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In this paper, we use the explicit Shimura Reciprocity Law to compute the cubic singular moduli α_n^*, which are used in the constructions of new rapidly convergent series for $1/\pi$. We also complete a table of values for the class invariant λ_n initiated by S. Ramanujan on page 212 of his Lost Notebook.

1. Introduction.

In his famous paper [**26**], S. Ramanujan offers several beautiful series representations for $1/\pi$, one of which is

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} \frac{(6k+1)\left(\frac{1}{2}\right)_k^3}{(k!)^3 4^k}, \tag{1.1}$$

where $(a)_0 = 1$ and for each positive integer k,

$$(a)_k = (a)(a+1)(a+2)\dots(a+k-1).$$

Motivated by Ramanujan's series, J.M. Borwein and P.B. Borwein [**10**] obtained many general representations for $1/\pi$. One generalization of (1.1) takes the form

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \left\{ \frac{\left(\frac{1}{2}\right)_k}{k!} \right\}^3 (a_n + b_n k)(G_n^{-12})^{2k}, \tag{1.2}$$

where n is a positive integer (usually odd) and a_n, b_n and G_n are certain special values of modular forms. It turns out that these special values can be expressed in terms of *the singular modulus* α_n, which is defined to be the unique positive number between 0 and 1 satisfying

$$\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha_n)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha_n)} = \sqrt{n}, \quad n \in \mathbb{Q},$$

where

$${}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad |z| < 1.$$

In his Notebooks, S. Ramanujan recorded many values of α_n, one of which is

$$\alpha_3 = \frac{2-\sqrt{3}}{4}.$$

This value, when substituted into the Borweins' formula (1.2) yields (1.1).[1] The proofs for all the singular moduli recorded in Ramanujan's Notebooks can now be found in [**9**] and [**6**].

Recently, B.C. Berndt, S. Bhargava and F.G. Garvan [**3**] succeeded in developing theories of elliptic functions to alternative bases vaguely mentioned by Ramanujan in [**26**]. As indicated in [**3**], Ramanujan's elliptic functions to alternative base 3 turns out to be the most interesting case of his theories. For this particular base, an analogue of the singular modulus, which we shall call "*cubic singular modulus*", is defined as the unique positive number α_n^* between 0 and 1 such that

$$\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\alpha_n^*)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\alpha_n^*)} = \sqrt{n}, \quad n \in \mathbb{Q}.$$

Although Ramanujan did not record any cubic singular moduli in his Notebooks or Lost Notebook, he must have computed some of them since these values (see [**10**]) are essential in his derivations of the series [**26**]

$$\frac{27}{4\pi} = \sum_{k=0}^{\infty}(2+15k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(\frac{2}{27}\right)^k$$

and

$$\frac{15\sqrt{3}}{2\pi} = \sum_{k=0}^{\infty}(4+33k)\frac{(\frac{1}{2})_k(\frac{1}{3})_k(\frac{2}{3})_k}{(k!)^3}\left(\frac{4}{125}\right)^k.$$

The first discussion of the computations of the cubic singular moduli was given by the Borweins [**10**]. They determined α_n^* for $n = 2, 3, 4, 5$ and 6 from known values of Ramanujan-Weber class invariants G_{3n} and g_{6n}, and deduced three new series for $1/\pi$ corresponding to $n = 2, 3$ and 6. Recently, Chan and Liaw [**16**] succeeded in evaluating α_n^* for $n = 2, 5, 7, 11$, and 23 using cubic Russell-type modular equations. From the values of α_7^* and α_{11}^*, they discovered that when $3n$ is an Euler convenient number, α_n^* can be determined using Kronecker's Limit Formula. Using these new α_n^*'s, they derived many new series for $1/\pi$. Their method, however, cannot be extended to include the computations of α_n^* when $3n$ is not convenient.

In Sections 2 and 3, we use an explicit version of the Shimura Reciprocity Law to extend the list of α_n^*. We show that when the class group of $\mathbb{Q}(\sqrt{-3n})$

[1]The determination of a_n from α_n is very challenging. It involves modular equations of degrees dividing n.

takes the form $\mathbb{Z}_2^t \oplus \mathbb{Z}_k$, with $t \in \mathbb{N}$ and $k = 4, 6$ and 8, α_n^* can be determined explicitly.

