



# Domb’s numbers and Ramanujan–Sato type series for $1/\pi$

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## Abstract

In this article, we construct a general series for  $\frac{1}{\pi}$ . We indicate that Ramanujan’s  $\frac{1}{\pi}$ -series are all special cases of this general series and we end the paper with a new class of  $\frac{1}{\pi}$ -series. Our work is motivated by series recently discovered by Takeshi Sato.

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## 1. Introduction

In [12], Ramanujan recorded a total of 17 series for  $1/\pi$ , one of which is

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} \frac{(6k+1) \left(\frac{1}{2}\right)_k^3}{(k!)^3 4^k}, \tag{1.1}$$

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where

$$(a)_k = (a)(a + 1) \cdots (a + k - 1).$$

Ramanujan did not indicate how he arrived at these series but instead hinted that some of these series belonged to what are now known, after the work of Berndt et al. [2], as “the theories of elliptic functions to alternative bases”.

Ramanujan’s series for  $1/\pi$  were not extensively studied until around 1987. The Chudnovskys succeeded in extending Ramanujan’s list of series and derived the following series which they used to compute over a billion digits of  $\pi$ :

$$\frac{640320^{3/2}}{\pi} = 12 \sum_{k=0}^{\infty} (-1)^k \frac{(6k)!}{(k!)^3(3k)!} \frac{(13591409 + 545140134k)}{640320^{3k}}. \tag{1.2}$$

For a recent discussion of the Chudnovskys’ series, see [3].

In [6], the Borweins provided rigorous proofs of Ramanujan’s series for the first time and derived many new series for  $1/\pi$ . Both the Borweins and the Chudnovskys admitted that Clausen’s identity, namely, [6, p. 188]

$$\left( {}_2F_1\left(a, b; a + b + \frac{1}{2}; z\right) \right)^2 = {}_3F_2\left(2a, 2b, a + b; a + b + \frac{1}{2}, 2a + 2b; z\right),$$

plays an important role in their derivations of such series.

In this article, we will derive a general series for  $1/\pi$  without using Clausen’s identity. We will show that all the existing series for  $1/\pi$  are special cases of this general series. We will then specialize our series further to derive new classes of series for  $1/\pi$ . In particular, we will show that

$$\frac{8}{\sqrt{3}\pi} = \sum_{k=0}^{\infty} \sum_{j=0}^k \binom{k}{j}^2 \binom{2(k-j)}{k-j} \binom{2j}{j} \left(\frac{1}{64}\right)^k (1 + 5k). \tag{1.3}$$

Our work is motivated by the following series discovered recently by Takeshi Sato [13]:<sup>2</sup>

$$\begin{aligned} & \frac{1}{\pi} \frac{\sqrt{15}}{120(4\sqrt{5} - 9)} \\ &= \sum_{k=0}^{\infty} \sum_{j=0}^k \binom{k}{j}^2 \binom{k+j}{j}^2 \left(\frac{1}{2} - \frac{3}{20}\sqrt{5} + k\right) \left(\frac{\sqrt{5} - 1}{2}\right)^{12k}. \end{aligned}$$

The above series shows that Clausen’s formula is probably not needed in the derivation of Ramanujan’s series since there are no Clausen-type transformation

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<sup>2</sup>The authors are grateful to S. Kanemitsu for sending the abstract of Sato’s talk which contains several new series for  $1/\pi$ .

