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Analogues of Jacobi's inversion formula for the incomplete elliptic integral of the first kind[☆]

Heng Huat Chan[a,b] and Zhi-Guo Liu[b,c,*]

[a] School of Mathematical Sciences, University of Sussex, BN1 9QH, Brighton, UK
[b] Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Singapore
[c] Department of Mathematics, East China Normal University, Shanghai 200062, P.R. China

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Abstract

In this article, we revisit Ramanujan's cubic analogue of Jacobi's inversion formula for the classical elliptic integral of the first kind. Our work is motivated by the recent work of Milne (Ramanujan J. 6(1) (2002) 7–149), Chan and Chua (Ramanujan J., to appear) on the representations of integers as sums of even squares.

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1. Introduction

Let

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \quad |q| < 1.$$

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*Corresponding author. Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Republic of Singapore.
E-mail addresses: chanhh@math.nus.edu.sg (H.H. Chan), matliuzg@nus.edu.sg (Z.-G. Liu).

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The function $\varphi^s(q)$ is the generating function of $r_s(n)$, the number of representations of an integer as a sum of s squares. Recently, Milne [8] discovered a new expression for $\varphi^{24}(q)$, namely,

$$\varphi^{24}(q) = \tfrac{1}{9}\{\mathscr{E}_4(q)\mathscr{E}_8(q) - 8\mathscr{E}_6^2(q)\}, \tag{1.1}$$

where

$$\mathscr{E}_4(q) = 1 + 16\sum_{k=1}^{\infty} \frac{k^3 q^k}{1-(-q)^k},$$

$$\mathscr{E}_6(q) = 1 - 8\sum_{k=1}^{\infty} \frac{k^5 q^k}{1-(-q)^k}$$

and

$$\mathscr{E}_8(q) = 17 + 32\sum_{k=1}^{\infty} \frac{k^7 q^k}{1-(-q)^k}.$$

Milne's formula (1.1) led Chan and Chua [5] to conjecture certain explicit expressions for $\varphi^{2s}(q)$, $s \in \mathbb{N}$, in terms of Eisenstein series analogous to that of $\mathscr{E}_4(q), \mathscr{E}_6(q)$ and $\mathscr{E}_8(q)$. For any fixed s, their conjectures can be verified through a table expressing the Eisenstein series in terms of two functions $z := \varphi^2(q)$ and

$$x = 16q\frac{\psi^4(q^2)}{\varphi^4(q)}$$

with

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

For example if

$$\mathscr{T}_{2k}(q) = \sum_{n=1}^{\infty} \frac{n^{2k-1} q^{2n}}{1-q^{4n}},$$

then we have Table 1.

Table 1 is used to show that the set $S := \{\mathscr{T}_{2u}(q)\mathscr{T}_{2v}(q) | u+v=16\}$ is linearly independent over $\mathbb{C}$. This in turn shows that $q^8\psi^{32}(q^2)$ is a linear combination of the elements in S, giving the formula

$$q^8\psi^{32}(q^2) = \frac{1}{75\,600}\left(\frac{21}{4}\mathscr{T}_8^2(q) + \mathscr{T}_4(q)\mathscr{T}_{12}(q) - \frac{25}{4}\mathscr{T}_{10}(q)\mathscr{T}_6(q)\right), \tag{1.2}$$

Table 1

$\mathcal{T}_4(q) = z^4\left(\frac{x}{16}\right)^2$
$\mathcal{T}_6(q) = z^6\left(\frac{x}{16}\right)^2\left(1-\frac{x}{2}\right)$
$\mathcal{T}_8(q) = z^8\left(\frac{x}{16}\right)^2\left(1-x+\frac{17}{32}x^2\right)$
$\mathcal{T}_{10}(q) = z^{10}\left(\frac{x}{16}\right)^2\left(1-\frac{x}{2}\right)\left(1-x+\frac{31}{16}x^2\right)$
$\mathcal{T}_{12}(q) = z^{12}\left(\frac{x}{16}\right)^2\left(1-2x+\frac{291}{32}x^2-\frac{259}{32}x^3+\frac{691}{256}x^4\right)$

which is equivalent to the formula (for more details, see [9, (2.12)])

$$\varphi^{32}(q) = \frac{1}{4725}\{-400\mathcal{E}_6(q)\mathcal{E}_{10}(q) + 16\mathcal{E}_{12}(q)\mathcal{E}_4(q) + 21\mathcal{E}_8^2(q)\}.$$

In [5], Chan and Chua show that Table 1 can be extended indefinitely by using a recurrence relation satisfied by $\mathcal{T}_{2k}(q)$. They show that the recurrence relation is a consequence of the differential equation satisfied by the classical Jacobian elliptic function $\mathrm{sn}^2(u, \mathbf{k})$ (here, $\mathbf{k}^2 = x$).

In this paper, we propose to study the functions

$$\mathcal{S}_{2k}(q) = \sum_{n=1}^{\infty} \frac{n^{2k}q^n}{1+q^n+q^{2n}}, \quad k \geqslant 1.$$

The functions $\mathcal{S}_{2k}(q)$ are perhaps first studied by Ramanujan [10, p. 257]. Let

$$z_3 := a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}$$

and

$$x_3 = \frac{c^3(q)}{a^3(q)},$$

where

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

Table 2 is given by Ramanujan.

