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QUADRATIC ITERATIONS TO π ASSOCIATED WITH ELLIPTIC FUNCTIONS TO THE CUBIC AND SEPTIC BASE

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ABSTRACT. In this paper, properties of the functions $A_d(q)$, $B_d(q)$ and $C_d(q)$ are derived. Specializing at $d = 1$ and 2, we construct two new quadratic iterations to π. These are analogues of previous iterations discovered by the Borweins (1987), J. M. Borwein and F. G. Garvan (1997), and H. H. Chan (2002). Two new transformations of the hypergeometric series ${}_2F_1(1/3, 1/6; 1; z)$ are also derived.

1. INTRODUCTION

The famous Jacobi identity states that if

$$\vartheta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}, \tag{1.1}$$

$$\vartheta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{1.2}$$

and

$$\vartheta_4(q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}, \tag{1.3}$$

then [1, p. 40, Entry 25 (vii)]

$$\vartheta_3^4(q) = \vartheta_4^4(q) + \vartheta_2^4(q). \tag{1.4}$$

Around 1991, J. M. Borwein and P. B. Borwein discovered the following cubic analogue of (1.4) [6]:

$$a^3(q) = b^3(q) + c^3(q), \tag{1.5}$$

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where

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$
$$b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{n-m} q^{n^2+mn+m^2},$$

and

$$c(q) = \sum_{m,n=-\infty}^{\infty} q^{(n+1/3)^2+(n+1/3)(m+1/3)+(m+1/3)^2},$$

ω being a primitive cube root of unity. The Borwein functions $a(q), b(q)$ and $c(q)$ and the Jacobi theta functions $\vartheta_1(q), \vartheta_2(q)$ and $\vartheta_3(q)$ share many similar features. For example, if we set

$$\alpha(q) = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)} \tag{1.6}$$

and

$$x_3(q) = \frac{c^3(q)}{a^3(q)}, \tag{1.7}$$

then [1, p. 125]

$$\alpha(q^2) = \left(\frac{1-\sqrt{1-\alpha(q)}}{1+\sqrt{1-\alpha(q)}}\right)^2 \tag{1.8}$$

and [2, (3.2)]

$$x_3(q^3) = \left(\frac{1-\sqrt[3]{1-x_3(q)}}{1+2\sqrt[3]{1-x_3(q)}}\right)^3. \tag{1.9}$$

The duplication formula (1.8) for $\alpha(q)$ plays an important role in proving the following iteration:

Iteration 1.1. Let $t_0 = \frac{1}{2}$, $s_0 = \frac{1}{\sqrt{2}}$,

$$s_{n+1} = \frac{1-\sqrt{1-s_n^2}}{1+\sqrt{1-s_n^2}}, \quad \text{and} \quad t_{n+1} = (1+s_{n+1})^2 t_n - 2^{n+1} s_{n+1}.$$

Then t_n^{-1} converges quadratically to π.

In a similar way, the triplication formula (1.9) leads to

Iteration 1.2. Let $t_0 = \frac{1}{3}$, $s_0 = \frac{\sqrt{3}-1}{2}$,

$$s_n = \frac{1-(1-s_{n-1}^3)^{1/3}}{1+2(1-s_{n-1}^2)^{1/3}}, \quad \text{and} \quad t_n = (1+2s_n)^2 t_{n-1} - 3^{n-1}((1+2s_n)^2-1).$$

Then t_n^{-1} converges cubically to π.

For an excellent account and proofs of Iterations 1.1 and 1.2, see the paper by J. M. Borwein and F. G. Garvan [8].

Recently, motivated by the approach given in [8], H. H. Chan [10] devised a new method for deriving iterations to π. He succeeded in proving the following respective analogues of Iterations 1.1 and 1.2:

Iteration 1.3. Let $k_0 = 0$ and $s_0 = \frac{1}{\sqrt{2}}$. Set

$$s_n = \frac{1-\sqrt{1-s_{n-1}^2}}{1+\sqrt{1-s_{n-1}^2}} \quad \text{and} \quad k_n = (1+s_n)^2 k_{n-1} + 2^n(1-s_n)s_n.$$

Then k_n^{-1} tends to π quadratically.

Iteration 1.4. Let $k_0 = 0$ and $s_0 = \frac{1}{\sqrt[3]{2}}$. Set

$$s_n = \frac{1-\sqrt[3]{1-s_{n-1}^3}}{1+2\sqrt[3]{1-s_{n-1}^3}} \quad \text{and} \quad k_n = (1+2s_n)^2 k_{n-1} + 8\cdot 3^{n-2}\sqrt{3}s_n\frac{1-s_n^3}{1+2s_n}.$$

Then k_n^{-1} tends to π cubically.

At the end of [10], Chan compared his method with that of Borwein and Garvan. He remarked that Borwein and Garvan's method appeared to work only when $N = p$, since their function $\epsilon_{N,p}$ [8, (3.13)] simplifies only in this case. In other words, for a function of "base p" (or a function associated with the modular group $\Gamma(p)$), the only iteration they obtained was a p-th order iteration to π. Chan's method, on the other hand, yields (at least theoretically) an N-th order iteration to π for suitably chosen functions of "base p". Unfortunately, except for those when $N = p$, the iterations obtained using Chan's method for $N \neq p$ are not as simple as those of Iterations 1.1-1.4. An example of a quadratic iteration to the septic base was recently given by Song Heng Chan [13]. It therefore appears that we have exhausted all possible simple iterations analogous to Iterations 1.1-1.4 using the functions motivated by Ramanujan's theories of elliptic functions to alternative bases [2], as well as the functions obtained from [8].