On page 212 of his Lost Notebook, Ramanujan defined a certain function λ_n (see (3.1)) and recorded its values for $n = 1, 9, 17, 25, 33, 41, 49, 73, 97,$ and 121. He also indicated that he could compute λ_n when $n = 57$, 65, 81, 89, 169, 193, 217, 241, 265, 289, and 361 but did not supply any values for these n's. Using cubic Russell-type modular equations, Kronecker's Limit Formulas and other techniques, Berndt, Chan, S.-Y. Kang and L.-C. Zhang [**7**] provided proofs of all these values except for $n = 73, 97, 193, 217,$ and 241. In Section 4, we modify our method in Sections 2 and 3 and determine rigorously these remaining values of λ_n.

2. Some properties of α_n^*.

Let

$$\eta(\tau) := q^{1/24} \prod_{k=1}^{\infty} (1 - q^k), \quad \text{where} \quad q = e^{2\pi i \tau} \quad \text{with} \quad \operatorname{Im} \tau > 0, \tag{2.1}$$

and

$$\mu_n = \frac{1}{3\sqrt{3}} \left\{ \frac{\eta\left(\sqrt{-n/3}\right)}{\eta\left(\sqrt{-3n}\right)} \right\}^6, \quad n \in \mathbb{Q}. \tag{2.2}$$

The relation between μ_n and the cubic singular moduli α_n^* is given by [**17**]

$$\frac{1}{\alpha_n^*} = 1 + \mu_n^2. \tag{2.3}$$

Identity (2.3) shows that in order to determine α_n^*, it suffices to compute μ_n^2. First, we need the following:

Theorem 2.1. *Suppose that n is squarefree so that $-12n$ is a fundamental imaginary quadratic discriminant. Then μ_n^2 is a real unit contained in K_1, the Hilbert class field of $K := \mathbb{Q}(\sqrt{-3n})$.*

To prove Theorem 2.1, we need the following lemmas:

Lemma 2.2 ([**23**, p. 159, Corollary])**.** *Let K be as defined in Theorem* 2.1*, and let $\mathfrak{O}_K$ be the ring of integers of K. Let $\mathfrak{a} = [\tau_1, \tau_2]$ be an $\mathfrak{O}_K$-ideal and define*

$$\Delta(\mathfrak{a}) := \tau_2^{-12} \eta^{24}(\tau), \tag{2.4}$$

where $\tau = \tau_1/\tau_2$ with $\operatorname{Im} \tau > 0$*. Then the value $\Delta(\mathfrak{a})/\Delta(\mathfrak{O}_K)$ lies in K_1, where K_1 is the Hilbert class field of K.*

Lemma 2.3 ([**23**, p. 166, Corollary])**.** *Let* $\mathbf{N}(\mathfrak{a})$ *denote the index* $(\mathfrak{O}_K : \mathfrak{a})$ *where* $\mathfrak{a}$ *is an* $\mathfrak{O}_K$*-ideal. Then the number*

$$\mathbf{N}(\mathfrak{a})^{12}\frac{|\Delta(\mathfrak{a})|^2}{|\Delta(\mathfrak{O}_K)|^2}$$

is a unit.

Lemma 2.4. *Recall that for* $\tau \in \mathbb{C}$*, with* $\operatorname{Im}\tau > 0$*, the* j*-function is defined by*

$$j(\tau) = 1728\frac{g_2^3(\tau)}{g_2^3(\tau) - 27g_3^2(\tau)},$$

with g_2 *and* g_3 *given by*

$$g_2(\tau) = 60 \sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty} (m+n\tau)^{-4}, \quad \textit{and}$$

$$g_3(\tau) = 140 \sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{\infty} (m+n\tau)^{-6}.$$

If

$$g(\tau) := \frac{1}{3\sqrt{3}}\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^6,$$

then

$$j(\tau) = 27\frac{(1+g^2(\tau))(9+g^2(\tau))^3}{g^6(\tau)}. \tag{2.5}$$

Lemma 2.4 follows from the fact that $g^2(\tau)$ generates the function field associated with the group $\Gamma_0(3)$, which implies the $j(\tau)$ is a rational function of $g^2(\tau)$. For a more elementary proof of this lemma using Ramanujan's identities, see [**14**] and [**5**].

Proof of Theorem 2.1. Let $\mathfrak{a} = [3, \sqrt{-3n}]$ with $n \equiv 3 \pmod 4$. By (2.4),

$$\mu_n^4 = 3^{-12}\frac{\eta^{24}(\sqrt{-n/3})}{\eta^{24}(\sqrt{-3n})} = \frac{\Delta(\mathfrak{a})}{\Delta(\mathfrak{O}_K)} = \mathbf{N}(\mathfrak{a})^6\frac{|\Delta(\mathfrak{a})|}{|\Delta(\mathfrak{O}_K)|}. \tag{2.6}$$

From the second equality of (2.6) and Lemma 2.2, we find that μ_n^4 belongs to K_1 and from the last equality of (2.6) and Lemma 2.3, we conclude that μ_n^2 is a real unit. To complete the proof of Theorem 2.1, it remains to show that μ_n^2 is in K_1.