Table 2 is clearly a cubic analogue of Table 1. For the proofs of these identities, see [3] or [7]. In Section 2, we will show that Table 2 can be extended indefinitely using a recurrence relation (see Corollary 2.4) satisfied by $\mathcal{S}_{2k}(q)$. This recurrence relation

Table 2

$\mathscr{S}_2(q) = z_3^3 \frac{x_3}{27}$
$\mathscr{S}_4(q) = z_3^5 \frac{x_3}{27}$
$\mathscr{S}_6(q) = z_3^7 \left(\frac{x_3(3+4x_3)}{81}\right)$
$\mathscr{S}_8(q) = z_3^9 \left(\frac{x_3(81+648x_3+80x_3^2)}{3^7}\right)$

is obtained from a differential equation (see (2.1)) satisfied by the function

$$L(u|\tau) := \sum_{k=1}^{\infty} (-1)^k \mathscr{S}_{2k}(q) \frac{(2u)^{2k}}{(2k)!}. \tag{1.3}$$

Note that the definition of $L(u|\tau)$ is motivated by the following representation of $\mathrm{sn}^2(u, \mathbf{k})$ [5]:

$$\mathrm{sn}^2(u, \mathbf{k}) = -\frac{8}{z^2\mathbf{k}^2} \sum_{k=1}^{\infty} (-1)^k \mathscr{T}_{2k+2}(\sqrt{q}) \frac{(2u/z)^{2k}}{(2k)!}.$$

The classical inversion formula for the incomplete elliptic integral of the first kind is the following:

Theorem 1.1. *Let* ${}_2F_1(a, b; c; z)$ *be the Gaussian hypergeometric series. If* $0 \leqslant q < 1,\ 0 \leqslant \phi \leqslant \pi/2$, *and*

$$\theta\varphi^2(q) = \int_0^{\phi} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; \mathbf{k}^2 \sin^2 t\right) dt,$$

then for $0 \leqslant \theta \leqslant \pi/2$,

$$\phi = \theta + 2 \sum_{n=1}^{\infty} \frac{q^n}{n(1+q^{2n})} \sin 2n\theta.$$

For a proof of this classical result, see [12].

On page 257 of his second Notebook, Ramanujan recorded the following analogue of Theorem 1.1:

Theorem 1.2. *If* $0 \leqslant q < 1,\ 0 \leqslant \phi \leqslant \pi/2$ *and*

$$\theta a(q) = \int_0^{\phi} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x_3 \sin^2 t\right) dt$$

then for $0 \leqslant \theta \leqslant \pi/2$,

$$\phi = \theta + 3 \sum_{n=1}^{\infty} \frac{q^n}{n(1+q^n+q^{2n})} \sin 2n\theta.$$

Proof of Theorem 1.2 can be found in [3, Theorem 8.1, 13]. In Section 3, we will sketch a proof of Theorem 1.2 based on the identity established in Section 2. We wish to emphasize that Theorems 1.1 and 1.2 can be derived from certain differential equations using suitable substitutions.

In the final section, we discuss a transformation formula satisfied by $L(u|\tau)$.

2. The main identity

The main aim of this section is to derive a differential equation satisfied by $L(u|\tau)$ defined in (1.3). Here and in subsequent sections, we set $q = e^{2\pi i\tau}$, with $\mathrm{Im}\,\tau > 0$.

Theorem 2.1.

$$\frac{d^2L(u|\tau)}{du^2} = -4\mathscr{S}_2(q) - 4a^2(q)L(u|\tau) - 24a(q)L^2(u|\tau) - 32L^3(u|\tau). \tag{2.1}$$

The differential equation (2.1) first appeared in a different form in [3, p. 4209]. We rediscover it in an attempt to construct a recurrence relation satisfied by $\mathscr{S}_{2k}(q)$. Note that Theorem 2.1 indeed provides us with such a relation by simply comparing the coefficients of u^{2k} on both sides. We emphasize here that our proof of (2.1) is new and can be used to construct differential equations satisfied by functions analogous to $L(u|\tau)$ and $\mathrm{sn}^2(u, \mathbf{k}^2)$.

We will prove (2.1) in several steps. Our first task is to find an alternative expression for $L(u|\tau)$. Recall that

$$\begin{aligned}\vartheta_1(z|\tau) &:= 2q^{1/8} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z \\ &= -iq^{1/8}e^{iz}(q;q)_\infty (qe^{2iz};q)_\infty (e^{-2iz};q)_\infty,\end{aligned}$$

where

$$(a;q)_\infty = \prod_{k=1}^{\infty} (1 - aq^k).$$

The second equality follows from the Jacobi triple product identity.

In [13, p. 129], Shen shows that

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi\tau|3\tau) - \frac{\vartheta_1'}{\vartheta_1}(u-\pi\tau|3\tau) = -2i - 4i \sum_{n=1}^{\infty} \frac{q^n \cos 2nu}{1+q^n+q^{2n}}. \tag{2.2}$$

Substituting the series expansion

$$\cos 2nu = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{(2nu)^{2k}}{(2k)!} \tag{2.3}$$

into (2.2), interchanging the order of summation, and using the identity [3, p. 4173]

$$\mathscr{S}_0(q) = \tfrac{1}{6}(a(q) - 1),$$

we deduce that

Lemma 2.2.

$$\frac{\vartheta_1'}{\vartheta_1}(u + \pi\tau|3\tau) - \frac{\vartheta_1'}{\vartheta_1}(u - \pi\tau|3\tau) = -\frac{2}{3}(2 + a(q))i - 4iL(u|\tau). \tag{2.4}$$