In this article, we introduce three new functions:

$$A_d(q) = \sum_{m,n\in\mathbb{Z}} q^{2(m^2+mn+dn^2)}, \tag{1.10}$$

$$B_d(q) = \sum_{m,n\in\mathbb{Z}} (-1)^{m-n} q^{m^2+mn+dn^2}, \tag{1.11}$$

and

$$C_d(q) = \sum_{m,n\in\mathbb{Z}} q^{2((m+1/2)^2+(m+1/2)n+dn^2)}. \tag{1.12}$$

Note that when $d = 1$, we are working with the function $a(q^2)$, which is associated with the cubic base. Surprisingly, in this case, we are able to obtain an analogue of Iterations 1.1-1.4, namely,

Iteration 1.5. Let $k_0 = 0$ and $y_0 = 8/9$. Set

$$y_n = 2\frac{(6y_{n-1}^2 - 5y_{n-1} + \sqrt{y_{n-1}(4-y_{n-1})})}{9y_{n-1}^2 - 6y_{n-1} + 1}$$

and

$$k_n = 2^n\sqrt{3}\frac{y_{n-1}(1-y_{n-1})}{\sqrt{4-3y_{n-1}}} + (4-3y_{n-1})k_{n-1}.$$

Then k_n^{-1} tends to π quadratically.

Note that this is a quadratic iteration in the cubic base. For $d = 2$, we have the following quadratic iteration associated with the septic base:

Iteration 1.6. Let $y_0 = \frac{4}{5}, k_0 = 0$. Let

$$y_n = \frac{2y_{n-1}^2 - y_{n-1} + \sqrt{4y_{n-1} - 3y_{n-1}^2}}{1 + y_{n-1}^2}.$$

Set

$$k_n = \frac{2^n}{\sqrt{7}}\frac{y_{n-1}(1-y_{n-1})}{2-y_{n-1}}\sqrt{(y_{n-1}+1)(4-3y_{n-1})(y_{n-1}^2 - 3y_{n-1} + 4)} + (2-y_{n-1})^2 k_{n-1}.$$

Then k_n^{-1} tends to π quadratically.

In Section 2, we quote several identities satisfied by $A_d(q), B_d(q)$ and $C_d(q)$, and we prove that

$$A_d^2(q) = B_d^2(q) + C_d^2(q). \tag{1.13}$$

The special case of (1.13) when $d = 2$ is proved in [11] using the existence of a unique 7-modular lattice of dimension 4 over $\mathbb{Z}$. Note that (1.13) is an analogue of (1.4) and (1.5).

In Section 3, we set $d = 2$. Motivated by (1.6) and (1.7), we define

$$Y(q) := B_2(q)/A_2(q). \tag{1.14}$$

We then establish a duplication formula satisfied by $Y(q)$, which is given by

$$Y(q^2) = \frac{2Y^2(q) - Y(q) + \sqrt{4Y(q) - 3Y^2(q)}}{1 + Y^2(q)}. \tag{1.15}$$

We end the section by deriving Iteration 1.6.

In Section 4, we set $d = 1$ and establish the duplication formula

$$\mathcal{Y}(q^2) = 2\left(\frac{6\mathcal{Y}^2(q) - 5\mathcal{Y}(q) + \sqrt{\mathcal{Y}(q)(4-3\mathcal{Y}(q))}}{9\mathcal{Y}^2(q) - 6\mathcal{Y}(q) + 1}\right), \tag{1.16}$$

where

$$\mathcal{Y}(q) := \frac{B_1^2(q)}{A_1^2(q)}. \tag{1.17}$$

We then prove several identities which eventually lead to Iteration 1.5. We also establish two transformation formulas for the hypergeometric series ${}_2F_1(1/3, 1/6; 1; z)$. In Section 5, we give another proof of the transformation formulas established in Section 4, using one of C. W. Borchardt's mean iterations studied by the Borweins and Garvan [5], [7].

2. Identities satisfied by A_d, B_d and C_d

Let $\vartheta_i(q), i = 2, 3, 4$, be defined as in (1.1)-(1.3). We have

Lemma 2.1.

$$A_d(q) = \sum_{m,n=-\infty}^{\infty} q^{2(m^2+mn+dn^2)} = \vartheta_3(q^2)\vartheta_3(q^{8d-2}) + \vartheta_2(q^2)\vartheta_2(q^{8d-2}), \tag{2.1}$$

$$B_d(q) = \sum_{m,n=-\infty}^{\infty} (-1)^{m-n} q^{m^2+mn+dn^2} = \vartheta_4(q)\vartheta_4(q^{4d-1}), \tag{2.2}$$

and

$$\begin{aligned} C_d(q) &= \sum_{m,n=-\infty}^{\infty} q^{2((m+1/2)^2+(m+1/2)n+dn^2)} \\ &= \vartheta_2(q^2)\vartheta_3(q^{8d-2}) + \vartheta_3(q^2)\vartheta_2(q^{8d-2}). \end{aligned} \tag{2.3}$$

Proof. The identity (2.1) can be found, for example, in [12, p. 1738]. For the proofs of the subsequent identities, we will need the following simple identity, namely, for any odd integer n,

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2} = 0. \tag{2.4}$$

This identity follows immediately from [15, p. 464].