Now, when $\tau = \sqrt{-n/3}$, $g(\tau) = \mu_n$ and

$$\mu_n^8 + 270\mu_n^4 + 3^6 = ((j(\sqrt{-n/3})/27 - 28)\mu_n^4 - 972)\mu_n^2,$$

by Lemma 2.4. Since both μ_n^4 and $j(\sqrt{-n/3})$ are in K_1, we can conclude that $\mu_n^2 \in K_1$ unless $(j(\sqrt{-n/3})/27 - 28)\mu_n^4 - 972 = 0$. If this is the case, we can deduce that $j(\sqrt{-n/3})$ satisfies the quadratic equation

$$X^2 + 8208X - 5832000 = 0.$$

But the two roots of this equation has numerical values 657.8 and -8865.8. This contradicts the fact that $j(\sqrt{-m}) \geq 1728$ for any $m \geq 1$. This completes the Proof of Theorem 2.1. $\square$

A *class invariant* γ of a field K is defined to be a generator for the Hilbert class field of K, i.e., $K_1 = K(\gamma)$. Theorem 2.1, Lemma 2.4 and the fact that $j(\sqrt{-n/3})$ is a class invariant [**20**] imply that μ_n^2 is a class invariant of $\mathbb{Q}(\sqrt{-3n})$ when $n \equiv 3 \pmod 4$. Hence, we conclude that α_n^* is also a class invariant of $\mathbb{Q}(\sqrt{-3n})$ by Theorem 2.1 and (2.3).

We remark here that our result given in this section is not "optimal". We have shown that μ_n^2 is a class invariant whenever $3 \nmid n$ and n squarefree. It is possible to show further that smaller powers of the η-quotients given in the definition of μ_n^2, namely, $\frac{1}{3\sqrt{3}}\frac{\eta(\sqrt{-n/3})^s}{\eta\left(\sqrt{-3n}\right)^s}$, with $s|12$ and $s < 12$, is a class invariant if we impose further congruence conditions on n. This can be established using Gee's results [**21**, Section 5].

3. The explicit Shimura reciprocity law and new values of α_n^*.

We have seen in Section 2 that μ_n^2 is a class invariant whenever n satisfies the hypothesis of Theorem 2.1. In this section, we identify μ_n^2 as a value of a modular function, construct the explicit action of $\mathrm{Gal}(K_1|K)$ on μ_n^2 and as a result, evaluate μ_n^2.

Let $\mathbb{M}_2^+(\mathbb{Z})$ denote the set of 2×2 matrices with integer coefficients and positive determinant. For each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2^+(\mathbb{Z})$, define the function

$$\frac{\eta \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}}{\eta} : \tau \mapsto \frac{\eta\left(\dfrac{a\tau + b}{c\tau + d}\right)}{\eta(\tau)}.$$

It is easy to see that μ_n is the value of $\mathfrak{g}_0(\tau)^6/(3\sqrt{3})$ at $\tau = \sqrt{-3n}$ where

$$\mathfrak{g}_0(\tau) := \frac{\eta \circ \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}{\eta}(\tau).$$

The function $\mathfrak{g}_0(\tau)$ is an element of F_{72}, the modular function field of level 72 defined over $\mathbb{Q}(\zeta_{72})$. This means that it is meromorphic on the completed upper half plane $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, admits a Laurent series expansion in the variable $q^{1/72} = e^{2\pi i\tau/72}$ centered at $q = 0$ having coefficients in $\mathbb{Q}(\zeta_{72})$ and invariant with respect to the matrix group

$$\Gamma(72) := \ker[SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/72\mathbb{Z})].$$

From (2.5), we find that the minimal polynomial for $\mathfrak{g}_0^{12}$ over the modular function field $\mathbb{Q}(j)$ is

$$X^4 + 36\,X^3 + 270\,X^2 + (756 - j)X + 3^6.$$

Over $\mathbb{Q}(j)$, the conjugates of $\mathfrak{g}_0^{12}$ are $\mathfrak{g}_1^{12}$, $\mathfrak{g}_2^{12}$ and $\mathfrak{g}_3^{12}$ defined by

$$\mathfrak{g}_1 := \zeta_{24}^{-1}\frac{\eta \circ \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}}{\eta}, \quad \mathfrak{g}_2 := \frac{\eta \circ \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}}{\eta}, \quad \text{and} \quad \mathfrak{g}_3 := \sqrt{3}\cdot \frac{\eta \circ \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}{\eta}.$$

