q)$, where $\alpha_{\Psi_s}(q)$ is a certain modular function associated with $\Psi_s(q)$.

3 Elliptic Functions in the Classical Base and a Proof of Iteration 1.5

In this section, we first state the main results arising from the classical theory of elliptic functions. Let

$$\varphi(q) := \sum_{k=-\infty}^{\infty} q^{k^2}.$$

It is known that [1, p. 102, Corollary] the function $\varphi^2(q)$ satisfies the transformation formula

$$\sqrt{n}\varphi^2\left(e^{-\pi\sqrt{n}}\right) = \varphi^2\left(e^{-\pi/\sqrt{n}}\right).$$

This implies that we may take $s = 4$ and $\Psi_s(q) = \varphi^2(q)$ where s and $\Psi_s(q)$ are given as in (2.1). Now, let

$$\frac{1}{\alpha_{\varphi^2}(q)} := \frac{1}{16q}\left(\frac{\varphi(q)}{\psi(q^2)}\right)^4 = 1 + \frac{1}{16q}\left(\frac{f(-q)}{f(-q^4)}\right)^8, \tag{3.1}$$

where

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}$$

and

$$f(-q) = \prod_{n=1}^{\infty}(1 - q^n).$$

Note that $f(-q) = q^{-1/24}\eta(\tau)$ when $q = e^{2\pi i\tau}$ and that the last equality of (3.1) follows from the famous Jacobi identity [1, p. 40, Entry 25(vii)]

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2)$$

and the product representations of $\varphi(q)$ and $\psi(q)$ [1, p. 36, Entry 22(i), (ii)]. It is known that [1, p. 120, Entry 9(i)]

$$q\frac{d\alpha_{\varphi^2}(q)}{dq} = \varphi^4(q)\alpha_{\varphi^2}(q)\{1 - \alpha_{\varphi^2}(q)\}. \tag{3.2}$$

Replacing q by q^N in (3.2), we deduce that

$$q\frac{d\alpha_{\varphi^2}\left(q^N\right)}{dq} = N\varphi^4\left(q^N\right)\alpha_{\varphi^2}\left(q^N\right)\left\{1 - \alpha_{\varphi^2}\left(q^N\right)\right\}. \tag{3.3}$$

From (3.2) and (3.3), we find that

$$m_{N,\varphi^2}^2(q) = N\frac{\alpha_{\varphi^2}\left(q^N\right)\left\{1-\alpha_{\varphi^2}\left(q^N\right)\right\}}{\alpha_{\varphi^2}(q)\left\{1-\alpha_{\varphi^2}(q)\right\}}\frac{d\alpha_{\varphi^2}(q)}{d\alpha_{\varphi^2}\left(q^N\right)}, \tag{3.4}$$

where $m_{N,\varphi^2}(q)$ is given by (2.5) with $\Psi_s(q) = \varphi^2(q)$.

Let N be a prime. From the second equality of (3.1) and Newman's criterion [8], we find that $\alpha_{\varphi^2}(q)$ is a modular function invariant under $\Gamma_0(4)$. It follows immediately from the theory of modular forms that there is a relation between $\alpha_{\varphi^2}(q)$ and $\alpha_{\varphi^2}(q^N)$. This relation, say $F_N(\alpha_{\varphi^2}(q), \alpha_{\varphi^2}(q^N)) = 0$, is known as a modular equation of degree N associated with $\alpha_{\varphi^2}(q)$. Differentiating F_N with respect to $\alpha_{\varphi^2}(q)$, we deduce from (3.4) that $m_{N,\varphi^2}(q)$ may be expressed in terms of $\alpha_{\varphi^2}(q)$ and $\alpha_{\varphi^2}(q^N)$ and that

$$\widetilde{m}_{N,\varphi^2} = q\frac{d\alpha_{\varphi^2(q)}}{dq}\frac{dm_{N,\varphi^2}}{d\alpha_{\varphi^2(q)}} = \varphi^4(q)\alpha_{\varphi^2}(q)\{1-\alpha_{\varphi^2}(q)\}\frac{dm_{N,\varphi^2}}{d\alpha_{\varphi^2}(q)}, \tag{3.5}$$