Next, note that

$$\begin{aligned} B_d(q) &= \sum_{m,n=-\infty}^{\infty} (-1)^{m-n} q^{m^2+mn+dn^2} \\ &= \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{(m+\frac{n}{2})^2+\frac{D}{4}n^2}, \quad \text{with } D = 4d-1, \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{Dn^2/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2} \\ &= \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} (-1)^n q^{Dn^2/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2} \\ &\quad + \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} (-1)^n q^{Dn^2/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2} \\ &= \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} (-1)^n q^{Dn^2/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{(m+\frac{n}{2})^2}, \quad \text{by (2.4)}, \\ &= \vartheta_4(q^D)\vartheta_4(q). \end{aligned}$$

Finally, rewrite $C_d(q)$ as

$$\begin{aligned} C_d(q) &= \sum_{m,n=-\infty}^{\infty} q^{2((m+\frac{n}{2})^2+\frac{D}{4}n^2)} \\ &= \sum_{\substack{m,n=-\infty \\ n \text{ even}}} q^{2((m+\frac{n}{2})^2+\frac{D}{4}n^2)} + \sum_{\substack{m,n=-\infty \\ n \text{ odd}}} q^{2((m+\frac{n}{2})^2+\frac{D}{4}n^2)} \\ &= \vartheta_2(q^2)\vartheta_3(q^{8d-2}) + \vartheta_3(q^2)\vartheta_2(q^{8d-2}). \end{aligned}$$

□

Our next task is to establish relations between $A_d(q), B_d(q)$, and $C_d(q)$. We have

Lemma 2.2.

(i) $A_d(q^2) = B_d(q) + C_d(q^2)$.
(ii) $A_d^2(q) = B_d^2(q) + C_d^2(q)$.

Proof. From (2.1) and (2.3), we find, after some simplification, that

$$\begin{aligned} A_d(q^2) - C_d(q^2) &= (\vartheta_3(q^4) - \vartheta_2(q^4))(\vartheta_3(q^{16d-4}) - \vartheta_2(q^{16d-4})) \\ &= \vartheta_4(q)\vartheta_4(q^{4d-1}) = B_d(q). \end{aligned}$$

This completes the proof of (i).

Next, by (2.1) and (2.3), we find that

$$A_d^2(q) - C_d^2(q) = (\vartheta_3^2(q^{8d-2}) - \vartheta_2^2(q^{8d-2}))(\vartheta_3^2(q^2) - \vartheta_2^2(q^2)). \tag{2.5}$$

On the other hand,

$$\begin{aligned} \vartheta_3^2(q^4) - \vartheta_2^2(q^4) &= (\vartheta_3(q^4) - \vartheta_2(q^4))(\vartheta_3(q^4) + \vartheta_2(q^4)) \\ &= \vartheta_3(q)\vartheta_4(q) = \vartheta_4^2(q^2), \end{aligned} \tag{2.6}$$

where the last equality follows from a well-known identity (see, for example, [1, p. 40, Entry 25(iii)]). Substituting (2.6) into (2.5) completes the proof of (ii). □

Lemma 2.3.

$$A_d(q) + B_d(q^2) = 2 \sum_{m,n=-\infty}^{\infty} q^{2(dm^2+2mn+4n^2)}. \tag{2.7}$$

Proof. We have

$$\begin{aligned} A_d(q) + B_d(q^2) &= \sum_{m,n=-\infty}^{\infty} (1+(-1)^{m-n}) q^{2(m^2+mn+dn^2)} \\ &= 2\left(\left\{ \sum_{\substack{m,n=-\infty \\ m,n \text{ both even}}}^{\infty} + \sum_{\substack{m,n=-\infty \\ m,n \text{ both odd}}}^{\infty} \right\} q^{2(m^2+mn+dn^2)}\right) \\ &= 2\left(\left\{ \sum_{n=-\infty}^{\infty} \sum_{\substack{m=-\infty \\ m \text{ even}}}^{\infty} + \sum_{\substack{m,n=-\infty \\ m,n \text{ both odd}}}^{\infty} - \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \sum_{\substack{m=-\infty \\ m \text{ even}}}^{\infty} \right\} q^{2(m^2+mn+dn^2)}\right). \end{aligned}$$

From (2.4), we deduce that

$$\begin{aligned}\left\{\sum_{\substack{m,n=-\infty\\ m,n \text{ both odd}}}^{\infty} - \sum_{\substack{n=-\infty\\ n \text{ odd}}}^{\infty}\sum_{\substack{m=-\infty\\ m \text{ even}}}^{\infty}\right\} q^{2(m^2+mn+dn^2)} &= -\sum_{\substack{n=-\infty\\ n \text{ odd}}}^{\infty}\sum_{m=-\infty}^{\infty}(-1)^m q^{2(m^2+mn+dn^2)}\\ &= -\sum_{\substack{n=-\infty\\ n \text{ odd}}}^{\infty}\sum_{m=-\infty}^{\infty}(-1)^m q^{2(m+\frac{n}{2})^2+\frac{D}{4}n^2} = 0.\end{aligned}$$

□

3. The case $d = 2$ and Iteration 1.6

We now specialize our results in the previous section to the case $d = 2$. Set $A(q) = A_2(q), B(q) = B_2(q)$ and $C(q) = C_2(q)$. We have

Corollary 3.1.