If K is an imaginary quadratic field of discriminant D, class field theory gives an isomorphism

$$\mathrm{Gal}\,(K_1/K) \simeq C(D)$$

between the Galois group for $K \subset K_1$ and the form class group of discriminant D. Among the primitive forms $[a, b, c]$ having discriminant $D = b^2 - 4ac$, one obtains a complete set of representatives in $C(D)$ by choosing the reduced forms

$$\mid b \mid \le a \le c \quad \text{and} \quad b \ge 0 \text{ if either } \mid b \mid = a \text{ or } a = c.$$

The class of $[a, -b, c]$ is the inverse of $[a, b, c]$ in $C(D)$, and the elements having order 2 in $C(D)$ correspond to *ambiguous forms.* These are the reduced forms $[a, b, c]$ for which $a = b$, $a = c$ or $b = 0$ occurs.

Given $h \in F_m$, if $h(\theta) \in K_1$ where θ is the generator of $\mathfrak{O}_K$ over $\mathbb{Z}$ (we assume here the algebraic closure of K is embedded in the complex plane such that θ lies in the upper half plane $\mathbb{H}$), there is an explicit formula for computing the action of $C(D)$ on $h(\theta)$ which is a consequence of the Shimura Recirpocity Law. This is given as follows:

Lemma 3.1. *Let K be an imaginary quadratic number field of discriminant D and let $h \in F_m$ be such that $h(\frac{\sqrt{D}}{2}) \in K_1$. Given a primitive quadratic form $[a, b, c]$ of discriminant D, let $M = M_{[a,b,c]} \in GL_2(\mathbb{Z}/m\mathbb{Z})$ be the matrix that satisfies the congruences*

$$M \equiv \begin{cases} \begin{pmatrix} a & \frac{b}{2} \\ 0 & 1 \end{pmatrix} & \pmod{p^r} \quad \textit{if } p \nmid a, \\ \begin{pmatrix} -\frac{b}{2} & -c \\ 1 & 0 \end{pmatrix} & \pmod{p^r} \quad \textit{if } p \mid a \textit{ and } p \nmid c, \\ \begin{pmatrix} -\frac{b}{2} - a & -\frac{b}{2} - c \\ 1 & -1 \end{pmatrix} & \pmod{p^r} \quad \textit{if } p \mid a \textit{ and } p \mid c. \end{cases}$$

at all prime power factors $p^r \mid m$. *The Galois action of the class of* $[a,-b,c]$ *in* $C(D)$ *with respect to the Artin map is given by*

$$\left(h\left(\frac{\sqrt{D}}{2}\right)\right)^{[a,-b,c]} = h^M\left(\frac{-b+\sqrt{D}}{2a}\right),$$

where h^M *denote the image of* h *under the action of* M.

For a proof of Lemma 3.1 and the description of the action of M on h, see [**21**].

In view of Lemma 3.1, we first need to discuss the action of $M \in GL_2(\mathbb{Z}/m\mathbb{Z})$ on functions $h \in F_m$. The action of such an M depends only on $M_{p^{r_p}}$ for all prime factors $p \mid m$ where $M_N \in GL_2(\mathbb{Z}/N\mathbb{Z})$ is the reduction modulo N of M and r_p is the largest power of p such that p^{r_p} divides m.

Now every M_N with determinant x decomposes as

$$M_N = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_N \begin{pmatrix} a & b \\ c & d \end{pmatrix}_N$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_N \in SL_2(\mathbb{Z}/N\mathbb{Z})$. Since $SL_2(\mathbb{Z}/N\mathbb{Z})$ is generated by S_N and T_N where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, it suffices to find the action of $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^{r_p}}$, $S_{p^{r_p}}$ and $T_{p^{r_p}}$ on h for all $p \mid m$.

For $\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_{p^{r_p}}$, the action on F_m is given by lifting the automorphism of $\mathbb{Q}(\zeta_m)$ determined by

$$\zeta_{p^{r_p}} \mapsto \zeta_{p^{r_p}}^x \qquad \text{and} \qquad \zeta_{q^{r_q}} \mapsto \zeta_{q^{r_q}}$$

for all prime factors $q|m$ such that $q \neq p$.