by (3.2). Hence we may rewrite (2.9) as

$$\begin{aligned}\mathcal{K}_{\varphi^2}(N^2r) = m_{N,\varphi^2}^2&\left(\alpha_{\varphi^2}\left(e^{-\pi\sqrt{r}}\right), \alpha_{\varphi^2}\left(e^{-\pi\sqrt{N^2r}}\right)\right)\\ &\times\Big(2\sqrt{r}\alpha_{\varphi^2}(q)\{1-\alpha_{\varphi^2}(q)\}\\ &\times\frac{m'_{N,\varphi^2}}{m_{N,\varphi^2}}\left(\alpha_{\varphi^2}\left(e^{-\pi\sqrt{r}}\right), \alpha_{\varphi^2}\left(e^{-\pi\sqrt{N^2r}}\right)\right) + \mathcal{K}_{\varphi^2}(r)\Big),\end{aligned} \tag{3.6}$$

where

$$m'_{N,\varphi^2} = \frac{dm_{N,\varphi^2}}{d\alpha_{\varphi^2}(q)}.$$

We are now ready to prove Iteration 1.5. First, recall that when $N = 2$, [1, p. 213, (24.11)]

$$m_{2,\varphi^2} = \frac{2-2\sqrt{1-\alpha_{\varphi^2}(q)}}{\alpha_{\varphi^2}(q)}. \tag{3.7}$$

Hence,

$$m'_{2,\varphi^2} = \frac{1}{\sqrt{1-\alpha_{\varphi^2}(q)}}\left(\frac{\sqrt{1-\alpha_{\varphi^2}(q)}-1}{\alpha_{\varphi^2}(q)}\right)^2. \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6), we deduce that

$$\begin{aligned}\mathcal{K}_{\varphi^2}(4r) = 4&\left(\frac{1-\sqrt{1-\alpha_{\varphi^2}\left(e^{-\pi\sqrt{r}}\right)}}{\alpha_{\varphi^2}\left(e^{-\pi\sqrt{r}}\right)}\right)^2\\ &\times\left\{\sqrt{r}\sqrt{1-\alpha_{\varphi^2}(e^{-\pi\sqrt{r}})}\left(1-\sqrt{1-\alpha_{\varphi^2}(e^{-\pi\sqrt{r}})}\right) + \mathcal{K}_{\varphi^2}(r)\right\}.\end{aligned} \tag{3.9}$$

This gives the most important relation leading to Iteration 1.5.

Next, recall that [1, p. 215, (24.21)]

$$\alpha_{\varphi^2}(q^2) = \left(\frac{1 - \sqrt{1 - \alpha_{\varphi^2}(q)}}{1 + \sqrt{1 - \alpha_{\varphi^2}(q)}} \right)^2 . \tag{3.10}$$

From (3.10), we easily compute $\alpha_{\varphi^2}(e^{-\pi\sqrt{4r}})$ from $\alpha_{\varphi^2}(e^{-\pi\sqrt{r}})$, which together with $\mathcal{K}_{\varphi^2}(4r)$ obtained from (3.9), determines $\mathcal{K}_{\varphi^2}(16r)$. Repeating the process using (3.10) and (3.9) shows that $\mathcal{K}_{\varphi^2}(4^n r)$ can be explicitly determined for any positive integer n.

We are now ready to state our generalization of Iteration 1.5:

Iteration 3.1. Define $\{s_n\}$ by $s_0 = \sqrt{\alpha_{\varphi^2}(e^{-\pi\sqrt{r}})}$ and

$$s_n = \sqrt{\alpha_{\varphi^2}(e^{-\pi\sqrt{4^n r}})} = \frac{1 - \sqrt{1 - s_{n-1}^2}}{1 + \sqrt{1 - s_{n-1}^2}},$$

and the sequence $\{k_n\}$ by $k_0 = \mathcal{K}_{\varphi^2}(r)$ and

$$\begin{aligned} k_n &= \mathcal{K}_{\varphi^2}(4^n r) \\ &= (1 + s_n^2) k_{n-1} + 2^n \sqrt{r}(1 - s_n) s_n. \end{aligned}$$

Then k_n tends to $\frac{1}{\pi}$ quadratically.