(i) $A^2(q) = B^2(q) + C^2(q)$.
(ii) $A(q) = 2A(q^2) - B(q^2)$.
(iii) $B(q) = A(q^2) - C(q^2)$.

Now, let $Y(q)$ be given as in (1.14). From Corollary 3.1(ii), we find that

$$\frac{A(q)}{A(q^2)} = 2 - Y(q^2). \tag{3.1}$$

From Corollary 3.1(iii), we deduce that

$$\begin{aligned}Y(q) &= \frac{A(q^2)}{A(q)} - \frac{C(q^2)}{A(q)}\\ &= \frac{A(q^2)}{A(q)} - \frac{\sqrt{A^2(q^2)-B^2(q^2)}}{A(q)} \quad \text{by Corollary 3.1(i)}\\ &= \frac{A(q^2)}{A(q)}\left(1-\sqrt{1-Y^2(q^2)}\right)\\ &= \frac{1}{2-Y(q^2)}\left(1-\sqrt{1-Y^2(q^2)}\right), \quad \text{by (3.1).}\end{aligned}$$

Solving for $Y(q^2)$, we obtain the duplication formula (1.15).

We are now ready to establish Iteration 1.6. It is well known that $A(q)$ satisfies the relation [14, p. 205, (5)]

$$\sqrt{n}A(e^{-\pi\sqrt{n/7}}) = A(e^{-\pi/\sqrt{7n}}).$$

Following the method illustrated in [10], define

$$\mathcal{K} := \frac{1}{\pi A^2(e^{-\pi\sqrt{n/7}})} - 2\sqrt{\frac{n}{7}}\frac{\widetilde{A}(e^{-\pi\sqrt{n/7}})}{A^3(e^{-\pi\sqrt{n/7}})},$$

where

$$\widetilde{f}(q) := q\frac{df(q)}{dq}.$$

As in [10], we find that

$$r\mathcal{K}\left(\frac{1}{r}\right) + \mathcal{K}(r) = 0 \tag{3.2}$$

and, for any positive integer N,

$$\mathcal{K}(N^2 r) = m_{_N}^2(q)\left(\frac{2}{A^2(e^{-\pi\sqrt{r/7}})}\sqrt{\frac{r}{7}}\frac{\widetilde{m_{_N}}}{m_{_N}}(e^{-\pi\sqrt{r/7}}) + \mathcal{K}(r)\right), \tag{3.3}$$

where

$$m_{_N}(q) = \frac{A(q)}{A(q^N)}.$$

Now set $N = 2$. In order to derive Iteration 1.6, we need to convert the derivative $\widetilde{m_2}$ with respect to q to a derivative with respect to $Y_2 := Y_2(q) = Y(q^2)$. As such, we set

$$F(q) := q\frac{dY_2}{dq},$$

and we find from (3.3) and (3.1) that

$$\begin{aligned}\mathcal{K}(2^2 r) = -&\frac{2}{A^2(e^{-\pi\sqrt{r/7}})}\sqrt{\frac{r}{7}}(2 - Y_2(e^{-\pi\sqrt{r/7}}))F(e^{-\pi\sqrt{r/7}}) \\ &+ (2 - Y_2(e^{-\pi\sqrt{r/7}}))^2\mathcal{K}(r).\end{aligned} \tag{3.4}$$

It remains now to evaluate $F(q)$, which we will complete in a few steps. First, we observe that

$$2\eta^3(\tau)\eta^3(7\tau) = B^2(q^2)(A(q^2) - B(q^2)), \tag{3.5}$$

where

$$\eta(\tau) = q^{1/12}\prod_{k=1}^{\infty}(1 - q^{2k}), \qquad q = e^{\pi i \tau}.$$

Identity (3.5) is obtained from the two representations of the theta series associated with the lattice A_6^2 (see [11, Section 5.1]). Dividing by $A^3(q^2)$, we find that

$$Y_2^2(1 - Y_2) = 2\frac{\eta^3(\tau)\eta^3(7\tau)}{A^3(q^2)} = 2\frac{A^3(q)}{A^3(q^2)}\frac{\eta^3(\tau)\eta^3(7\tau)}{A^3(q)}.$$

By (3.1) and the identity (see, for example, [12, Lemma 2.2])

$$\frac{A^3(q)}{\eta^3(\tau)\eta^3(7\tau)} = 49\frac{\eta^4(7\tau)}{\eta^4(\tau)} + 13 + \frac{\eta^4(\tau)}{\eta^4(7\tau)},$$

we find that

$$Y_2^2(1 - Y_2) = 2(2 - Y_2)^3\frac{\chi}{7 - \chi}, \tag{3.6}$$

where

$$\chi := \chi(q) = x(1 - x), \tag{3.7}$$

with

$$\frac{1}{x} = \frac{1}{x(q)} = 1 + \frac{1}{7}\frac{\eta^4(\tau)}{\eta^4(7\tau)}.$$

Solving for χ yields

$$\chi = 7\frac{Y_2^2(Y_2-1)}{(3Y_2-4)(Y_2^2-3Y_2+4)}. \tag{3.8}$$

Differentiating (3.8) with respect to q, we find that

$$q\frac{d\chi}{dq} = -14\frac{Y_2(5Y_2-4)(Y_2-2)^2}{(3Y_2-4)^2(Y_2^2-3Y_2+4)^2}q\frac{dY_2}{dq}. \tag{3.9}$$