formula for series such as

$$\sum_{k=0}^{\infty} \sum_{j=0}^k \binom{k}{j}^2 \binom{k+j}{j}^2 x^k.$$

## 2. A general series for $1/\pi$

Suppose the function  $Z(q)$  satisfies the relation

$$rZ(e^{-2\pi\sqrt{r/s}}) = Z(e^{-2\pi/\sqrt{rs}}). \quad (2.1)$$

By differentiating (2.1) with respect to  $r$  and simplifying, we find that

$$\frac{\sqrt{rs}}{\pi} = e^{-2\pi/\sqrt{rs}} \frac{Z'(e^{-2\pi/\sqrt{rs}})}{Z(e^{-2\pi/\sqrt{rs}})} + re^{-2\pi\sqrt{r/s}} \frac{Z'(e^{-2\pi\sqrt{r/s}})}{Z(e^{-2\pi\sqrt{r/s}})}, \quad (2.2)$$

where

$$Z'(q) = \frac{dZ(q)}{dq}.$$

Next, set

$$M_N(q) = \frac{Z(q)}{Z(q^N)}. \quad (2.3)$$

By differentiating (2.3) with respect to  $q$ , we find that

$$\frac{M'_N(q)}{M_N(q)} = \frac{Z'(q)}{Z(q)} - Nq^{N-1} \frac{Z'(q^N)}{Z(q^N)},$$

which implies that

$$q \frac{M'_N(q)}{M_N(q)} = q \frac{Z'(q)}{Z(q)} - Nq^N \frac{Z'(q^N)}{Z(q^N)}. \quad (2.4)$$

Substituting  $q = e^{-2\pi/\sqrt{Ns}}$  into (2.4) and using (2.1), we find that

$$\frac{e^{-2\pi/\sqrt{Ns}} M'_N(e^{-2\pi/\sqrt{Ns}})}{N} = e^{-2\pi/\sqrt{Ns}} \frac{Z'(e^{-2\pi/\sqrt{Ns}})}{Z(e^{-2\pi/\sqrt{Ns}})} - Ne^{-2\pi\sqrt{N/s}} \frac{Z'(e^{-2\pi\sqrt{N/s}})}{Z(e^{-2\pi\sqrt{N/s}})}. \quad (2.5)$$

Solving (2.5) and (2.2) with  $r = N$ , we conclude that

$$\frac{\sqrt{Ns}}{\pi} = 2Ne^{-2\pi\sqrt{N/s}} \frac{Z'(e^{-2\pi\sqrt{N/s}})}{Z(e^{-2\pi\sqrt{N/s}})} + \frac{e^{-2\pi/\sqrt{Ns}} M'_N(e^{-2\pi/\sqrt{Ns}})}{N}. \tag{2.6}$$

At this stage, we suppose that there exist functions  $X(q)$  and  $U(q)$  such that

$$Z(q) = \sum_{k=0}^{\infty} A_k X^k(q), \quad A_k \in \mathbf{Q} \tag{2.7}$$

and

$$q \frac{dX(q)}{dq} = U(q)X(q)Z(q). \tag{2.8}$$

Relations (2.3) and (2.7) also imply that  $M_N$  may be expressed as a function of two variables, namely,

$$M_N = M_N(X(q), X(q^N)). \tag{2.9}$$

We may also assume that  $X(q)$  and  $X(q^N)$  satisfy a polynomial relation and hence, using (2.8), (2.1) and (2.7), we deduce that

$$\begin{aligned} & \frac{e^{-2\pi/\sqrt{Ns}} M'_N(e^{-2\pi/\sqrt{Ns}})}{N} \\ &= \frac{1}{N} q \frac{dX(q)}{dq} \frac{dM_N(X(q), X(q^N))}{dX(q)} \Big|_{q=e^{-2\pi/\sqrt{Ns}}} \\ &= \frac{1}{N} U(e^{-2\pi/\sqrt{Ns}}) X(e^{-2\pi/\sqrt{Ns}}) Z(e^{-2\pi/\sqrt{Ns}}) \frac{dM_N(X(q), X(q^N))}{dX(q)} \Big|_{q=e^{-2\pi/\sqrt{Ns}}} \\ &= U(e^{-2\pi/\sqrt{Ns}}) X(e^{-2\pi/\sqrt{Ns}}) Z(e^{-2\pi\sqrt{N/s}}) \frac{dM_N(X(q), X(q^N))}{dX(q)} \Big|_{q=e^{-2\pi/\sqrt{Ns}}} \\ &= U(e^{-2\pi/\sqrt{Ns}}) X(e^{-2\pi/\sqrt{Ns}}) \frac{dM_N(X(q), X(q^N))}{dX(q)} \Big|_{q=e^{-2\pi/\sqrt{Ns}}} \\ & \quad \times \sum_{k=0}^{\infty} A_k X^k(e^{-2\pi\sqrt{N/s}}). \end{aligned} \tag{2.10}$$

