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Representations of Integers as Sums of 32 Squares

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In memory of Robert A. Rankin

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Abstract. In this paper, we derive a new explicit formula for $r_{32}(n)$, where $r_k(n)$ is the number of representations of n as a sum of k squares. For a fixed integer k, our method can be used to derive explicit formulas for $r_{8k}(n)$. We conclude the paper with various conjectures that lead to explicit formulas for $r_{2k}(n)$, for any fixed positive integer $k > 4$.

Key words: Sums of squares, Eisenstein series

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1. Introduction

Following Ramanujan's notation, let

$$\varphi(q) := \sum_{k=-\infty}^{\infty} q^{k^2}$$

and

$$\psi(q) := \sum_{k=0}^{\infty} q^{k(k+1)/2},$$

where $|q| < 1$. It is immediate from the above definitions that if

$$\varphi^{2s}(q) = \sum_{n=0}^{\infty} r_{2s}(n) q^n$$

and

$$\psi^{2s}(q) = \sum_{n=0}^{\infty} t_{2s}(n) q^n,$$

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then $r_{2s}(n)$ and $t_{2s}(n)$ are the number of representations of n as a sum of $2s$ squares and $2s$ triangular numbers, respectively.

The famous Four Squares Theorem states that every positive integer is a sum of four squares and was first proved by Lagrange in 1772. Lagrange's proof, however, contains no further quantitative information about $r_4(n)$ besides the fact that $r_4(n) > 0$ for all $n \in \mathbb{N}$.

An explicit formula for $r_4(n)$ was first discovered by C.G.J. Jacobi around 1829. Using the identity,

$$\varphi^4(q) = 1 + 8\sum_{k=1}^{\infty} \frac{kq^k}{1+(-q)^k}, \tag{1.1}$$

he deduced that

$$r_4(n) = 8\sigma'(n),$$

where

$$\sigma'(n) = \sum_{\substack{d|n \\ d\not\equiv 0 \pmod 4}} d.$$

Jacobi's identity clearly implies that $r_4(n) > 0$ and his method suggests that the problem of finding explicit formulas for $r_{2s}(n)$ is equivalent to finding an analogue of (1.1) for $\varphi^{2s}(q)$.

For $s = 1$, 3, and 4, Jacobi proved that

$$\varphi^2(q) = 1 + 4\sum_{k=1}^{\infty}(-1)^{k-1}\frac{q^{2k-1}}{1-q^{2k-1}}, \tag{1.2}$$

$$\varphi^6(q) = 1 + 16\sum_{k=1}^{\infty}\frac{k^2q^k}{1+q^{2k}} + 4\sum_{k=1}^{\infty}(-1)^k\frac{(2k-1)^2q^{2k-1}}{1-q^{2k-1}}, \tag{1.3}$$

and

$$\varphi^8(q) = 1 + 16\sum_{k=1}^{\infty}\frac{k^3q^k}{1-(-q)^k}. \tag{1.4}$$

For $s > 4$, identities analogous to (1.1)–(1.4) are also known. The identity for $s = 5$ is given by Liouville as follows:

$$\varphi^{10}(q) = 1 + \frac{4}{5}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(2k-1)^4q^{2k-1}}{1-q^{2k-1}} + \frac{64}{5}\sum_{k=1}^{\infty}\frac{k^4q^k}{1+q^{2k}} + \frac{32}{5}q\varphi^2(q)\varphi^4(-q)\psi^4(q^2). \tag{1.5}$$

Liouville's identity (1.5) is the first instance where an even power of $\varphi(q)$ is expressed as a sum of a Generalized Lambert series[1] and a *cusp form*. This phenomenon turns out to be true for any $s > 4$ and was first established in Ramanujan's famous paper "*On Certain Arithmetical Functions*" [10]. In [11, Chapter 7], R. A. Rankin explains, using the theory of modular forms, a uniform way to construct identities associated with $\varphi^{2s}(q)$, $s > 1$.