On the other hand, from (3.7), we find that

$$q\frac{d\chi}{dq} = (1-2x)q\frac{dx}{dq} = 2(1-2x)A^2(q)\chi, \tag{3.10}$$

using the identity [12, (2.23)]

$$q\frac{dx}{dq} = 2A^2(q)x(1-x).$$

Since

$$x = \frac{1}{2}\left(1-\sqrt{1-4\chi}\right),$$

we deduce from (3.10) that

$$q\frac{d\chi}{dq} = 2A^2(q)\chi\sqrt{1-4\chi}. \tag{3.11}$$

Substituting (3.11) into (3.9) and simplifying using (3.8), we conclude that

$$F(q) = q\frac{dY_2}{dq} = A^2(q)\frac{Y_2(1-Y_2)}{(Y_2-2)^2}\sqrt{(Y_2+1)(4-3Y_2)(Y_2^2-3Y_2+4)}. \tag{3.12}$$

Now, substituting (3.12) into (3.4), and setting $r=1$, $k_n = \mathcal{K}(4^n)$ and $y_n = Y_2(e^{-2^{n-1}\pi/\sqrt{7}})$, we deduce that

$$k_n = \frac{2^n}{\sqrt{7}}\frac{y_{n-1}(1-y_{n-1})}{2-y_{n-1}}\sqrt{(y_{n-1}+1)(4-3y_{n-1})(y_{n-1}^2-3y_{n-1}+4)} + (2-y_{n-1})^2k_{n-1}.$$

To complete our proof, it remains to compute k_0 and y_0.

By (3.2), $k_0 = \mathcal{K}(1) = 0$. To compute y_0, note that $x(e^{-\pi/\sqrt{7}}) = 1/2$ [12, proof of (1.9)]. Hence $\chi(e^{-\pi/\sqrt{7}}) = 1/4$, and, solving the equation (3.6), we deduce that $y_0 = 4/5$.

We end this section by giving another proof of the key identity (3.5). The second proof connects our work with Ramanujan's modular equations of degree 7.

First, we observe that (3.5) is equivalent to

$$4\eta^3(\tau)\eta^3(7\tau) = B(q)(B(q)-A(q))^2, \qquad q = e^{\pi i\tau}. \tag{3.13}$$

This follows from the identity

$$2B^2(q^2)(A(q^2)-B(q^2)) = B(q)(A(q)-B(q))^2,$$

which is a consequence of the duplication formula (1.15).

By using the identities [1, p. 122, Entry 10(i), (vii), p. 123, Entry 10(iii), p. 124, Entry 12 (ii)], we may write (3.13) (with q replaced by $\sqrt{q}$) in the form

$$2\sqrt{(1-\alpha)(1-\beta)}(\alpha\beta)^{1/8} = \sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})}\left((\alpha\beta)^{1/4}+1-(1-\sqrt{\beta})(1-\sqrt{\alpha})\right)^2, \tag{3.14}$$

where $\alpha = \alpha(q)$ and $\beta = \alpha(q^7)$, $\alpha(q)$ being given as in (1.6). Now, from [1, pp. 316-317], it is known that, with $t = (\alpha\beta)^{1/8}$,

$$\sqrt{\alpha} := (1-t+t^2)(t(2-3t+2t^2))^{1/2} + (1-t)(t(1-t+2t^2)(2-t+t^2))^{1/2}$$

and

$$\sqrt{\beta} := (1-t+t^2)(t(2-3t+2t^2))^{1/2} - (1-t)(t(1-t+2t^2)(2-t+t^2))^{1/2}.$$

Note that

$$(1-\sqrt{\alpha})(1-\sqrt{\beta}) = 1 + 2(t(2-3t+2t^2))^{1/2}t - 2(t(2-3t+2t^2))^{1/2} - 2(t(2-3t+2t^2))^{1/2}t^2 + t^4)^{1/2}.$$

Hence,

$$\delta := \sqrt{(1-\sqrt{\alpha})(1-\sqrt{\beta})} = 1-t+t^2-\sqrt{t(2-3t+2t^2)}.$$

Hence the proof of (3.13) is reduced to showing the algebraic expression

$$2t(1-t)^4 = \delta(\delta-1-t^2),$$

which is straightforward.

4. Case $d = 1$ and a quadratic iteration to π

In this section, we construct an iteration to π arising from the case $d = 1$, or in other words, the cubic base. Set $\mathcal{A}(q) = A_1(q), \mathcal{B}(q) = B_1(q)$ and $\mathcal{C}(q) = C_1(q)$. From Lemma 2.2, we have

Corollary 4.1.

(i) $\mathcal{A}^2(q) = \mathcal{B}^2(q) + \mathcal{C}^2(q)$.

(ii) $\mathcal{B}(q) = \mathcal{A}(q^2) - \mathcal{C}(q^2)$.

We need a third relation which plays the role of Corollary 4.1(ii). The third relation, unlike that of the case $d = 2$, turns out to be another quadratic relation.