This simplifies the second term on the right-hand side of (2.6). Next, by (2.7) and (2.8), we find that

$$\begin{aligned} qZ'(q) &= \sum_{k=0}^{\infty} kA_k X^{k-1}(q)qX'(q) \\ &= \sum_{k=0}^{\infty} kA_k X^{k-1}(q)U(q)X(q)Z(q). \end{aligned}$$

Hence,

$$q \frac{Z'(q)}{Z(q)} = U(q) \sum_{k=0}^{\infty} k A_k X^k(q). \tag{2.11}$$

Substituting  $q = e^{-2\pi\sqrt{N/s}}$  into (2.11) and combining with (2.10) and (2.6), we deduce the following result:

**Theorem 2.1.** *Suppose  $Z(q)$  satisfies (2.1) and let  $M_N(q)$ ,  $X(q)$ ,  $A_k$  and  $U(q)$  be defined as in (2.3), (2.7) and (2.8). Then*

$$\sqrt{\frac{s}{N}} \frac{1}{2\pi} = \sum_{k=0}^{\infty} (b_N k + a_N) A_k X_N^k, \tag{2.12}$$

where

$$a_N = U(e^{-2\pi/\sqrt{Ns}}) \frac{X(e^{-2\pi/\sqrt{Ns}})}{2N} \frac{dM_N}{dX(q)}(X(q), X(q^N)) \Big|_{q=e^{-2\pi/\sqrt{Ns}}},$$

$$b_N = U(e^{-2\pi\sqrt{N/s}}),$$

and

$$X_N = X(e^{-2\pi\sqrt{N/s}}).$$

We end this section by listing  $s$ ,  $x$ ,  $Z$ ,  $U$  and  $A_k$  which correspond to Ramanujan’s series for  $1/\pi$ . In Table 1,  $x$  is given by the relation

$$X = 4x(1 - x),$$

$$f(-q) = \prod_{k=1}^{\infty} (1 - q^k),$$

and

$$j(q) = 1728 \frac{Q^3}{q f^{24}(-q)},$$

where

$$Q := 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1 - q^k}.$$

In the next section, we will illustrate how we derive the fourth class of series without using Clausen’s identity. We will then give a new proof of (1.1).

Table 1

$s$	$x$	$Z$	$U$	$A_k$
1	$\frac{1}{2} \left( 1 - \sqrt{1 - \frac{1728}{j(q)}} \right)$	$\sqrt{Q}$	$1 - 2x$	$\frac{(\frac{1}{2})_k (\frac{1}{6})_k (\frac{5}{6})_k}{(k!)^3}$
2	$\left( 1 + \frac{1}{64q} \left( \frac{f(-q)}{f(-q^2)} \right)^{24} \right)^{-1}$	$\frac{f^8(-q)}{f^4(-q^2)} \sqrt{1 + 64q \left( \frac{f(-q^2)}{f(-q)} \right)^{24}}$	$1 - 2x$	$\frac{(\frac{1}{2})_k (\frac{1}{4})_k (\frac{3}{4})_k}{(k!)^3}$
3	$\left( 1 + \frac{1}{27q} \left( \frac{f(-q)}{f(-q^3)} \right)^{12} \right)^{-1}$	$\left( \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \right)^2$	$1 - 2x$	$\frac{(\frac{1}{2})_k (\frac{1}{3})_k (\frac{2}{3})_k}{(k!)^3}$
4	$\left( 1 + \frac{1}{16q} \left( \frac{f(-q)}{f(-q^4)} \right)^8 \right)^{-1}$	$\left( \sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} \right)^2$	$1 - 2x$	$\frac{(\frac{1}{2})_k^3}{(k!)^3}$