In order that the actions of the matrices at different primes commute with each other, we have to lift $S_{p^{r_p}}$ and $T_{p^{r_p}}$ to matrices in $SL_2(\mathbb{Z}/m\mathbb{Z})$ such that they reduce to the identity matrix in $SL_2(\mathbb{Z}/q^{r_q}\mathbb{Z})$ for all $q \neq p$. In our case for $m = 72$, the prime powers are 8 and 9 and we have

$$S_8 \mapsto \begin{pmatrix} -8 & 9 \\ -9 & -8 \end{pmatrix}_{72}, \qquad T_8 \mapsto \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}_{72},$$
$$S_9 \mapsto \begin{pmatrix} 9 & -8 \\ 8 & 9 \end{pmatrix}_{72}, \qquad T_9 \mapsto \begin{pmatrix} 1 & -8 \\ 0 & 1 \end{pmatrix}_{72}.$$

When $h \in F_m$ is an η-quotient, we can use the transformation rule

$$\eta \circ S_m(\tau) = \sqrt{-i\tau}\,\eta(\tau) \qquad \text{and} \qquad \eta \circ T_m(\tau) = \zeta_{24}\eta(\tau)$$

to determine the action of any $M_m \in SL_2(\mathbb{Z}/m\mathbb{Z})$. In particular, we have

$$(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ S_{72} = (\mathfrak{g}_3, \zeta_{24}^{10}\mathfrak{g}_2, \zeta_{24}^{14}\mathfrak{g}_1, \mathfrak{g}_0),$$

and

$$(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ T_{72} = (\mathfrak{g}_1, \zeta_{24}^{-2}\mathfrak{g}_2, \mathfrak{g}_0, \zeta_{24}^{2}\mathfrak{g}_3).$$

Consequently, we derive the following actions:

	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_8$	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
S_8	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
T_8	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9, \quad 3\|(x-1)$	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9, \quad 3\|(x-2)$	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_3^{12}$
S_9	$\mathfrak{g}_3^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_0^{12}$
T_9	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_3^{12}$

Using this, together with Lemma 3.1, we have:

Theorem 3.2. *The action of a reduced primitive quadratic form* $[a, b, c]$ *with discriminant* D *in* $C(D)$ *on* $\mathfrak{g}_0(\frac{\sqrt{D}}{2})^{12}$ *is given by*

$$\left\{\mathfrak{g}_0\left(\frac{\sqrt{D}}{2}\right)^{12}\right\}^{[a,-b,c]} = \begin{cases} \mathfrak{g}_0(\frac{-b+\sqrt{D}}{2a})^{12} & \textit{if } b \equiv 0, a \not\equiv 0 \pmod 3, \\ \mathfrak{g}_1(\frac{-b+\sqrt{D}}{2a})^{12} & \textit{if } ab \equiv -1 \pmod 3, \\ \mathfrak{g}_2(\frac{-b+\sqrt{D}}{2a})^{12} & \textit{if } ab \equiv 1 \pmod 3, \\ \mathfrak{g}_3(\frac{-b+\sqrt{D}}{2a})^{12} & \textit{if } a \equiv 0 \pmod 3. \end{cases}$$

Proof. The above result follows from the observation that the action of M_8 on $\mathfrak{g}_0^{12}$ is trivial. Hence, it suffices to consider the action of M_9 on $\mathfrak{g}_0^{12}$. When $3 \nmid a$,

$$M_9 = \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{b-1}{2a} \\ 0 & 1 \end{pmatrix} \equiv S_9 \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} S_9 T_9^{\frac{b-1}{2a}}.$$

When $3 \mid a$, then $3 \nmid c$, so

$$M_9 = \begin{pmatrix} \frac{-b-1}{2} & -c \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} c & \frac{-b-1}{2} \\ 0 & 1 \end{pmatrix} S_9 \equiv \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{-b-1}{2c} \\ 0 & 1 \end{pmatrix} S_9$$
$$\equiv S_9 \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} S_9 T_9^{\frac{-b-1}{2c}} S_9.$$

□

Theorem 3.2 should be viewed as a cubic analogue of the results of N. Yui and D. Zagier [**28**, Proposition, Section 2] and it indicates that all the conjugates of μ_n^2 can be computed numerically once we determine the class group of $\mathbb{Q}(\sqrt{-3n}), n \equiv 3 \pmod 4$. Using these numerical values, we could then determine the minimal polynomial satisfied by μ_n^2. If the degree of the minimal polynomial is at most 4, we could solve the minimal polynomial and determine μ_n^2 explicitly. In order to calculate μ_n^2 for which the class number of $\mathbb{Q}(\sqrt{-3n})$ is greater than 4, we need the following lemma, which essentially tells us the action of the ambiguous forms (see the remarks before Lemma 3.1 for the definition of ambiguous forms) on μ_n^2.