Recently, S. Milne [8] revisited the problem of representing an integer as a sum of an even number of squares. He succeeded in expressing $\varphi^{2s}(q)$, where $s = 2n^2$ or $2n(n+1)$, as an $n \times n$ determinant and also a sum of determinants of certain Generalized Lambert series. His most elegant identity is perhaps the following:

$$\begin{aligned}\varphi^{24}(q) &= \frac{1}{3^2}\det\begin{vmatrix}16\mathcal{S}_4(q)+1 & 16\mathcal{S}_6(q)-2\\ 32\mathcal{S}_6(q)-4 & 32\mathcal{S}_8(q)+17\end{vmatrix}\\ &= 1+\frac{16}{9}(17\mathcal{S}_4(q)+8\mathcal{S}_6(q)+2\mathcal{S}_8(q))+\frac{512}{9}\big(\mathcal{S}_4(q)\mathcal{S}_8(q)-\mathcal{S}_6^2(q)\big), \end{aligned}\tag{1.6}$$

where

$$\mathcal{S}_{2s}(q)=\sum_{j=1}^{\infty}\frac{j^{2s-1}q^j}{1-(-q)^j}.$$

Milne's identity (1.6) differs from the classical formula $\varphi^{24}(q)$ discovered by Ramanujan [10], namely,

$$\varphi^{24}(q)=1+\frac{16}{691}\mathcal{S}_{12}(q)+\frac{33152}{691}qf^{24}(q)-\frac{65536}{691}q^2f^{24}(-q^2),$$

where

$$f(-q)=\prod_{k=1}^{\infty}(1-q^k).$$

One key advantage of Milne's representations is that his identities yield immediately the exact order of $r_{24}(n)$.

It is not an overstatement to say that the present paper is motivated by (1.6). We first observe that if we define the "normalization" of $\mathcal{S}_{2m}(q)$, $m = 2, 3, 4$ by

$$\begin{aligned}\mathcal{E}_4(q) &= 1+16\sum_{k=1}^{\infty}\frac{k^3q^k}{1-(-q)^k},\\ \mathcal{E}_6(q) &= 1-8\sum_{k=1}^{\infty}\frac{k^5q^k}{1-(-q)^k},\end{aligned}$$

and

$$\mathcal{E}_8(q) = 17 + 32\sum_{k=1}^{\infty} \frac{k^7 q^k}{1-(-q)^k},$$

then we can rewrite (1.6) as

$$\varphi^{24}(q) = \frac{1}{9}\left\{\mathcal{E}_4(q)\mathcal{E}_8(q) - 8\mathcal{E}_6^2(q)\right\}. \tag{1.7}$$

This example suggests that perhaps $\varphi^{8t}(q)$ is a linear combination (over $\mathbb{Q}$) of $\mathcal{E}_{2u}(q)\,\mathcal{E}_{2v}(q)$, such that $u + v = 2t$, with $u, v \geq 2$. We record this observation as follow:

Conjecture 1.1. Let

$$\sec^2 u = \sum_{k=0}^{\infty} a_{2k} \frac{u^{2k}}{2k!},$$

and suppose that

$$\frac{A_k}{B_k} = \frac{a_{2k}}{2^{2k+3}},$$

where $k > 1$ and $\gcd(A_k, B_k) = 1$. Then for any positive integer $t \geq 1$,

$$\varphi^{8t+8}(q) = \sum_{l=1}^{t} \alpha_l \mathcal{E}_{4t+2-2l}(q)\mathcal{E}_{2l+2}(q),$$

where

$$\mathcal{E}_{2k+2}(q) = A_k - (-1)^k B_k \sum_{j=1}^{\infty} \frac{j^{2k+1} q^j}{1-(-q)^j}, \quad \textit{and} \quad \alpha_l \in \mathbb{Q}.$$

Conjecture 1.1 can be shown to be equivalent [9] to the following conjecture associated with $\psi(q)$:

Conjecture 1.2. For $k > 1$, let

$$\mathcal{T}_{2k}(q) := \sum_{n=1}^{\infty} \frac{n^{2k-1} q^{2n}}{1-q^{4n}}.$$

Then for any positive integer $t > 1$,

$$q^{2t}\psi^{8t}(q^2) = \sum_{\substack{m+n=2t \\ m \geq n > 1}} \alpha_{m,n} \mathcal{T}_{2m}(q)\mathcal{T}_{2n}(q).$$

where $\alpha_{m,n} \in \mathbb{Q}$.

In this short note, we will give a proof of (1.7) by proving that

$$q^6\psi^{24}(q^2) = \frac{1}{72}\big(\mathcal{T}_8(q)\mathcal{T}_4(q) - \mathcal{T}_6^2(q)\big). \tag{1.8}$$

We will also establish the new formula

$$q^8\psi^{32}(q^2) = \frac{1}{75600}\left(-\frac{25}{4}\mathcal{T}_{10}(q)\mathcal{T}_6(q) + \frac{21}{4}\mathcal{T}_8^2(q) + \mathcal{T}_4(q)\mathcal{T}_{12}(q)\right). \tag{1.9}$$