Lemma 4.2.

$$\mathcal{A}^2(q) = 4\mathcal{A}^2(q^2) - 3\mathcal{B}^2(q^2).$$

Proof. To prove Lemma 4.2, we first replace q^2 by $\sqrt{q}$ and observe that by [1, p. 122, Entry 10(i),(vi), p. 123, Entry 11(i), (iii)], it suffices to verify that

$$(\sqrt{(1+\sqrt{\alpha})(1+\sqrt{\gamma})} + 2\sqrt{s})^2 = 4(1+s)^2 - 3(1-\sqrt{\alpha})(1-\sqrt{\gamma}), \tag{4.1}$$

where $\alpha = \alpha(q)$, $\gamma = \alpha(q^3)$, and

$$s^4 = \alpha\gamma. \tag{4.2}$$

The definition of s is motivated by the definition of t in the previous section and the following modular equation of degree 3 [1, p. 230, Entry 5(ii)]:

$$(\alpha\gamma)^{1/4} + \{(1-\alpha)(1-\gamma)\}^{1/4} = 1. \tag{4.3}$$

The modular equation (4.3) also shows that

$$\{(1-\alpha)(1-\gamma)\} = (1-s)^4.$$

Solving for α and β yields

$$\sqrt{\alpha} = \sqrt{s}(1-s+\sqrt{s^2-s+1}), \tag{4.4}$$

and

$$\sqrt{\gamma} = \sqrt{s}(s-1+\sqrt{s^2-s+1}). \tag{4.5}$$

Using (4.4) and (4.5), we find that

$$(1+\sqrt{\alpha})(1+\sqrt{\gamma}) = (\sqrt{s}+\sqrt{s^2-s+1})^2,$$

and the left-hand side of (4.1) becomes

$$s^2+8s+1+6\sqrt{s}\sqrt{s^2-s+1}. \tag{4.6}$$

Substituting (4.4) and (4.5) into the right-hand side of (4.1), we obtain the same expression (4.6) and complete the proof of Lemma 4.2. □

Dividing the identity in Lemma 4.2 by $\mathcal{A}^2(q^2)$, we find that

$$\frac{\mathcal{A}^2(q)}{\mathcal{A}^2(q^2)} = 4 - 3\mathcal{Y}(q^2), \tag{4.7}$$

where $\mathcal{Y}(q)$ is given by (1.17). Now, substituting (1.17) into Corollary 3.1(ii), we deduce that

$$\mathcal{Y}(q) = \frac{1}{4-3\mathcal{Y}(q^2)}\left(1-\sqrt{1-\mathcal{Y}(q^2)}\right)^2,$$

and solving for $\mathcal{Y}(q^2)$ gives the duplication formula (1.16).

We are now ready to derive the quadratic iteration. Following the argument as in the previous section, we find that if

$$\mathcal{K}_{\mathcal{A}} := \frac{1}{\pi\mathcal{A}^2(e^{-\pi\sqrt{n/3}})} - 2\sqrt{\frac{n}{3}}\frac{\widetilde{\mathcal{A}}(e^{-\pi\sqrt{n/3}})}{\mathcal{A}^3(e^{-\pi\sqrt{n/3}})},$$

then

$$\mathcal{K}_{\mathcal{A}}(4r) = \mathbf{m}_2^2(e^{-\pi\sqrt{r/3}})\left(\frac{2}{\mathcal{A}^2(e^{-\pi\sqrt{r/3}})}\sqrt{\frac{r}{3}}\frac{\widetilde{\mathbf{m}_2}}{\mathbf{m}_2}(e^{-\pi\sqrt{r/3}}) + \mathcal{K}_{\mathcal{A}}(r)\right), \tag{4.8}$$

where

$$\mathbf{m}_2(q) = \frac{\mathcal{A}(q)}{\mathcal{A}(q^2)}.$$

Simplifying (4.8) using (4.7), we find that

$$\mathcal{K}_{\mathcal{A}}(4r) = -3\sqrt{\frac{r}{3}}\frac{\mathcal{F}(q)}{\mathcal{A}^2(e^{-\pi\sqrt{r/3}})} + \left(4-3\mathcal{Y}_2(e^{-\pi\sqrt{r/3}})\right)\mathcal{K}_{\mathcal{A}}(r), \tag{4.9}$$

where

$$\mathcal{Y}_2(q) := \mathcal{Y}(q^2) \quad \text{and} \quad \mathcal{F}(q) = q\frac{d\mathcal{Y}_2(q)}{dq}.$$

We first establish the identity

$$16\eta^6(\tau)\eta^6(3\tau) = \mathcal{B}^2(q)\left(\mathcal{B}^2(q)-\mathcal{A}^2(q)\right)^2. \tag{4.10}$$

Replacing q by $\sqrt{q}$, we see that (4.10) is equivalent to

$$4s(1-s)^4 = (1-\sqrt{\alpha})(1-\sqrt{\gamma})\left((1-\sqrt{\alpha})(1-\sqrt{\gamma})-(1+s)^2\right)^2, \tag{4.11}$$

where s is given by (4.2). Using (4.4) and (4.5), we show that both sides of (4.11) are equal and complete the proof.