**3. Ramanujan’s series for  $1/\pi$  associated with the Jacobian theta functions**

Let  $Z(q) = \varphi^4(q)$  where

$$\varphi(q) = \sum_{k=-\infty}^{\infty} q^{k^2}.$$

Note that  $Z(q)$  satisfies the transformation formula [1, p. 43, Entry 27(i)]

$$NZ(e^{-\pi\sqrt{N}}) = Z(e^{-\pi/\sqrt{N}}), \tag{3.1}$$

and hence,  $s = 4$ .

It is known that if<sup>3</sup>

$$x = x(q) = 16q \frac{\psi^4(q^2)}{\varphi^4(q)}, \tag{3.2}$$

with

$$\psi(q) = \sum_{k=0}^{\infty} q^{k(k+1)/2},$$

then  $z = z(q) = \varphi^2(q)$  and  $x$  satisfy the differential equation [15, p. 54, (3.83)]

$$x(1-x) \frac{d^2z}{dx^2} + (1-2x) \frac{dz}{dx} - \frac{z}{4} = 0. \tag{3.3}$$

Substituting  $z = \sqrt{Z}$  and  $X = 4x(1-x)$  into (3.3), we deduce that<sup>4</sup>

$$\mathfrak{g}_X^3(Z) = X \left( \mathfrak{g}_X + \frac{1}{2} \right)^3 (Z), \tag{3.4}$$

<sup>3</sup>Our  $x$  given here is equal to the  $x$  given in the Table 1.

<sup>4</sup>The calculations are tedious but straightforward.

where the operator

$$\vartheta_X = X \frac{d}{dX}.$$

If

$$Z = \sum_{k=0}^{\infty} A_k X^k,$$

then (3.4) yields a recurrence relation satisfied by  $A_k$  and we deduce immediately that

$$A_k = \frac{\left(\frac{1}{2}\right)_k^3}{(k!)^3}.$$

In other words, we find that

$$Z = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1; X\right). \quad (3.5)$$

It is also known that [1, p. 120, Entry 9(i)]

$$q \frac{dX}{dq} = ZX(1 - 2x), \quad (3.6)$$

which means that

$$U(q) = 1 - 2x(q).$$

By (3.6), we find that if  $Y = X(q^N)$  and  $y = x(q^N)$ , then

$$\frac{dX}{dY} = \frac{1}{N} M_N \frac{X(1 - 2x)}{Y(1 - 2y)}. \quad (3.7)$$

Squaring (3.7), we deduce that

$$M_N^2 = N^2 \frac{Y^2(1 - Y)}{X^2(1 - X)} \left(\frac{dX}{dY}\right)^2. \quad (3.8)$$

From the above relation,  $\frac{dM_N}{dX}$  can be computed explicitly once we compute the modular relation satisfied by  $X$  and  $Y$ .

We now give an explicit series for  $1/\pi$ . First, from (3.1) and [1, p. 43, Entry 27(ii)], we find that

$$\alpha\varphi^4(e^{-\alpha}) = \beta\varphi^4(e^{-\beta}),$$

and

$$16\alpha e^{-\alpha} \psi^4(e^{-2\alpha}) = \beta\varphi^4(-e^{-\beta}),$$

if  $\alpha\beta = \pi^2$ . By (3.2) and the above relations,

$$x(e^{-\pi/\sqrt{N}}) = \frac{\varphi^4(-e^{-\pi\sqrt{N}})}{\varphi^4(e^{-\pi\sqrt{N}})}. \tag{3.9}$$