Note that our sequences are obtained directly from (3.10) and (3.9). To complete the proof of Iteration 3.1, it suffices to show that

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} \mathcal{K}_{\varphi^2}(n) = \frac{1}{\pi}$$

quadratically. However, this follows from the fact that

$$0 < \frac{1}{\pi} - \mathcal{K}_{\varphi^2}(t) = \left(\frac{8}{\pi} + 4\sqrt{t} \right) e^{-\pi\sqrt{t}} + \mathrm{O}\left(e^{-2\pi\sqrt{t}} \right) \leq 8\sqrt{t} e^{-\pi\sqrt{t}},$$

which implies that

$$0 < \frac{1}{\pi} - k_n \leq 8 \cdot 2^n \sqrt{r} e^{-2^n \pi \sqrt{r}}.$$

Iteration 1.5 is Iteration 3.1 when $r = 1$. Note that when $r = 1$,

$$\mathcal{K}_{\varphi^2}(1) = 0 \tag{3.11}$$

by (2.4). Furthermore, by using the transformation formula [1, p. 43, Entry 27(iii)]

$$e^{-\pi/(12\tau)}f(-e^{-2\pi/\tau}) = \sqrt{\tau}e^{-\pi\tau/12}f(-e^{-2\pi\tau}), \tag{3.12}$$

with $\tau = 2$ and the second equality of (3.1), we deduce that

$$\alpha_{\varphi^2}(e^{-\pi}) = \frac{1}{2}. \tag{3.13}$$

This completes the proof of Iteration 1.5.

Theoretically, when p is a prime, a p^{th}-order iteration associated with $\varphi^2(q)$ may be obtained as long as we know the modular equation of order p, namely the relation between $\alpha_{\varphi^2}(q)$ and $\alpha_{\varphi^2}(q^p)$, and the expression $m_{p,\varphi^2}(q)$ in terms of $\alpha_{\varphi^2}(q)$ and $\alpha_{\varphi^2}(q^p)$. In practice we find it difficult to construct such iterations unless we are able to write $\alpha_{\varphi^2}(q^p)$ explicitly in terms of $\alpha_{\varphi^2}(q)$.

We end this section with a quartic iteration associated with $\varphi^2(q)$. Note that although 4 is not a prime, the fact that $\alpha_{\varphi^2}(q^4)$ can be expressed in terms of $\alpha_{\varphi^2}(q)$ allows us to construct such an iteration. The relations which we need in order to construct a quartic iteration are [1, p. 215, (24.22)]

$$\alpha_{\varphi^2}(q^4) = \left(\frac{1-(1-\alpha_{\varphi^2}(q))^{1/4}}{1+(1-\alpha_{\varphi^2}(q))^{1/4}}\right)^4 \tag{3.14}$$

and [1, p. 216, (24.23)]

$$m_{4,\varphi^2} = \frac{4}{\{1+(1-\alpha_{\varphi^2}(q))^{1/4}\}^2}. \tag{3.15}$$

Using (3.14) and (3.15) and following exactly the same steps illustrated in our proof of Iteration 1.5, we deduce the following generalization of Iteration 1.6:

Iteration 3.2. Let $s_0 = \sqrt[4]{\alpha_{\varphi^2}(e^{-\pi\sqrt{r}})}$ and $k_0 = \mathcal{K}_{\varphi^2}(r)$. Suppose that

$$s_n = \frac{1-\sqrt[4]{1-s_{n-1}^4}}{1+\sqrt[4]{1-s_{n-1}^4}}$$

and

$$k_n = k_n = (1+s_n)^4 k_{n-1} + 4^n\sqrt{r}s_n\frac{1-s_n^4}{1+s_n}.$$

Then k_n tends to $\frac{1}{\pi}$ quartically.

Iteration 1.6 follows easily from Iteration 3.2 by setting $r = 1$ and using (3.11) and (3.13).