Using the method as illustrated in [9, (2.12)], we deduce the new formula

$$\varphi^{32}(q) = \frac{1}{4725}\big\{-400\mathcal{E}_6(q)\mathcal{E}_{10}(q) + 16\mathcal{E}_{12}(q)\mathcal{E}_4(q) + 21\mathcal{E}_8^2(q)\big\}.$$

2. Modular properties of $\mathcal{T}_{2n}(q)$

As usual, let $\Gamma(1) = SL_2(\mathbb{Z})$ and define

$$\Gamma_0(2) := \left\{S \in \Gamma(1) : S \equiv \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} \pmod 2\right\}.$$

Let $\{\Gamma_0(2), 4t, 1\}$ be the space of entire modular forms of weight $4t$ with multiplier system 1. We first show that

Lemma 2.1. *If $u + v = 2t$ then*

$$\mathcal{T}_{2u}(q)\mathcal{T}_{2v}(q) \in \{\Gamma_0(2), 4t, 1\}.$$

Proof: First, note that

$$\mathcal{T}_{2m}(q) = \frac{(2m-1)!}{2(2\pi i)^{2m}}\{G_{2m}(\tau) - G_{2m}(2\tau)\},$$

where

$$G_{2m}(\tau) = 2\zeta(2m) + \frac{2(2\pi i)^{2m}}{(2m-1)!}\sum_{n=1}^{\infty}\frac{n^{2m-1}q^n}{1-q^n}$$

is the classical Eisenstein series. Let $S = \begin{pmatrix}a & b\\ 2c & d\end{pmatrix} \in \Gamma_0\,(2)$. Then

$$\begin{aligned}G_{2m}(S\tau) - G_{2m}(2S\tau) &= (2c\tau+d)^{2m}G_{2m}(\tau) - G_{2m}\left(\frac{2a\tau+2b}{2c\tau+d}\right)\\ &= (2c\tau+d)^{2m}G_{2m}(\tau) - G_{2m}\left(\frac{a(2\tau)+2b}{c(2\tau)+d}\right)\\ &= (2c\tau+d)^{2m}(G_{2m}(\tau) - G_{2m}(2\tau)).\end{aligned}$$

Hence,

$$\mathcal{T}_{2u}(q)\mathcal{T}_{2v}(q) \in \{\Gamma_0(2), 4t, 1\}. \qquad \square$$

It is known [11, p. 218] that the function $z^4 := \varphi^8(q)$ is in $\{\Gamma(2), 4, 1\}$, where

$$\Gamma(2) := \left\{ S \in \Gamma(1) : S \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}.$$

Since $\mathcal{T}_{2u}(q)\mathcal{T}_{2v}(q) \in \{\Gamma_0(2), 4t, 1\}$, $\mathcal{T}_{2u}(q)\mathcal{T}_{2v}(q) \in \{\Gamma(2), 4t, 1\}$. Hence,

$$\frac{\mathcal{T}_{2u}(q)\mathcal{T}_{2v}(q)}{z^{4t}} \in \{\Gamma(2), 0, 1\},$$

since z^4 is never zero on Im $\tau > 0$. It is well known that elements in $\{\Gamma(2), 0, 1\}$ are generated by

$$x := 16q\frac{\psi^4(q^2)}{\varphi^4(q)}.$$

Therefore, we have

Lemma 2.2.

(a) $\mathcal{T}_4(q) = z^4(\frac{x}{16})^2$,
(b) $\mathcal{T}_6(q) = z^6(\frac{x}{16})^2(1 - \frac{x}{2})$,
(c) $\mathcal{T}_8(q) = z^8(\frac{x}{16})^2(1 - x + \frac{17}{32}x^2)$,
(d) $\mathcal{T}_{10}(q) = z^{10}(\frac{x}{16})^2(1 - \frac{x}{2})(1 - x + \frac{31}{16}x^2)$,
(e) $\mathcal{T}_{12}(q) = z^{12}(\frac{x}{16})^2(1 - 2x + \frac{291}{32}x^2 - \frac{259}{32}x^3 + \frac{691}{256}x^4)$.

Remark. Lemma 2.2(a)–(d) can also be found in [2, p. 132, Entry 15(v)–(viii)]. We also mention here that in [3], B.C. Berndt gave a proof of the sum of six triangular numbers using tables of identities similar to Lemma 2.2. Our method of deriving identities for sums of squares and triangular numbers from Lemma 2.2 is motivated by his approach.