Using the famous Jacobi identity [1, p. 40, Entry 25(vii)],

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2),$$

we conclude that

$$x(e^{-\pi/\sqrt{N}}) = 1 - x(e^{-\pi\sqrt{N}}). \tag{3.10}$$

Now, let  $N = 3$ , then from Ramanujan’s modular equation [1, p. 230, Entry 5(ii)]

$$(x(q)x(q^3))^{1/4} + ((1 - x(q))(1 - x(q^3)))^{1/4} = 1,$$

and (3.10), we find that

$$X(e^{-\pi/\sqrt{3}}) = \frac{1}{4}. \tag{3.11}$$

Solving (3.11) for  $x$ , we deduce that

$$x(e^{-\pi\sqrt{3}}) = \frac{1}{2} - \frac{\sqrt{3}}{4}. \tag{3.12}$$

Next, one can show that  $X = X(q)$  and  $Y = X(q^3)$  satisfy the relation<sup>5</sup>

$$\begin{aligned} & -4096XY + 4608X^2Y + 4608XY^2 - 900X^3Y + 28422X^2Y^2 \\ & - 900XY^3 + 4608X^3Y^2 + 4608X^2Y^3 - 4096X^3Y^3 + X^4 + Y^4 = 0. \end{aligned}$$

Differentiating the above relation with respect to  $Y$  and substituting the result into (3.8), we obtain an expression of  $M_3$  in terms of  $X$  and  $Y$ . Differentiating the resulting relation with respect to  $X$  and using (3.11) and (3.12), we conclude that

$$a_3 = \frac{\sqrt{3}}{12} \quad \text{and} \quad b_3 = \frac{\sqrt{3}}{2}.$$

This gives Ramanujan’s series (1.1).

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<sup>5</sup>This modular equation can be proved using [1, p. 231, Entry 5(xii)].



**4. A new class of series for  $1/\pi$**

From the last section, we find that in order to derive a new class of series different from that of Ramanujan, one needs to seek new pair of functions  $X$  and  $Z$ . In this section, we will construct a new class of series for  $1/\pi$  and derive (1.3). Series similar to (1.3) where all the coefficients in the sum are rational are very rare. It turns out that Ramanujan nearly exhausted all these series in [12]. The Chudnovskys<sup>6</sup> and the Borweins<sup>7</sup> constructed several such series for  $1/\pi$  associated with the first and second type of series in Table 1, respectively. Recently, Chan et al. added a total of six such series [8] associated essentially to the third type of series in Table 1.

Let

$$Z = \frac{(f(-q)f(-q^3))^4}{(f(-q^2)f(-q^6))^2}$$

and

$$X = q \left( \frac{f(-q^2)f(-q^6)}{f(-q)f(-q^3)} \right)^6.$$

From our discussion in Section 3, we need to determine  $U(q)$  and  $A_k$ . It follows from [4, Theorem 10.6]

$$q \frac{dX}{dq} = ZX \sqrt{(4X + 1)(16X + 1)} \tag{4.1}$$

that

$$U(q) = \sqrt{(4X + 1)(16X + 1)}. \tag{4.2}$$

To determine  $A_k$ , we need to establish the differential equation satisfied by  $Z$  and  $X$ .

Let

$$\frac{1}{t} = 1 + \frac{1}{27q} \frac{f^{12}(-q)}{f^{12}(-q^3)}. \tag{4.3}$$

The differential equation satisfied by

$$u = 4t(1 - t) \quad \text{and} \quad f = \left( \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \right)^2$$

<sup>6</sup>The Chudnovskys' series (1.2) yields the fastest convergent series for  $1/\pi$  which involves only rational coefficients.