Now, since $\vartheta_1(u + \pi\tau|3\tau)$ and $\vartheta_1(u - \pi\tau|3\tau)$ appears in the expression of $L(u|\tau)$, we are led to the function

$$f(z, u|\tau) = e^{2iz} \frac{\vartheta_1(z + u + \pi\tau|3\tau)\vartheta_1(z - u + \pi\tau|3\tau)\vartheta_1(z + \pi\tau|3\tau)}{\vartheta_1^3(z|3\tau)},$$

where $|u| < \pi|\tau|$. The construction of $f(z, u|\tau)$ is guided by the transformation formulas of $\vartheta_1(z|\tau)$ given by

$$\vartheta_1(z + \pi|\tau) = -\vartheta_1(z|\tau) \quad \text{and} \quad \vartheta_1(z + \pi\tau|\tau) = -q^{-1/2}e^{-2iz}\vartheta_1(z|\tau).$$

Basically, we need an elliptic function $f(z, u|\tau)$ in the variable z with periods π and $3\pi\tau$ such that its logarithmic derivative at $z = 0$ involves the left-hand side of (2.4).

Since $\vartheta_1(z|\tau)$ has a simple zero at $z = 0$, the function $f(z, u|\tau)$ has a pole of order 3 at $z = 0$. The residue of f at $z = 0$ is zero since the sum of residues of an elliptic function is always zero. This implies, by logarithmically differentiating $z^3 f(z, u|\tau)$, that

$$\begin{aligned} \operatorname{res}(f; 0) &= \left[\frac{1}{2}\frac{d^2}{dz^2}(z^3 f(z, u|\tau))\right]_{z=0} \\ &= \frac{1}{2}[z^3 f(z, u|\tau)(\phi^2(z, u|\tau) + \phi'(z, u|\tau))]_{z=0} = 0, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \phi(z, u|\tau) = 2i + \frac{3}{z} - 3\frac{\vartheta_1'}{\vartheta_1}(z|3\tau) + \frac{\vartheta_1'}{\vartheta_1}(z + \pi\tau|3\tau) \\ + \frac{\vartheta_1'}{\vartheta_1}(z + u + \pi\tau|3\tau) + \frac{\vartheta_1'}{\vartheta_1}(z - u + \pi\tau|3\tau). \end{aligned} \tag{2.6}$$

Since

$$\lim_{z\to 0} z^3 f(z,u|\tau) = -\frac{\vartheta_1(u+\pi\tau|3\tau)\vartheta_1(u-\pi\tau|3\tau)\vartheta_1(\pi\tau|3\tau)}{\vartheta_1'(0|3\tau)^3} \neq 0,$$

we conclude from (2.5) that

$$\phi^2(0,u|\tau) + \phi'(0,u|\tau) = 0. \tag{2.7}$$

We will first compute $\phi(0,u)$. It is known that [14, p. 489]

$$\frac{\vartheta_1'}{\vartheta_1}(z|\tau) = \cot z + 4\sum_{k=1}^{\infty} \frac{q^n}{1-q^n}\sin 2nz = \frac{1}{z} + \frac{z}{3}\left(1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}\right) + O(z^3). \tag{2.8}$$

By substituting $u=0$ in Lemma 2.2, we find that

$$\frac{\vartheta_1'}{\vartheta_1}(\pi\tau|3\tau) = -\frac{1}{3}(2+a(q))i. \tag{2.9}$$

Hence,

$$\phi(0,u|\tau) = -a(q)i - 4iL(u|\tau). \tag{2.10}$$

Next, differentiating both sides of (2.6), using (2.8) and evaluating at $z=0$, we find that

$$\phi'(0,u|\tau) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^{3k}}{1-q^{3k}} + \left(\frac{\vartheta_1'}{\vartheta_1}\right)'(\pi\tau|3\tau) + M(u|\tau), \tag{2.11}$$

where

$$M(u|\tau) = \left(\frac{\vartheta_1'}{\vartheta_1}\right)'(u+\pi\tau|3\tau) + \left(\frac{\vartheta_1'}{\vartheta_1}\right)'(u-\pi\tau|3\tau).$$

Now, since

$$M(0|\tau) = 2\left(\frac{\vartheta_1'}{\vartheta_1}\right)'(\pi\tau|3\tau)$$

and

$$\phi'(0,0|\tau) = -\phi^2(0,0|\tau) = a^2(q)$$

by (2.7) and (2.10), we conclude that

$$\left(\frac{\vartheta_1'}{\vartheta_1}\right)'(\pi\tau|3\tau) = \frac{1}{3}(a^2(q) - P(q^3)), \tag{2.12}$$

where

$$P(q) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}.$$

Hence, by (2.7), (2.10) and (2.11), we conclude that

Lemma 2.3.

$$M(u|\tau) = -\frac{2P(q^3)+a^2(q)}{3} + (a(q)+4L(u|\tau))^2.$$

Our final step is to find another relation between $M(u|\tau)$ and $L(u|\tau)$. To achieve our aim, we construct a second function

$$g(z,u|\tau) = \frac{\vartheta_1(z+u+\pi\tau|3\tau)\vartheta_1(z-u+\pi\tau|3\tau)\vartheta_1^2(z-\pi\tau|3\tau)}{\vartheta_1^4(z|3\tau)},$$

where $|u|<\pi|\tau|$. The function $g(z,u|\tau)$ is an elliptic function with periods π and $3\pi\tau$ having only one pole of order 4 at $z=0$. This implies that

$$\operatorname{res}(g;0) = \left[\frac{1}{24}\frac{d^4}{dz^4}(z^4 g(z,u|\tau))\right]_{z=0} = 0. \tag{2.13}$$