Lemma 3.3. *Let $n \equiv 3 \pmod 4$ and $K = \mathbb{Q}(\sqrt{-3n})$, where n is squarefree. Then*

$$(\mu_n^2)^{[2,2,\frac{3n+1}{2}]} = -\lambda_n^2,$$

where

$$\lambda_n = \frac{1}{3\sqrt{3}} \left\{ \frac{\eta(\frac{1+\sqrt{-n/3}}{2})}{\eta(\frac{1+\sqrt{-3n}}{2})} \right\}^6. \tag{3.1}$$

If $n = p_1 p_2 \dots p_k$ then

$$(\mu_n^2)^{[p_1p_2\dots p_j, 0, \frac{3n}{p_1p_2\cdots p_j}]} = \mu^2_{n/(p_1p_2\dots p_j)^2},$$

where $j \le k$.

Proof. We apply Theorem 3.2 with $ab \equiv 1 \pmod 3$ and $b \equiv 0 \pmod 3$, respectively and note that

$$\lambda_n^2 = -\frac{1}{27}\mathfrak{g}_1^{12}\left(\frac{1+\sqrt{-3n}}{2}\right) \qquad \text{and}$$

$$\mu^2_{n/(p_1p_2\dots p_j)^2} = \frac{1}{27}\mathfrak{g}_0^{12}\left(\sqrt{-\frac{3n}{(p_1p_2\dots p_j)^2}}\right).$$

□

We can now explicitly determined μ_n^2 by first collecting in a symmetric way the products of the real conjugates of μ_n^2.

For example, when $n = 23$, $C(-276) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_4$, generated by $a = [2, 2, 35]$ and $b = [5, 2, 14]$. Now define

$$P_{23} := (\mu_{23}\lambda_{23})^2 + (\mu_{23}\lambda_{23})^{-2} \quad \text{and} \quad Q_{23} := (\mu_{23}/\lambda_{23})^2 + (\mu_{23}/\lambda_{23})^{-2}.$$

These numbers are fixed by the Galois action of a^2 and b and since P_{23} and Q_{23} are algebraic integers, one concludes that $P_{23} + P_{23}^a$, $P_{23}P_{23}^a$, $Q_{23} + Q_{23}^a$, and $Q_{23}Q_{23}^a$ are integers. These integers can be found by approximating the numerical values of the $\mathfrak{g}_i^{12}$ at the corresponding arguments given by Theorem 3.2. Hence, we obtain

$$\begin{aligned} P_{23} + P_{23}^a &= 296143772, \\ P_{23}P_{23}^a &= -389054012, \\ Q_{23} + Q_{23}^a &= 5980, \end{aligned}$$

and

$$Q_{23}Q_{23}^a = -17852.$$

Solving the quadratic polynomials satisfied by P_{23} and Q_{23} and simplifying, we deduce that

$$\mu_{23}^2 = (5\sqrt{3}+24)^{1/2}(13\sqrt{23}+36\sqrt{3})^{1/2}\left(\sqrt{84+48\sqrt{3}}+\sqrt{83+48\sqrt{3}}\right)^3.$$

Substituting the value μ_{23}^2 into (2.3), we easily determine α_{23}^*, which is crucial in the derivation of the following series.

$$\frac{1}{\pi} = \sum_{m=0}^{\infty}(a_{23}+b_{23}m)\frac{(\frac{1}{2})_m(\frac{1}{3})_m(\frac{2}{3})_m}{(m!)^3}H_{23}^m,$$

$$\begin{aligned} z_{23} &= \frac{1}{23}\left(\sqrt{-83+48\sqrt{3}}(444+252\sqrt{3}) - 56 + 54\sqrt{3}\right), \\ a_{23} &= -\frac{1}{6\sqrt{3}}\left(z_{23} + (8\alpha_{23}^* - 4)\sqrt{23}\right), \\ b_{23} &= \frac{2\sqrt{23}}{\sqrt{3}}\frac{\mu_{23}^2-1}{\mu_{23}^2+1}, \quad \text{and} \\ H_{23} &= \frac{1}{2^4 23^3}\left(6\sqrt{-83+48\sqrt{3}} + 9\sqrt{3}\sqrt{-83+48\sqrt{3}} - 2 - 3\sqrt{3}\right)^3. \end{aligned}$$

For methods of deriving series of the above type, and the relation between μ_n^2 and series for $1/\pi$, see [**17**] and [**18**].