By considering the parametrizations in the above lemma, we conclude immediately that for $t = 3$ and 4, the elements in $\mathcal{C}_t := \{\mathcal{T}_{2u}(q)\mathcal{T}_{2v}(q) \mid u + v = 2t\}$ are linearly independent modular forms in $\{\Gamma_0(2), 4t, 1\}$. Note that the number of elements in $\mathcal{C}_t$ is $t - 1$, which is exactly the dimension of the space of modular forms with q-expansion beginning with q^{2l}, $l \geq 2$. [11, p. 222]. Hence, every modular form with q-expansion beginning with $q^{2l}, l \geq 2$, can be written as a linear combination of the elements in $\mathcal{C}_t$. This completes the proof of (1.8) and (1.9).

We remark here that the linear independence of the elements in $\mathcal{C}_t$ for $t = 3$ and 4 can be verified directly by considering the first few terms of their series expansions. The purpose of stating Lemma 2.2 is to indicate that the proof of Conjecture 1.2 may be derived if one can prove the linear independence of the products of polynomials which appear in the parametrizations of $\mathcal{T}_{2k}(q)$.

If we set $\mathcal{C}_t' := \mathcal{C}_t \cup \{\mathcal{T}_{4t}(q)\}$, then we see this gives a new basis for the space of modular forms with q-expansion beginning with $q^{2l}, l \geq 1$. An application of this observation in the case $t = 3$ gives the following new identity:

$$q^2 \prod_{n=1}^{\infty}(1-q^{2n})^{24} = \mathcal{T}_{12}(q) - \frac{13112}{9}\mathcal{T}_6^2(q) - \frac{5536}{9}\mathcal{T}_4(q)\mathcal{T}_8(q).$$

This yields immediately the following congruence due to Bambah, Chowla and Gupta [1]:

$$\tau(n) \equiv \begin{cases} 0 \pmod 8 & \text{if } n \text{ is even} \\ \sigma_{11}(n) \equiv \sigma(n) \pmod 8 & \text{otherwise.} \end{cases}$$

We conclude this section with a formula which allows us to express $\mathcal{T}_{2m}(q)$ in terms of z and x for any $m \geq 2$. This will allow us to extend the table in Lemma 2.2.

Theorem 2.3. *Let*

$$\mathcal{T}_2(q) = 1 + 24\sum_{j=1}^{\infty}\frac{jq^{2j}}{1+q^{2j}}.$$

Then for $n \geq 0$,

$$\mathcal{T}_{2n+8}(q) = \mathcal{T}_2(q)\mathcal{T}_{2n+6}(q) + 12\sum_{j=0}^{n}\binom{2n+4}{2j+2}\mathcal{T}_{2j+4}(q)\mathcal{T}_{2n-2j+4}(q). \tag{2.1}$$

Proof: It is known that [8, (2.14)]

$$\mathrm{sn}^2(u,\mathbf{k}) = \frac{K-E}{\mathbf{k}^2K} - \frac{2\pi^2}{\mathbf{k}^2K^2}\sum_{j=1}^{\infty}\frac{jq^j}{1-q^{2j}}\cos\frac{2ju}{z}, \tag{2.2}$$

where

$$\begin{aligned} K &= K(\mathbf{k}) = \frac{\pi}{2}{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\mathbf{k}^2\right), \\ E &= \frac{\pi}{2}{}_2F_1\left(\frac{1}{2};-\frac{1}{2};1;\mathbf{k}^2\right), \\ q &= e^{-\pi K(\sqrt{1-\mathbf{k}^2})/K(\mathbf{k})} \end{aligned}$$

and

$$z = 2K/\pi. \tag{2.3}$$

First, write

$$\sum_{j=1}^{\infty}\frac{jq^j}{1-q^{2j}}=\sum_{j=1}^{\infty}\frac{jq^j}{1+q^j}+\sum_{j=1}^{\infty}\frac{jq^{2j}}{1-q^{2j}}. \tag{2.4}$$

From [2, Entry 13(viii)] and [4, p. 164, Ex. 15(a)], we find respectively that

$$1+24\sum_{j=1}^{\infty}\frac{jq^j}{1+q^j}=z^2(1+\mathbf{k}^2), \tag{2.5}$$

and

$$1-24\sum_{j=1}^{\infty}\frac{jq^{2j}}{1-q^{2j}}=z^2\left(3\frac{E}{K}-2+\mathbf{k}^2\right), \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), we deduce that

$$\sum_{j=1}^{\infty}\frac{jq^j}{1-q^{2j}}=\frac{z^2}{8}\left(1-\frac{E}{K}\right)$$

and this implies that

$$\frac{K-E}{\mathbf{k}^2K}=\frac{2\pi^2}{\mathbf{k}^2K^2}\sum_{j=1}^{\infty}\frac{jq^j}{1-q^{2j}},$$

by (2.3). Hence, we may define $D_{2n+2}(\mathbf{k})$ by writing

$$\mathrm{sn}^2(u,\mathbf{k})=\sum_{n=0}^{\infty}D_{2n+2}(\mathbf{k})\frac{u^{2n+2}}{(2n+2)!}. \tag{2.7}$$

This is the representation of $\mathrm{sn}^2(u,\mathbf{k})$ that Brillhart and Lomont used in [7, p. 119, (8.26a)].