On the other hand, by logarithmically differentiating $z^4 g(z,u|\tau)$, we find that

$$\frac{d^4}{dz^4}(z^4 g(z,u|\tau)) = z^4 g(z,u|\tau)(\xi^3(z,u|\tau) + 3\xi(z,u|\tau)\xi'(z,u|\tau) + \xi''(z,u|\tau)),$$

where

$$\begin{aligned}\xi(z,u|\tau) = {} & \frac{4}{z} - 4\frac{\vartheta_1'}{\vartheta_1}(z|3\tau) + 2\frac{\vartheta_1'}{\vartheta_1}(z-\pi\tau|3\tau) \\ & + \frac{\vartheta_1'}{\vartheta_1}(z+u+\pi\tau|3\tau) + \frac{\vartheta_1'}{\vartheta_1}(z-u+\pi\tau|3\tau).\end{aligned} \tag{2.14}$$

Since

$$\lim_{z\to 0}(z^4 g(z,u|\tau)) = -\frac{\vartheta_1(u+\pi\tau|3\tau)\vartheta_1(u-\pi\tau|3\tau)\vartheta_1^2(\pi\tau|3\tau)}{\vartheta_1'(0|3\tau)^4} \neq 0,$$

we conclude from (2.13) that

$$\xi^3(0,u|\tau) + 3\xi(0,u|\tau)\xi'(0,u|\tau) + \xi''(0,u|\tau) = 0. \tag{2.15}$$

From (2.8) and (2.14), we find that

$$\xi(z,u|\tau) = 2\frac{\vartheta_1'}{\vartheta_1}(z-\pi\tau|3\tau) + \frac{\vartheta_1'}{\vartheta_1}(z+u+\pi\tau|3\tau) + \frac{\vartheta_1'}{\vartheta_1}(z-u+\pi\tau|3\tau) + \frac{4}{3}P(q^3)z + O(z^3). \tag{2.16}$$

Substituting $z=0$ into (2.16) and using (2.4) and (2.9), we find that

$$\xi(0,u|\tau) = -4iL(u|\tau). \tag{2.17}$$

Next, differentiating both sides of (2.16) with respect to z, substituting $z=0$ and simplifying with (2.12), we deduce that

$$\xi'(0,u|\tau) = \tfrac{2}{3}(a^2(q) + P(q^3)) + M(u|\tau). \tag{2.18}$$

By Lemma 2.3, we find that from (2.18)

$$\xi'(0,u|\tau) = \tfrac{4}{3}a^2(q) + 8a(q)L(u|\tau) + 16L^2(u|\tau). \tag{2.19}$$

Finally, by differentiating (2.16) twice with respect to z and setting $z=0$, we conclude that

$$\xi''(0,u|\tau) = -2\left(\frac{\vartheta_1'}{\vartheta_1}\right)''(\pi\tau|3\tau) + \left(\frac{\vartheta_1'}{\vartheta_1}\right)''(u+\pi\tau|3\tau) - \left(\frac{\vartheta_1'}{\vartheta_1}\right)''(u-\pi\tau|3\tau). \tag{2.20}$$

Differentiating (2.2) twice with respect to u and substituting $u=0$, we deduce that

$$\left(\frac{\vartheta_1'}{\vartheta_1}\right)''(\pi\tau|3\tau) = 8i\mathscr{S}_2(q).$$

Furthermore, observing the fact that the expression we obtain from differentiating twice the function

$$\frac{\vartheta_1'}{\vartheta_1}(z+u+\pi\tau|3\tau) + \frac{\vartheta_1'}{\vartheta_1}(z-u+\pi\tau|3\tau)$$

with respect to z is the same as differentiating it twice with respect to u, we conclude that

$$\left(\frac{\vartheta_1'}{\vartheta_1}\right)''(u+\pi\tau|3\tau) - \left(\frac{\vartheta_1'}{\vartheta_1}\right)''(u-\pi\tau|3\tau) = -4i\frac{d^2L(u|\tau)}{du^2}.$$

Hence, we may rewrite (2.20) as

$$\xi''(0,u|\tau) = -16i\mathscr{S}_2(q) - 4i\frac{d^2L(u|\tau)}{du^2}. \tag{2.21}$$

Substituting (2.17), (2.19) and (2.21) into (2.15), we conclude the proof of Theorem 2.1.

Let

$$V(u|\tau) = z_3 + 6L(u|\tau). \tag{2.22}$$

Then the differential equation (2.1) is equivalent to the differential equation given in [3, (8.32)].

From Theorem 2.1, we obtain the following recurrence for $\mathscr{S}_{2k} := \mathscr{S}_{2k}(q)$ by comparing coefficients of u on both sides of (2.1):

Corollary 2.4. *Let k be any positive integer and*

$$\mathscr{S}_{2k} := \mathscr{S}_{2k}(q) = \sum_{n=1}^{\infty} \frac{n^{2k} q^n}{1 + q^n + q^{2n}}.$$

Then

$$\mathscr{S}_{2k+2} = z_3^2 \mathscr{S}_{2k} + 6z_3 \sum_{\substack{u+v=k \\ u,v\geqslant 1}} \frac{(2k)!}{(2u)!(2v)!} \mathscr{S}_{2u}\mathscr{S}_{2v} + 8 \sum_{\substack{u+v+w=k \\ u,v,w\geqslant 1}} \frac{(2k)!}{(2u)!(2v)!(2w)!} \mathscr{S}_{2u}\mathscr{S}_{2v}\mathscr{S}_{2w}.$$

Note that Corollary 2.4 shows that Table 2 can be extended indefinitely and that $\mathscr{S}_{2k}$ is a function of z_3 and x_3 for $k \geqslant 1$. This result is a cubic analogue of the well-known recurrence relations for the classical Eisenstein series [1, p. 13] and the functions $\mathscr{T}_{2k}$ [5].