4 Ramanujan's Quartic Theory of Elliptic Functions and Iteration 1.7

The function that plays the key role in Ramanujan's quartic theory of elliptic functions [3] is

$$\mathcal{A}(q) := \left(\varphi^4(q) + 16q\psi^4(q^2)\right)^{1/2}.$$

This function is the quartic analogue of $\varphi^2(q)$ defined in Section 3, and it satisfies the transformation formula [3, (4.23)]

$$\mathcal{A}\left(e^{-2\pi/\sqrt{2n}}\right) = \sqrt{n}\mathcal{A}\left(e^{-2\pi\sqrt{n/2}}\right).$$

This implies that we may take $s = 2$ and $\Psi_s(q) = \mathcal{A}(q)$ where s and $\Psi_s(q)$ are given as in (2.1). In this theory, the function that plays the role of $\alpha_{\varphi^2}(q)$ is the function defined by

$$\frac{1}{\alpha_{\mathcal{A}}(q)} := 1 + \frac{f^{24}(-q)}{64qf^{24}(-q^2)}. \tag{4.1}$$

It turns out that the functions $\mathcal{A}(q)$ and $\alpha_{\mathcal{A}}(q)$ satisfy a relation similar to that of (3.2), namely [3, (4.13)],

$$q\frac{d\alpha_{\mathcal{A}}(q)}{dq} = \mathcal{A}^2(q)\alpha_{\mathcal{A}}(q)\{1 - \alpha_{\mathcal{A}}(q)\}. \tag{4.2}$$

Using identity (4.2) and a similar argument outlined in Section 3, we deduce that

$$\widetilde{m}_{N,\mathcal{A}}(q) = \mathcal{A}^2(q)\alpha_{\mathcal{A}}(q)\{1 - \alpha_{\mathcal{A}}(q)\}\frac{dm_{N,\mathcal{A}}}{d\alpha_{\mathcal{A}}(q)}. \tag{4.3}$$

From (4.3), we may rewrite (2.9) as

$$\begin{aligned}\mathcal{K}_{\mathcal{A}}(N^2r) = m^2_{N,\mathcal{A}}&\left(\alpha_{\mathcal{A}}\left(e^{-\pi\sqrt{2r}}\right), \alpha_{\mathcal{A}}\left(e^{-\pi\sqrt{2N^2r}}\right)\right)\\ &\times \Big(2\sqrt{2r}\alpha_{\mathcal{A}}(q)\{1 - \alpha_{\mathcal{A}}(q)\}\\ &\times \frac{m'_{N,\mathcal{A}}}{m_{N,\mathcal{A}}}\left(\alpha_{\mathcal{A}}\left(e^{-\pi\sqrt{2r}}\right), \alpha_{\mathcal{A}}\left(e^{-\pi\sqrt{2N^2r}}\right)\right) + \mathcal{K}_{\mathcal{A}}(r)\Big).\end{aligned} \tag{4.4}$$

It is known that when $N = 2$ [3, (2.20)],

$$m^2_{2,\mathcal{A}} = \frac{4}{1 + 3\sqrt{1 - \alpha_{\mathcal{A}}(q)}} \tag{4.5}$$

and [3, (2.18)]

$$\alpha_{\mathcal{A}}(q^2) = \left(\frac{1 - \sqrt{1 - \alpha_{\mathcal{A}}(q)}}{1 + 3\sqrt{1 - \alpha_{\mathcal{A}}(q)}} \right)^2 . \tag{4.6}$$

Using (4.5) and (4.6) and following exactly the steps illustrated in Section 3, we deduce the following generalization of Iteration 1.7:

Iteration 4.1. Let $s_0 = \sqrt{\alpha_{\mathcal{A}}(e^{-\pi\sqrt{2r}})}$ and $k_0 = \mathcal{K}_{\mathcal{A}}(r)$. Suppose that

$$s_n = \frac{1 - \sqrt{1 - s_{n-1}^2}}{1 + 3\sqrt{1 - s_{n-1}^2}}$$

and

$$k_n = (1 + 3s_n)k_{n-1} + 3 \cdot 2^{n-1}\sqrt{2r}s_n \frac{1 - s_n^2}{1 + 3s_n}.$$

Then k_n tends to $\frac{1}{\pi}$ quadratically.

Iteration 1.7 follows immediately from Iteration 4.1 by observing that

$$\mathcal{K}_{\mathcal{A}}(1) = 0$$

by (2.4), and that

$$\alpha_{\mathcal{A}}\left(e^{-\sqrt{2}\pi}\right) = \frac{1}{2},$$

which follows from (4.1) and (3.12) with $\tau = \sqrt{2}$.