It is known from [9, (2,1), (2.2), (2.3)] that if $x_3(q^2)$ is given by (1.7), where $q = e^{\pi i \tau}$, then

$$\boldsymbol{\chi}(q) := x_3(q^2)(1-x_3(q^2)) = 27\frac{\eta^6(\tau)\eta^6(3\tau)}{\mathcal{A}^6(q)} = \frac{27}{16}\mathcal{Y}(q)\left(\mathcal{Y}(q)-1\right)^2, \tag{4.12}$$

where the last equality follows from (4.10). Using (1.16), we find that

$$\boldsymbol{\chi}(q) = \frac{27}{4}\frac{\mathcal{Y}_2^2(q)(1-\mathcal{Y}_2(q))}{(4-3\mathcal{Y}_2(q))^3}. \tag{4.13}$$

Differentiating (4.13) with respect to q and using the fact that [9]

$$q\frac{dx_3}{dq} = 2\mathcal{A}^2(q)\boldsymbol{\chi}(q),$$

we deduce that

$$\mathcal{F}(q) = q\frac{d\mathcal{Y}_2}{dq} = \mathcal{A}^2(q)\frac{2\mathcal{Y}_2(q)(\mathcal{Y}_2(q)-1)}{\sqrt{4-3\mathcal{Y}_2(q)}}. \tag{4.14}$$

Substituting (4.14) into (4.9), and setting $r=1$, $k_n = \mathcal{K}_{\mathcal{A}}(4^n)$ and $y_n = \mathcal{Y}_2(e^{-2^{n-1}\pi/\sqrt{3}})$, we deduce that

$$k_n = 2^n\sqrt{3}\frac{y_{n-1}(1-y_{n-1})}{\sqrt{4-3y_{n-1}}} + (4-3y_{n-1})k_{n-1}.$$

Now, again $k_0 = \mathcal{K}_{\mathcal{A}}(1) = 0$. It is also known [10, Proof of Iteration 5.1] that $\boldsymbol{\chi}(e^{-\pi/\sqrt{3}}) = 1/4$. Solving for y_0 using (4.13), we deduce that $y_0 = 8/9$. Together with the duplication formula (1.16), we obtain Iteration 1.5.

We end this section with two apparently new transformation formulas for the hypergeometric series

$${}_2F_1\left(\frac{1}{3},\frac{1}{6};1;z\right) = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{3}\right)_n\left(\frac{1}{6}\right)_n}{(1)_n}\frac{z^n}{n!},$$

where $(a)_n = (a)(a+1)\cdots(a+n-1)$. First, let $0<q<1$ and recall from [2, Lemma 2.6] or [9, (4.3)], and [4, p. 178] (due to Kummer), that

$$\mathcal{A}(q) = {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x_3(q^2)\right) = {}_2F_1\left(\frac{1}{3},\frac{1}{6};1;4\boldsymbol{\chi}(q)\right). \tag{4.15}$$

From (4.12), we find that

$$\boldsymbol{\chi}(q) = \frac{27}{16}\mathbf{x}^2(q)(1-\mathbf{x}(q)), \tag{4.16}$$

where

$$\mathbf{x}(q) = 1-\mathcal{Y}(q).$$

Using the new variable $\mathbf{x}(q)$, we write (4.13) as

$$\boldsymbol{\chi}(q) = \frac{27}{4}\frac{(1-\mathbf{x}(q^2))^2\mathbf{x}(q^2)}{(1+3\mathbf{x}(q^2))^3} \tag{4.17}$$

and (4.7) as

$$\frac{\mathcal{A}(q)}{\mathcal{A}(q^2)} = \sqrt{1+3\mathbf{x}(q^2)}. \tag{4.18}$$

Now, by (4.15) and (4.16),

$$\mathcal{A}(q^2) = {}_2F_1\left(\frac{1}{3},\frac{1}{6};1;\frac{27}{4}\mathbf{x}^2(q^2)(1-\mathbf{x}(q^2))\right). \tag{4.19}$$

On the other hand,

$$\mathcal{A}(q) = {}_2F_1\left(\frac{1}{3},\frac{1}{6};1;27\frac{(1-\mathbf{x}(q^2))^2\mathbf{x}(q^2)}{(1+3\mathbf{x}(q^2))^3}\right). \tag{4.20}$$

Finally, using (4.18) and replacing $\mathbf{x}(q^2)$ by u, we deduce the transformation formula

$$ {}_2F_1\left(\frac{1}{3},\frac{1}{6};1;27\frac{(1-u)^2u}{(1+3u)^3}\right) = \sqrt{1+3u}\,{}_2F_1\left(\frac{1}{3},\frac{1}{6};1;\frac{27}{4}(1-u)u^2\right). \tag{4.21}$$

The formula (4.21) is only valid when $0<u<1/9$. Note that the formula is not valid beyond the given range, since $u=1/9$ satisfies the equation

$$27\frac{(1-u)^2u}{(1+3u)^3} = 1$$

and that ${}_2F_1(a,b;c;z)$ is divergent at $z=1$.

If we now replace u by $\dfrac{u}{4-3u}$, we immediately obtain the following companion of (4.21):

$$ {}_2F_1\left(\frac{1}{3},\frac{1}{6};1;27\frac{u^2(1-u)}{(4-3u)^3}\right) = \frac{\sqrt{4-3u}}{2}\,{}_2F_1\left(\frac{1}{3},\frac{1}{6};1;\frac{27}{4}u(1-u)^2\right). \tag{4.22}$$

This is valid for $0<u<1/3$.

5. Borchardt's iteration revisited

The last transformation formula (4.22) can also be derived from one of C. W. Borchardt's mean iteration formulas [3], considered in [5]. We now describe a second proof of (4.22) (which certainly gives a second proof of (4.21)), using an explicit description of this mean iteration.