Comparing the coefficients of u^{2n}, $n\geq 2$ in (2.2) and (2.7), we conclude that

$$\sum_{j=1}^{\infty}\frac{j^{2n+1}q^j}{1-q^{2j}}=(-1)^{n+1}2^{-2n-3}z^{2n+2}D_{2n}(\mathbf{k})\mathbf{k}^2. \tag{2.8}$$

From [7, p. 123] and [7, p. 51, (4.33)], we find that

$$D_{2n+6}=-4(\mathbf{k}^2+1)D_{2n+4}+6\mathbf{k}^2\sum_{j=0}^{n}\binom{2n+4}{2j+2}D_{2k+2}D_{2n-2j+2}, \tag{2.9}$$

where $D_{2m} := D_{2m}(\mathbf{k})$. Substituting (2.5) and (2.8) into (2.9), and replacing q by q^2, we deduce (2.1). □

Remarks

1. The first few cases of (2.1) are used by Liu recently [6] to derive new proofs of the classical formulas for the sums of 8, 12, 16,20 and 24 squares.
2. We mention here another proof of (2.1). From the well-known differential equation satisfied by $U = \mathrm{sn}(u, \mathbf{k})$ [5, p. 121], namely,

$$\left(\frac{dU}{du}\right)^2 = (1 - U^2)(1 - \mathbf{k}^2 U^2),$$

we find that

$$4U^2\left(\frac{dU}{du}\right)^2 = 4U^2(1 - U^2)(1 - \mathbf{k}^2 U^2).$$

This implies that

$$\left(\frac{dV}{du}\right)^2 = 4V(1 - V)(1 - \mathbf{k}^2 V),$$

with $V = U^2 = \mathrm{sn}^2(u, \mathbf{k})$. Differentiating the last identity with respect to u again using (2.7), we deduce (2.9), and hence (2.1).

3. Conjectures for sums of $8m + 2$, $8m + 4$ and $8m + 6$ squares

We conclude this paper with conjectures associated with the formulas for sums of $8m + 2$, $8m + 4$ and $8m + 6$ squares. These are all analogues of Conjecture 1.1.

Conjecture 3.1. Let A_k, B_k and $\mathcal{E}_{2k}(q)$ be as defined in Conjecture 1.1. Then for any positive integer $t \geq 1$,

$$\varphi^{8t+4}(q) = \sum_{l=1}^{t} \beta_l \mathcal{E}_{4t+2-2l}(q)\mathcal{F}_{2l}(q),$$

where

$$\mathcal{F}_{2k+2}(q) = A_k + (-1)^k B_k \sum_{j=1}^{\infty} \frac{j^{2k+1} q^j}{1 + (-q)^j}, \quad \text{and} \quad \beta_l \in \mathbb{Q}.$$

Conjecture 3.2. Let $\mathcal{F}_{2k}(q)$ and $\mathcal{E}_{2k}(q)$ be as defined in Conjecture 3.1 and Conjecture 1.1 respectively. Set

$$\sec u = \sum_{k=0}^{\infty} C_k \frac{u^{2k}}{(2k)!}$$

and let

$$\mathcal{G}_{2k+1}(q) = C_k - (-1)^k 2^{2k+2} \sum_{j=1}^{\infty} \frac{j^{2k} q^j}{1+q^{2j}} - (-1)^k 4 \sum_{j=1}^{\infty} \frac{(-1)^j (2j-1)^{2k} q^{2j-1}}{1-q^{2j-1}}.$$

Then for any positive integer $t \geq 1$,

$$\varphi^{8t+2}(q) = \sum_{l=1}^{t} \gamma_l \mathcal{G}_{2l+1}(q) \mathcal{F}_{4t-2l}(q),$$

and

$$\varphi^{8t+6}(q) = \sum_{l=1}^{t} \delta_l \mathcal{G}_{2l+1}(q) \mathcal{E}_{4t+2-2l}(q),$$

where $\gamma_l, \delta_l \in \mathbb{Q}$.

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Note

1. Lambert series are usually defined as series of the form $\sum_{k=1}^{\infty} a_k \frac{q^k}{1-q^k}$, where $a_k(q)$ is a polynomial in q. We shall refer to the linear combinations of Lambert series as Generalized Lambert series.

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