<sup>7</sup>See [5, p. 145] for the two series missed by Ramanujan.

is

$$\vartheta_u^3 f = u \left( \vartheta_u + \frac{1}{2} \right) \left( \vartheta_u + \frac{1}{3} \right) \left( \vartheta_u + \frac{2}{3} \right) (f). \tag{4.4}$$

The proof of (4.4) is similar to that of (3.4) but instead of using (3.3), we use the differential equation [7, p. 200]

$$t(1-t)\frac{d^2z}{dt^2} + (1-2t)\frac{dz}{dt} - \frac{2}{9}z = 0,$$

where  $z^2 = f$ . Incidentally (4.4) is the differential equation needed to determine  $A_k$  in the third type of series in Table 1. Our method of deriving a differential equation satisfied by  $Z$  and  $X$  is to first establish the identities

$$f = Z(1 + 16X) \tag{4.5}$$

and

$$u = \frac{108X}{(1 + 16X)^3}. \tag{4.6}$$

Using (4.5) and (4.6), we conclude that

$$\vartheta_u f = \frac{(1 + 16X)^2 \vartheta_X Z + 16X(1 + 16X)Z}{1 - 32X}, \tag{4.7}$$

$$\begin{aligned} \vartheta_u^2 f &= \frac{1 + 16X}{(1 - 32X)^3} \{ (1 - 768X^2 - 8192X^3) \vartheta_X^2 Z \\ &\quad + (80X + 256X^2 - 16384X^3) \vartheta_X Z + (16X + 512X^2 - 8192X^3) Z \} \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} \vartheta_u^3 f &= \frac{1 + 16X}{(1 - 32X)^5} \\ &\quad \times \{ (1 - 16X - 1280X^2 + 4096X^3 + 524288X^4 + 4194304X^5) \vartheta_X^3 Z \\ &\quad + (192X - 1536X^2 - 147456X^3 - 393216X^4 + 12582912X^5) \vartheta_X^2 Z \\ &\quad + (96X + 8448X^2 - 12288X^3 - 1179648X^4 + 12582912X^5) \vartheta_X Z \\ &\quad + (16X + 2560X^2 + 24576X^3 - 524288X^4 + 4194304X^5) Z \}. \end{aligned} \tag{4.9}$$

Substituting (4.7)–(4.9) into (4.4) and simplifying, we immediately obtain the differential equation

$$(1 + 20X + 64X^2)\vartheta_X^3 Z + (192X^2 + 30X)\vartheta_X^2 Z + (192X^2 + 18X)\vartheta_X Z + (64X^2 + 4X)Z = 0, \tag{4.10}$$

where  $\vartheta_X = X \frac{d}{dX}$ . For an alternative proof of (4.10), see [16].

It remains to establish (4.5) and (4.6). From definition (4.3), we find that

$$\frac{1}{u} = 2 + \frac{1}{27}P^6 + \frac{27}{P^6},$$

where

$$P = q^{-1/6} \frac{f^2(-q)}{f^2(-q^3)}.$$

Identity (4.6) then follows immediately from [4, Lemma 10.3, (10.4)]

$$\left(P^3 + \frac{27}{P^3}\right)^2 = \frac{(1 + 16X)^3}{X}.$$

To prove (4.5), recall from [7, p. 200] that

$$q \frac{du}{dq} = fu\sqrt{1-u}. \tag{4.11}$$

This is the analogue of (3.6). From (4.6), we find that

$$\frac{du}{dX} = 108 \frac{1 - 32X}{(1 + 16X)^4}. \tag{4.12}$$

Combining (4.12), (4.11), (4.6) and (4.1), we conclude that

$$f = Z(1 + 16X),$$

which is (4.5).

Now, from (4.10), we conclude that if

$$Z = \sum_{k=0}^{\infty} A_k X^k,$$

then  $A_k$  satisfies the difference equation

$$k^3 A_k + 2(2k - 1)(5k^2 - 5k + 2)A_{k-1} + 64(k - 1)^3 A_{k-2} = 0. \tag{4.13}$$

A check with Sloane’s Online sequences [14] shows that

$$A_k = (-1)^k \sum_{j=0}^k \binom{k}{j}^2 \binom{2(k-j)}{k-j} \binom{2j}{j}. \tag{4.14}$$