Remarks.

(a) The method illustrated above for the case $n = 23$ works for any n such that $C(-12n)$ is of the type $\mathbb{Z}_2 \oplus \mathbb{Z}_{2s}$, where $s = 1, 2, 3$, or 4.

(b) If $C(-12n)$ is of the type $\mathbb{Z}_2^t \oplus \mathbb{Z}_{2s}$ with $s = 1, 2, 3$ or 4 and $t \in \mathbb{N}$, we need to construct more numbers analogous to P_n and Q_n. Examples of such constructions can be found in [**15**] and Section 4.

(c) One can modify the method in [**15**] to evaluate the corresponding μ_n^2 whenever the class group is of the form $\mathbb{Z}_2^t \oplus Z_{2s}$, where $s = 2, 3$, or 4. The method used there avoids the use of the explicit Shimura Reciprocity Law but it cannot be extended to compute μ_n^2 when the associated class groups are different from those mentioned above.

(d) Gee and M. Honsbeek [**22**] have recently devised a method of computing class invariants without solving their minimal polynomials. Their method involves determining the Lagrange resolvents of these minimal polynomials by determining the conjugates of the corresponding class invariants explicitly.

4. The class invariant λ_n^2 and the missing entries in the Lost Notebook.

We first note that

$$\lambda_n^2 = -\frac{1}{27}\mathfrak{g}_2^{12}\left(\frac{-1+\sqrt{-3n}}{2}\right).$$

To compute λ_n, it suffices to determine the action of the elements in the corresponding class groups. This is given by the following analogue of Theorem 3.2:

Theorem 4.1. *The action of a reduced primitive quadratic form* $[a,b,c]$ *with discriminant* D *in* $C(D)$ *on* $\mathfrak{g}_2\left(\frac{-1+\sqrt{D}}{2}\right)^{12}$ *is given by*

$$\left\{\mathfrak{g}_2\left(\frac{-1+\sqrt{D}}{2}\right)^{12}\right\}^{[a,-b,c]} = \begin{cases} \mathfrak{g}_0(\frac{-b+\sqrt{D}}{2a})^{12} & \text{if } b\equiv 0, a\not\equiv 0 \pmod 3,\\ \mathfrak{g}_1(\frac{-b+\sqrt{D}}{2a})^{12} & \text{if } ab\equiv -1 \pmod 3,\\ \mathfrak{g}_2(\frac{-b+\sqrt{D}}{2a})^{12} & \text{if } ab\equiv 1 \pmod 3,\\ \mathfrak{g}_3(\frac{-b+\sqrt{D}}{2a})^{12} & \text{if } a\equiv 0 \pmod 3.\end{cases}$$

To facilitate the computations of λ_n we need the analogue of Lemma 3.3.

Lemma 4.2. *Let* $n\equiv 1 \mod 4$ *and* $K=\mathbb{Q}(\sqrt{-3n})$, *where* n *is squarefree. If* $n=p_1p_2\ldots p_k$ *then*

$$(\lambda_n^2)^{\left[p_1p_2\ldots p_j, p_1p_2\ldots p_j, \frac{3n+(p_1p_2\ldots p_j)^2}{4p_1p_2\ldots p_j}\right]} = \lambda^2_{n/(p_1p_2\ldots p_j)^2},$$

where $j\le k$.

Lemma 4.2 indicates that instead of calculating the conjugates of λ_n^2 directly, it suffices to calculate the conjugates of symmetric combinations of all the real conjugates of λ_n^2.

We may now proceed to complete the table of λ_n initiated by Ramanujan on page 212 of his Lost Notebook. For $p=73, 97$, and 241, all of which are primes, set

$$P_p = \lambda_p^2 + \frac{1}{\lambda_p^2}. \tag{4.1}$$

Since the class groups corresponding to these p's are of the form $\mathbb{Z}_4$, we conclude that P_p each satisfies a quadratic polynomial. We now derive the polynomial satisfied by P_{73}.

Now the class group of $\mathbb{Q}(\sqrt{-219})$ is generated by the form $[5,1,11]$. By Theorem 4.1, we easily deduce that

$$P_{73} + P_{73}^{[5,1,11]} = 199044,$$

and

$$P_{73}P_{73}^{[5,1,11]} = 287491,$$

where $P_{73}^{[5,1,11]}$ denotes the image of P_{73} under the action of $[5,1,11]$. Hence, P_{73} satisfies the quadratic polynomial

$$x^2 - 199044x + 287491 = 0.$$

Solving and simplifying, we deduce that

$$\lambda_{73} = \left(\sqrt{\frac{11+\sqrt{73}}{8}} + \sqrt{\frac{3+\sqrt{73}}{8}}\right)^6.$$

The cases for $n = 97$ and 241 are similar.