We end this section with another proof of (2.1) and show how this differential equation is related to the Weiestrass $\wp$-function.

Using (2.2) and the fact that [14, p. 465]

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi|\tau) = \frac{\vartheta_1'}{\vartheta_1}(u|\tau) \quad \text{and} \quad \frac{\vartheta_1'}{\vartheta_1}(u+\pi\tau|\tau) = \frac{\vartheta_1'}{\vartheta_1}(u|\tau) - 2i,$$

we find that

$$\frac{\vartheta_1'}{\vartheta_1}(u+\pi\tau|3\tau) - \frac{\vartheta_1'}{\vartheta_1}(u-\pi\tau|3\tau)$$

is an even elliptic function with periods π and $3\pi\tau$.

Now, $\vartheta_1(z|\tau)$ has only one zero in the any period parallelogram. This implies that the function $L(u|\tau)$ has two simple poles in any period parallelogram. Now the expansion of $L(u|\tau)$ shows that it has a double zero at $u = 0$ and these are the only zeros since the number of zeros is equal to the number of poles for any elliptic functions in a period parallelogram. Using the fact that any even elliptic function is a rational function of $\wp$, we conclude that

$$\wp(u|3\tau) = \frac{A}{L(u|\tau)} + B$$

or

$$\wp(u|3\tau)L(u|\tau) = A + BL(u|\tau), \tag{2.23}$$

for some functions A and B independent of u.

It is known [13, (2.4)] that the expansion of $\wp(u|\tau)$ is of the form

$$\wp(u|\tau) = \csc^2 u - 8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\cos 2nu - \frac{1}{3}P(q),$$

where

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}.$$

Substituting this expansion of $\wp$ and the expansion of $L(u|\tau)$ into (2.23), we deduce that

$$\begin{aligned}&\left(\frac{1}{u^2} + \frac{1}{15}\left(1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}\right)u^2 + \cdots\right)\left(-2\mathscr{S}_2(q)u^2 + \frac{2}{3}\mathscr{S}_4(q)u^4 + \cdots\right)\\ &\quad = A + B\left(-2\mathscr{S}_2(q)u^2 + \frac{2}{3}\mathscr{S}_4(q)u^4 + \cdots\right).\end{aligned} \tag{2.24}$$

Comparing coefficients of both sides of (2.24), we conclude from Table 2 that

$$A = -2\mathscr{S}_2(q) = -\frac{2}{27}z_3^3x_3 \quad \text{and} \quad B = -\frac{\mathscr{S}_4(q)}{3\mathscr{S}_2(q)} = -\frac{z_3^2}{3}.$$

Hence, we obtain the identity

$$\wp(u|3\tau) = -\frac{2}{27}\frac{z_3^3x_3}{L(u|\tau)} - \frac{z_3^2}{3}. \tag{2.25}$$

We now give another proof of Theorem 2.1. Recall that the elliptic function $\wp$ satisfies the differential equation

$$\begin{aligned}(\wp'(u|3\tau))^2 &= 4\wp^3(u|3\tau) - \frac{4}{3}\left(1 + 240\sum_{n=1}^{\infty}\frac{n^3q^{3n}}{1-q^{3n}}\right)\wp(u|3\tau)\\ &\quad - \frac{8}{27}\left(1 - 504\sum_{n=1}^{\infty}\frac{n^5q^{3n}}{1-q^{3n}}\right).\end{aligned} \tag{2.26}$$

By Berndt et al. [3, Theorem 4.4], we may rewrite (2.26) as

$$(\wp'(u|3\tau))^2 = 4\wp^3(u|3\tau) - \frac{4}{3}z_3^4\left(1-\frac{8}{9}x_3\right)\wp(u|3\tau) - \frac{8}{27}z_3^6\left(1-\frac{4}{3}x_3+\frac{8}{27}x_3^2\right). \quad (2.27)$$

Substituting (2.25) into (2.27), we find that

$$\left(\frac{dL}{du}\right)^2 = -\frac{8}{27}z_3^3x_3L - 4z_3^2L^2 - 16z_3L^3 - 16L^4. \quad (2.28)$$

Differentiating with respect to u, we conclude the second proof of Theorem 2.1.

We now make a few remarks about (2.25). This should be viewed as a cubic analogue of the relation between $\wp$ and the Jacobian elliptic function $\mathrm{sn}^2(u,\mathbf{k})$ [14, p. 505, 22.351]. The identity in cite [p. 505, 22.351] Watson–Whittaker can also be written as

$$\wp(u|2\tau) = -\frac{z_4^4x_4}{32L_4(u|\tau)} - \frac{z_4^2}{3},$$

where

$$L_4(u|\tau) = \sum_{k=1}^{\infty}(-1)^k\mathscr{T}_{2k+2}(\sqrt{q})\frac{(2u)^{2k}}{(2k)!},$$

$$z_4^2 = \varphi^4(q) + 16q\psi^4(q^2)$$

and

$$\frac{1}{x_4} = 1 + \frac{1}{64q}\prod_{k=1}^{\infty}\left(\frac{1-q^k}{1-q^{2k}}\right)^{24}.$$

The functions z_4 and x_4 belong to the quartic theory of elliptic functions [4].