Once again, let $0<q<1$. Recall from Lemma 4.2 that if we set $a_n := a_n(q) = \mathcal{A}^2(q^{2^n})$ and $b_n := b_n(q) = \mathcal{B}^2(q^{2^{n+1}})$, then

$$a_{n+1} = \frac{a_n+3b_n}{4}. \tag{5.1}$$

From Corollary 4.1(ii), we find that

$$\sqrt{b_n} = \sqrt{a_{n+2}} - \sqrt{a_{n+2}-b_{n+1}},$$

which yields

$$\sqrt{\frac{b_n}{b_{n+1}}} = \sqrt{\frac{a_{n+2}}{b_{n+1}}} - \sqrt{\frac{a_{n+2}}{b_{n+1}}-1}. \tag{5.2}$$

From (5.2), we deduce that

$$\sqrt{\frac{b_{n+1}}{b_n}} = \sqrt{\frac{a_{n+2}}{b_{n+1}}} + \sqrt{\frac{a_{n+2}}{b_{n+1}}-1}. \tag{5.3}$$

Adding (5.2) and (5.3) and simplifying, we deduce that

$$b_{n+1} + b_n = 2\sqrt{a_{n+2}b_n}. \tag{5.4}$$

Using (5.1) twice and squaring, we find from (5.4) that

$$b_{n+1}^2 - b_{n+1}b_n - \frac{a_nb_n - b_n^2}{4} = 0. \tag{5.5}$$

Solving (5.5) for b_{n+1} and determining the square root using the expansions of $\mathcal{A}(q)$ and $\mathcal{B}(q)$, we deduce that

$$b_{n+1} = \frac{b_n + \sqrt{a_nb_n}}{2}. \tag{5.6}$$

Note that the iteration

$$(a,b) \to \left(\frac{a+3b}{4}, \frac{b+\sqrt{ab}}{2}\right)$$

gives the iteration considered by the Borweins [5].

It is known [5, (1.3)] that the sequences $\{a_n\}$ and $\{b_n\}$ have a common limit, $\mathbf{B}(a_0,b_0)$, when $0<q<1$. This common limit is the analogue of Gauss' AGM and satisfies (by observing that $\mathbf{B}(a_0,b_0) = \mathbf{B}(a_1,b_1)$) the transformation formula [5, (1.10)]

$$\mathbf{B}(x) = \frac{1+3x}{4}\mathbf{B}\left(\frac{2(\sqrt{x}+x)}{1+3x}\right), \tag{5.7}$$

where $\mathbf{B}(x) = \mathbf{B}(1,x)$. On the other hand, from our parametrization above, we find that, for $0<q<1$,

$$\mathbf{B}(\mathcal{A}^2(q), \mathcal{B}^2(q^2)) = \mathbf{B}(\mathcal{A}^2(q^2), \mathcal{B}^2(q^4)) = \cdots = \mathbf{B}(\mathcal{A}^2(0), \mathcal{B}^2(0)) = 1.$$

Hence,

$$\frac{1}{\mathbf{B}\left(1, \frac{\mathcal{B}^2(q^2)}{\mathcal{A}^2(q)}\right)} = \mathcal{A}^2(q) = \left({}_2F_1\left(\frac{1}{3}, \frac{1}{6}; 1; 27\frac{(1-\mathbf{x}(q^2))^2\mathbf{x}(q^2)}{(1+3\mathbf{x}(q^2))^3}\right)\right)^2, \tag{5.8}$$

by (4.20). Next, note that by (4.7), we may write

$$\frac{\mathcal{B}^2(q^2)}{\mathcal{A}^2(q)} = \frac{\mathcal{B}^2(q^2)}{\mathcal{A}^2(q^2)}\frac{\mathcal{A}^2(q^2)}{\mathcal{A}^2(q)} = \frac{1-\mathbf{x}(q^2)}{1+3\mathbf{x}(q^2)}. \tag{5.9}$$

Substituting (5.9) into (5.8) and replacing $\mathbf{x}(q^2)$ by x, we deduce that, for $0<x<1/9$,

$$\frac{1}{\mathbf{B}\left(1, \frac{1-x}{1+3x}\right)} = \left({}_2F_1\left(\frac{1}{3}, \frac{1}{6}; 1; 27\frac{(1-x)^2x}{(1+3x)^3}\right)\right)^2. \tag{5.10}$$

Replacing x by $\dfrac{1-x}{1+3x}$ in (5.10), we deduce that

$$\frac{1}{\mathbf{B}(1,x)} = \left({}_2F_1\left(\frac{1}{3}, \frac{1}{6}; 1; \frac{27}{4}(1-x)x^2\right)\right)^2, \tag{5.11}$$

valid for $2/3 < x < 1$. Using (5.11) in (5.7) and taking the positive square root, we obtain

$$ {}_2F_1\left(\frac{1}{3},\frac{1}{6};1;\frac{27x(1-x)^2}{(1+3x)^3}\right) = \sqrt{\frac{1+3x}{4}}\,{}_2F_1\left(\frac{1}{3},\frac{1}{6};1;\frac{27}{4}(1-x)x^2\right), $$

valid for $2/3 < x < 1$. Replacing x by $1-x$ yields (4.22).

The Borweins and Garvan [7, Theorem 5] obtained the identity (5.11) without stating the range for which it is valid.

6. Conclusion

We have seen that functions such as $A_d(q)$, $B_d(q)$ and $C_d(q)$ possess many interesting properties. In the cases $d = 1$ and $d = 2$, we have shown that they are good candidates for furthering the study of Ramanujan's elliptic functions to alternative bases. Using other values of d, we will probably encounter new functions analogous to those studied in this article.

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