We now turn to the case $n = 217$. Here 217 is divisible by two primes, namely, 7 and 31. In this case we consider two numbers Q_{217} and R_{217} defined by

$$Q_{217} = \lambda_{217}^2\lambda_{31/7}^2 + \frac{1}{\lambda_{217}^2\lambda_{31/7}^2}$$

and

$$R_{217} = \frac{\lambda_{217}^2}{\lambda_{31/7}^2} + \frac{\lambda_{31/7}^2}{\lambda_{217}^2}.$$

Note that the class group of $\mathbb{Q}(\sqrt{-651})$ is generated by $a := [5,3,33]$ and $b := [3,3,55]$. The order of a is 4 and the group generated by a^2 and b fixes Q_{217} and R_{217}. Hence it suffices to determine the action of a on Q_{217} and R_{217}, which can be easily done by Theorem 4.1. The value of λ_{217} which results from this consideration is a product of two units, given by

$$\lambda_{217} = \left(\sqrt{\frac{1901+129\sqrt{217}}{8}} + \sqrt{\frac{1893+129\sqrt{217}}{8}}\right)^{3/2} \cdot \left(\sqrt{\frac{1597+108\sqrt{217}}{4}} + \sqrt{\frac{1593+108\sqrt{217}}{4}}\right)^{3/2}.$$

Finally, consider the case $n = 193$. This is the case which we cannot evaluate using the previous method given in [**15**]. Here the class group of $\mathbb{Q}(\sqrt{-579})$ is generated by $a := [5,1,29]$ and it is of order 8. We consider the expression P_{193} where P_p is given by (4.1). To determine P_{193} we compute the image of P_{193} under a, a^2, and a^3. Our computations show that if

$$\begin{aligned}
&\alpha := P_{193},\\
&\beta := P_{193}^{a} = -\frac{1}{27}\mathfrak{g}_2^{12}\left(\frac{1+\sqrt{-579}}{10}\right) - 27\mathfrak{g}_2^{-12}\left(\frac{1+\sqrt{-579}}{10}\right)\\
&\gamma := P_{193}^{a^2} = -\frac{1}{27}\mathfrak{g}_0^{12}\left(\frac{3+\sqrt{-579}}{14}\right) - 27\mathfrak{g}_0^{-12}\left(\frac{3+\sqrt{-579}}{14}\right)
\end{aligned}$$

and

$$\delta := P_{193}^{a^3} = -\frac{1}{27}\mathfrak{g}_0^{12}\left(\frac{-9+\sqrt{-579}}{22}\right) - 27\mathfrak{g}_0^{-12}\left(\frac{-9+\sqrt{-579}}{22}\right),$$

then

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= 3251132424,\\
\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= 82707128352,\\
\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\alpha\gamma\delta &= 9465475096,
\end{aligned}$$

and

$$\alpha\beta\gamma\delta = 176664526832.$$

Solving the quartic polynomial satisfied by P_{193} and simplifying, we deduce that

$$\lambda_{193}^{1/3} + \frac{1}{\lambda_{193}^{1/3}} = \frac{1}{4}\left(39 + 3\sqrt{193} + \sqrt{2690 + 194\sqrt{193}}\right).$$

It was not clear to us what motivated Ramanujan to construct the table of values for λ_n. Perhaps he intended to set up a table for λ_n similar to that for the Ramanujan-Weber class invariants G_n and g_{2n} (see [**26**]). Recently, Chan, Liaw and Tan offered another reason for the existence of Ramanujan's table. They succeeded in deriving a new class of series for $1/\pi$ associated with λ_n. Two of such series are

$$\frac{4}{\pi\sqrt{3}} = \sum_{k=0}^{\infty}(5k+1)\frac{\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k\left(\frac{1}{2}\right)_k}{(k!)^3}\left(-\frac{9}{16}\right)^k,$$

and

$$\frac{12\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}(51k+7)\frac{\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k\left(\frac{1}{2}\right)_k}{(k!)^3}\left(-\frac{1}{16}\right)^k.$$

These simple series came as a surprise as it was thought that all the possible simple series should have been exhausted after the work of Ramanujan, the Chudnovskys [**19**] and the Borweins.

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