3. Proof of Theorem 1.2

Throughout this section, we let q to be a real number between 0 and 1. From (2.28), we find that

$$\left(\frac{dL}{du}\right)^2 = -\frac{4}{27}2z_3^3x_3L\left(1+\frac{27Lz_3^2+108L^2z_3+108L^3}{2z_3^3x_3}\right).$$

This prompted us to use the substitution

$$\sin^2 \Phi = 1 + \frac{27Lz_3^2 + 108L^2 z_3 + 108L^3}{2z_3^3 x_3} =: 1 + \Psi. \tag{3.1}$$

The validity of this substitution, namely the fact that $-1<\Psi<0$ for $0<u<\pi/2$ is shown in [3, pp. 4209–4211] and will not be repeated here. Our main aim is to illustrate the possible origin of this substitution. From (3.1), we find that

$$2 \sin \Phi \cos \Phi \frac{d\Phi}{du} = \frac{27}{2} \frac{(z_3 + 6L)(z_3 + 2L)}{z_3^3 x_3} \frac{dL}{du}.$$

Squaring both sides and using (2.28), we deduce that

$$\frac{d\Phi}{du} = -(z_3 + 6L). \tag{3.2}$$

Note that the right-hand side of the last equation is just V by (2.22). Substituting (2.22) into (3.1), we obtain the equation

$$4(1 - x_3 \cos^2 \Phi)S^3 - 3S - 1 = 0, \tag{3.3}$$

where

$$S = \frac{z_3}{V}.$$

The unique solution to the equation $4(1-y)T^3(y) - 3T(y) - 1 = 0$ with initial condition $T(0) = 1$ is

$$T = {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; y).$$

Hence, we deduce from (3.3) that

$$\frac{z_3}{V(u|\tau)} = {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; \tfrac{1}{2}; x_3 \cos^2 \Phi). \tag{3.4}$$

Substituting (3.4) back into (3.2), we conclude that

$$z_3 = -{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x_3 \cos^2 \Phi\right) \frac{d\Phi}{du}. \tag{3.5}$$

Hence,

$$z_3 = -\int_{\frac{\pi}{2}}^{\frac{\pi}{2} - \int_0^u V\,du} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; x_3 \cos^2\Phi\right) d\Phi.$$

Substituting $\theta = \frac{\pi}{2} - \Phi$, we deduce Theorem 1.2.

In a similar fashion, we can construct a quartic analogue of Theorems 1.1 and 1.2. We first note that the quartic version of (2.28) is

$$\left(\frac{dL_4}{du}\right)^2 = -\frac{L_4}{8}(z_4^4x_4 + 32z_4^2L_4 + 256L_4^2).$$

This prompted us to set

$$\sin^2 \Phi = 1 + \frac{32z_4^2L_4 + 256L_4^2}{z_4^4x_4}. \tag{3.6}$$

This implies, by direct computation, that

$$\frac{d\Phi}{du} = -\frac{z_4^2 + 16L_4}{\sqrt{8L_4 + z_4^2}}. \tag{3.7}$$

Setting $V_4 := z_4^2 + 8L_4$, we deduce from (3.6) that

$$\frac{V_4}{z_4^2} = \frac{1 + \sqrt{1 - x_4\cos^2\Phi}}{2},$$

which implies that

$$\frac{z_4^2 + 16L_4}{\sqrt{8L_4 + z_4^2}} = \frac{2V_4 - z_4^2}{\sqrt{V_4}} = \frac{z_4}{{}_2F_1(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; x_4\cos^2\Phi)}.$$

Substituting the last identity back into (3.7) and setting $\phi = \frac{\pi}{2} - \Phi$ we deduce that

$$\int_0^{\phi} {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; \frac{1}{2}; x_4\sin^2 t\right) dt = z_4u,$$

where

$$\phi = \int_0^u 2\sqrt{V_4(t|\tau)} - \frac{z_4^2}{\sqrt{V_4(t|\tau)}}\,dt,$$

and

$$V_4(u|\tau) = z_4^2 - 8\sum_{k=1}^{\infty}\frac{kq^k}{1-q^{2k}} + 8\sum_{k=1}^{\infty}\frac{kq^k}{1-q^{2k}}\cos 2ku.$$

By using the relation in [3, Theorem 9.11, 4, Theorem 2.9], we have

$$V_4(u|\tau) = 1 + 24\sum_{k=1}^{\infty}\frac{kq^k}{1+q^k} + L_4(u|\tau).$$

This integral should be compared to Shen's result [12, Theorem C]. We are unable to simplify further the expression for ϕ.