5 Ramanujan's Cubic Theory of Elliptic Functions and Iteration 1.8

The two important functions in Ramanujan's cubic theory of elliptic functions are $a(q)$ and $\alpha_a(q)$ defined by [2],[7],

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}$$

and [7, (2.7)]

$$\frac{1}{\alpha_a(q)} = 1 + \frac{f^{12}(-q)}{27qf^{12}(-q^3)}. \tag{5.1}$$

These functions satisfy the relation [7, (4.7)]

$$q\frac{d\alpha_a(q)}{dq} = a^2(q)\alpha_a(q)\{1 - \alpha_a(q)\}, \tag{5.2}$$

which is clearly an analogue of (3.2) and (4.2). Identity (5.2) then implies that $m_{N,a}(q)$ may be expressed in terms of $\alpha_a(q)$ and $\alpha_a(q^N)$ and that the relation (2.9) may be written as

$$\begin{aligned}\mathcal{K}_a(N^2 r) = m_{N,a}^2 &\left(\alpha_a\left(e^{-2\pi\sqrt{r/3}}\right), \alpha_{\mathcal{A}}\left(e^{-2\pi\sqrt{N^2 r/3}}\right)\right) \\ &\times \left(\frac{4}{\sqrt{3}}\sqrt{r}\alpha_a(q)\{1-\alpha_a(q)\} \right. \\ &\times \left. \frac{m'_{N,a}}{m_{N,a}}\left(\alpha_a\left(e^{-2\pi\sqrt{r/3}}\right), \alpha_a\left(e^{-2\pi\sqrt{N^2 r/3}}\right)\right) + \mathcal{K}_a(r)\right).\end{aligned} \tag{5.3}$$

When $N = 3$, it is known that [2, Lemma 7.4]

$$m_{3,a} = \frac{3}{1 + 2\sqrt[3]{1-\alpha_a(q)}} \tag{5.4}$$

and

$$\alpha_a(q^3) = \left(\frac{1-\sqrt[3]{1-\alpha_a(q)}}{1+2\sqrt[3]{1-\alpha_a(q)}}\right)^3. \tag{5.5}$$

Using (5.4), (5.5) and (5.3), and following the argument as in Section 3, we deduce the following iteration:

Iteration 5.1. Let $s_0 = \sqrt[3]{\alpha_a(e^{-2\pi\sqrt{r/3}})}$ and $k_0 = \mathcal{K}_a(r)$. Suppose that

$$s_n = \frac{1-\sqrt[3]{1-s_{n-1}^3}}{1+2\sqrt[3]{1-s_{n-1}^3}}$$

and

$$k_n = (1+2s_n)^2 k_{n-1} + 8\cdot 3^{n-2}\sqrt{3r}\, s_n \frac{1-s_n^3}{1+2s_n}.$$

Then k_n tends to $\frac{1}{\pi}$ cubically.

Once again, when $r = 1$, $\mathcal{K}_a(1) = 0$ by (2.4) and

$$\alpha_a\left(e^{-2\pi/\sqrt{3}}\right) = \frac{1}{2},$$

where the last equality follows from (5.1) and (3.12) with $\tau = \sqrt{3}$. Substituting these values into Iteration 5.1 yields Iteration 1.8.

6 Conclusion

Iterations 1.5–1.8 are some of the simplest iterations which we obtain using our new class of functions $\mathcal{K}_{\Psi_s}(t)$. They are clearly the analogues of the Borweins' Iterations 1.2–1.4. One common feature in our examples is that the initial values k_0 are all equal to zero and that our s_0's are relatively simple. This feature is not present in the Borweins-Garvan iterations.

Another feature of this method is that, unlike the Borwein-Garvan method described in Section 1, our method allows us to derive p-th order iterations even when $p \neq s$, where s is given as in (2.1) (see also the remarks after the proof of Iteration 1.5). This may be difficult in the Borwein-Garvan method since no simple expression is known for $\epsilon_{N,p}(r)$ when $N \neq p$.

A final feature of this method is that, with appropriate functions $\Psi_s(q)$ and $\alpha_{\Psi_s}(q)$, we could easily derive iterations from (2.9) in a uniform manner as shown in the previous sections.

An interesting future project will be to study other functions satisfying (2.1), derive their corresponding $\alpha(q)$'s, and construct new iterations to π.

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