4. Behavior of $L(u|\tau)$ under the imaginary transformation $\tau \to -\frac{1}{3\tau}$

First, recall that the imaginary transformation for $\vartheta_1(z|\tau)$ [2, p. 339, (0.54)]:

$$\vartheta_1\left(\frac{z}{\tau}\Big| -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}e^{iz^2/(\pi\tau)}\vartheta_1(z|\tau). \tag{4.1}$$

Differentiating (4.1) logarithmically with respect to z, we deduce that

$$\frac{\vartheta_1'}{\vartheta_1}\left(\frac{z}{\tau}\Big| -\frac{1}{\tau}\right) = \frac{2iz}{\pi} + \tau\frac{\vartheta_1'}{\vartheta_1}(z|\tau). \tag{4.2}$$

Replacing z by $z+\frac{\pi}{3}$ in (4.2), we have

$$\frac{\vartheta_1'}{\vartheta_1}\left(\frac{z}{\tau}+\frac{\pi}{3\tau}\Big| -\frac{1}{\tau}\right) = \frac{2iz}{\pi} + \frac{2i}{3} + \tau\frac{\vartheta_1'}{\vartheta_1}\left(z+\frac{\pi}{3}\Big|\tau\right). \tag{4.3}$$

Next, we replace z by $-z$ in (4.3) and find that

$$\frac{\vartheta_1'}{\vartheta_1}\left(\frac{z}{\tau}-\frac{\pi}{3\tau}\Big| -\frac{1}{\tau}\right) = \frac{2iz}{\pi} - \frac{2i}{3} + \tau\frac{\vartheta_1'}{\vartheta_1}\left(z-\frac{\pi}{3}\Big|\tau\right). \tag{4.4}$$

Subtracting (4.4) from (4.3), we deduce that

$$\begin{aligned}\frac{\vartheta_1'}{\vartheta_1}&\left(\frac{z}{\tau}+\frac{\pi}{3\tau}\Big| -\frac{1}{\tau}\right) - \frac{\vartheta_1'}{\vartheta_1}\left(\frac{z}{\tau}-\frac{\pi}{3\tau}\Big| -\frac{1}{\tau}\right)\\ &= \frac{4i}{3} + \tau\left(\frac{\vartheta_1'}{\vartheta_1}\left(z+\frac{\pi}{3}\Big|\tau\right) - \frac{\vartheta_1'}{\vartheta_1}\left(z-\frac{\pi}{3}\Big|\tau\right)\right).\end{aligned} \tag{4.5}$$

Recall from (2.4) and (2.22) that

$$V(u|\tau) = -2 + \frac{3}{2}i\left(\frac{\vartheta_1'}{\vartheta_1}(z+\pi\tau|3\tau) - \frac{\vartheta_1'}{\vartheta_1}(z-\pi\tau|3\tau)\right). \tag{4.6}$$

Hence, (4.5) translates to a transformation formula for $V(u|\tau)$. Writing the right-hand side of (4.5) using [14, p. 489]

$$\frac{\vartheta_1'}{\vartheta_1}(z|\tau) = \cot(z) + 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz,$$

we deduce that

$$V\left(\frac{z}{\tau}\bigg|-\frac{1}{3\tau}\right)=-\sqrt{3}i\tau U(z|\tau), \tag{4.7}$$

where

$$\begin{aligned} U(z|\tau) &= \frac{\sqrt{3}}{2}\left(\frac{\vartheta_1'}{\vartheta_1}\left(z+\frac{\pi}{3}\bigg|\tau\right)-\frac{\vartheta_1'}{\vartheta_1}\left(z-\frac{\pi}{3}\bigg|\tau\right)\right) \\ &= \frac{\sqrt{3}}{2}\left(\cot\left(z+\frac{\pi}{3}\right)-\cot\left(z-\frac{\pi}{3}\right)\right) \\ &\quad + 6\sum_{n=1}^{\infty}\chi_3(n)\frac{q^n}{1-q^n}\cos 2nz, \end{aligned} \tag{4.8}$$

with $\chi_3(n)=(\frac{n}{3})$ being the Legendre symbol. Substituting (2.3) and the following expansion

$$\cot\left(z+\frac{\pi}{3}\right)-\cot\left(z-\frac{\pi}{3}\right)=2\sum_{k=0}^{\infty}\frac{\cot^{(2k)}(\frac{\pi}{3})}{(2k)!}z^{2k} \tag{4.9}$$

into (4.8) and simplifying, we find that

$$\frac{\vartheta_1'}{\vartheta_1}\left(z+\frac{\pi}{3}\bigg|\tau\right)-\frac{\vartheta_1'}{\vartheta_1}\left(z-\frac{\pi}{3}\bigg|\tau\right)=2\sum_{k=0}^{\infty}\frac{\cot^{(2k)}(\frac{\pi}{3})}{(2k)!}E_{2k}(\chi_3;q)z^{2k}, \tag{4.10}$$

where

$$E_{2k}(\chi_3;q)=1+(-1)^k\frac{2^{2k+1}\sqrt{3}}{\cot^{(2k)}\left(\frac{\pi}{3}\right)}\sum_{n=1}^{\infty}\chi_3(n)\frac{n^{2k}q^n}{1-q^n}. \tag{4.11}$$

Therefore, we have

$$\begin{aligned} U(z|\tau) &= \frac{\sqrt{3}}{2}\left(\frac{\vartheta_1'}{\vartheta_1}\left(z+\frac{\pi}{3}\bigg|\tau\right)-\frac{\vartheta_1'}{\vartheta_1}\left(z-\frac{\pi}{3}\bigg|\tau\right)\right) \\ &= \sqrt{3}\sum_{k=0}^{\infty}\frac{\cot^{(2k)}(\frac{\pi}{3})}{(2k)!}E_{2k}(\chi_3;q)z^{2k} \\ &= a(q)+\sqrt{3}\sum_{k=1}^{\infty}\frac{\cot^{(2k)}(\frac{\pi}{3})}{(2k)!}E_{2k}(\chi_3;q)z^{2k}, \end{aligned} \tag{4.12}$$

since [2, p. 346, Entry 1(v)]

$$a(q) = 1 + 6\sum_{n=1}^{\infty} \chi_3(n)\frac{q^n}{1-q^n}.$$

The first few $E_{2k}(\chi_3; q)$ for $k \geqslant 1$ are

$$E_2(\chi_3; q) = 1 - 9\sum_{n=1}^{\infty} \chi_3(n)\frac{n^2 q^n}{1-q^n},$$

$$E_4(\chi_3; q) = 1 + 3\sum_{n=1}^{\infty} \chi_3(n)\frac{n^4 q^n}{1-q^n},$$

$$E_6(\chi_3; q) = 1 - \frac{3}{7}\sum_{n=1}^{\infty} \chi_3(n)\frac{n^6 q^n}{1-q^n}.$$

These are clearly the cubic analogue of the classical Eisenstein series. The series $E_{2k}(\chi_3; q)$ were studied in [7], but the relation between $E_{2k}(\chi_3; q)$ and $S_{2k}(q)$ was not established there.

Recall from (2.22) that

$$V(z|\tau) = a(q) + 6\sum_{k=1}^{\infty} (-1)^k \frac{(2z)^{2k}}{(2k)!}\mathscr{S}_{2k}(q). \tag{4.13}$$

Using (4.12) and (4.13) in (4.7) and then equating the coefficients we deduce that

$$a(e^{-2\pi i/(3\tau)}) = -\sqrt{3}\tau i a(e^{2\pi i\tau}), \tag{4.14}$$

$$\mathscr{S}_{2k}(e^{-2\pi i/(3\tau)}) = (-1)^{k+1} i\frac{\tau^{2k+1}\cot^{(2k)}(\frac{\pi}{3})}{2^{2k+1}} E_{2k}(\chi_3; e^{2\pi i\tau}), \quad k \geqslant 1. \tag{4.15}$$

The transformation formula (4.14) is a special case of the formula given in [11, p. 205].

Identities (4.14) and (4.15) have interesting applications. For example, when $k = 1$ we have

$$\mathscr{S}_2(e^{-2\pi i/(3\tau)}) = i\frac{1}{8}\tau^3 \cot^{(2)}\left(\frac{\pi}{3}\right) E_2(\chi_3; e^{2\pi i\tau}). \tag{4.16}$$

From the first identity of Table 2, we have

$$\mathscr{S}_2(e^{-2\pi i/(3\tau)}) = \frac{1}{27}c^3(e^{-2\pi i/(3\tau)}) = \frac{1}{3\sqrt{3}} i\,\tau^3\, b^3(e^{2\pi i\tau}), \tag{4.17}$$

where

$$b(q) = \sum_{m,n=-\infty}^{\infty} \omega^{n-m} q^{n^2+mn+m^2}, \quad \omega = e^{2\pi i/3}.$$

The last equality of (4.17) follows from the η-product representations of $b(q)$ and $c(q)$ [3, Lemma 5.1] and the transformation formula

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau).$$

Therefore, we have

$$\frac{8}{3\sqrt{3}} b^3(e^{2\pi i\tau}) = \cot^{(2)}\left(\frac{\pi}{3}\right) E_2(\chi_3; e^{2\pi i\tau}). \tag{4.18}$$

We do not need to know in advance the value of $\cot^{(2)}(\frac{\pi}{3})$. By setting $q = 0$ in (4.18), we deduce that

$$\cot^{(2)}\left(\frac{\pi}{3}\right) = \frac{8}{3\sqrt{3}}. \tag{4.19}$$

Hence,

$$E_2(\chi_3; q) = 1 - 9 \sum_{n=1}^{\infty} \chi_3(n) \frac{n^2 q^n}{1-q^n} = b^3(q). \tag{4.20}$$

Using exactly the same idea, one can obtain a representation of $E_{2k}(\chi_3; q)$ in terms of x_3 and z_3 for any $k \geqslant 1$. This shows that there is a one to one correspondence between the identities associated with the triplets $(\mathscr{S}_{2k}(q), z_3^{2k+1}, x_3)$ and $(E_{2k}(\chi_3; q), z_3^{2k+1}, x_3)$.

Formula (4.7) is equivalent to the formula

$$L\left(\frac{z}{\tau}\middle| -\frac{1}{3\tau}\right) = -\frac{1}{2\sqrt{3}} \tau i H(z|\tau), \tag{4.21}$$

where

$$H(z|\tau) = \sqrt{3} \sum_{k=1}^{\infty} \frac{\cot^{(2k)}(\frac{\pi}{3})}{(2k)!} E_{2k}(\chi_3; q) z^{2k}.$$

Using the transformation formulas for $a(q), b(q), c(q)$ and the formula

$$\wp\left(\frac{z}{\tau}\middle| -\frac{1}{\tau}\right) = \tau^2 \wp(z|\tau),$$

we deduce from (2.25) that

$$\wp(z|\tau) = \frac{4}{3}\frac{z_3^3(1-x_3)}{H(z|\tau)} + z_3^2.$$

Remark. The generalization of (4.7) can be found in the excellent article by Kolberg [6, (2.20)].

5. Conclusion

It is clear that there are many identities associated with the cubic theory which are yet to be discovered. Functions such as $H(z|\tau)$ and $L(z|\tau)$ provides us with some insights in constructing cubic analogues of the Jacobian elliptic functions. However, the study of such functions is far from complete. For example, we are still unable to find a cubic analogue of the famous Landen's transformation formula [14, p. 507] despite the fact that we now have a better understanding of this theory.

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