This sequence appears to be first discovered by Domb [9]. The fact that this sequence satisfies the recurrence (4.13) can be verified using the method of creative telescoping implemented in D. Zeilberger’s MAPLE programme EKHAD [11, Chapter 7]. More precisely, Zeilberger’s programme produces the function

$$R(k,j) = - (12k^3 + 62k^2 + 104k + 56 - 26k^2j - 89kj - 74j + 18kj^2 + 30j^2 - 4j^3)j^3 \frac{4(k+1)^2(2k-2j+1)}{(k+1-j)^3(k+2-j)^3},$$

which satisfies the relation

$$(k+2)^3 f(k+2,j) + 2(2(k+2)-1)(5(k+2)^2 - 5(k+2)+2)f(k+1,j) + 64(k+1)^3 f(k,j) = R(k,j+1)f(k,j+1) - R(k,j)f(k,j), \tag{4.15}$$

where

$$f(k,j) = (-1)^k \binom{k}{j}^2 \binom{2(k-j)}{k-j} \binom{2j}{j}.$$

Summing (4.15) for  $j$  from 0 to  $k+2$  completes the proof that  $A_k$  satisfies (4.13).

We now turn to the proof of (1.3). First, we let  $\mathbf{Z} = \mathbf{Z}(-q)$ . Then from the transformation formulas [1, Entry 27(iii), (iv)]

$$e^{-\alpha/12} \sqrt[4]{\alpha} f(-e^{-2\alpha}) = e^{-\beta/12} \sqrt[4]{\beta} f(-e^{-2\beta}) \tag{4.16}$$

and

$$e^{-\alpha/24} \sqrt[4]{\alpha} f(e^{-\alpha}) = e^{-\beta/24} \sqrt[4]{\beta} f(e^{-\beta}), \tag{4.17}$$

where  $\alpha\beta = \pi^2$ , we deduce that

$$\mathbf{Z}(e^{-\pi/\sqrt{3N}}) = N\mathbf{Z}(e^{-\pi\sqrt{N/3}}). \tag{4.18}$$

This shows that  $s = 12$ . We may now apply Theorem 2.1 with the functions  $\mathbf{Z}$  and  $\mathbf{X} = -X(-q)$ . Note that

$$\mathbf{Z} = \sum_{k=0}^{\infty} A_k (-\mathbf{X})^k,$$

and hence our corresponding

$$\mathbf{A}_k = (-1)^k A_k = \sum_{j=0}^k \binom{k}{j}^2 \binom{2(k-j)}{k-j} \binom{2j}{j}.$$

From (4.2), we have

$$\mathbf{U}(q) = \sqrt{(1-4\mathbf{X})(1-16\mathbf{X})}.$$

In addition, if

$$\mathbf{M}_N = \frac{\mathbf{Z}(q)}{\mathbf{Z}(q^N)},$$

then the analogue of (3.8) is

$$\mathbf{M}_N^2 = N^2 \frac{\mathbf{Y}^2(1-4\mathbf{Y})(1-16\mathbf{Y}) \left(\frac{d\mathbf{X}}{d\mathbf{Y}}\right)^2}{\mathbf{X}^2(1-4\mathbf{X})(1-16\mathbf{X})}, \quad (4.19)$$

where  $\mathbf{Y} = \mathbf{X}(-q^N)$ .

Now let  $N = 5$ . We have<sup>8</sup>

$$\begin{aligned} \Phi(\mathbf{X}, \mathbf{Y}) &= \mathbf{X}^6 + \mathbf{Y}^6 + 30\mathbf{X}\mathbf{Y}^2 + 30\mathbf{X}^2\mathbf{Y} - 285\mathbf{X}\mathbf{Y}^3 - 285\mathbf{X}^3\mathbf{Y} + 970\mathbf{X}^4\mathbf{Y} \\ &\quad + 970\mathbf{X}\mathbf{Y}^4 - 990\mathbf{X}\mathbf{Y}^5 - 990\mathbf{X}^5\mathbf{Y} - 930\mathbf{X}^2\mathbf{Y}^3 - 930\mathbf{X}^3\mathbf{Y}^2 - 25665\mathbf{X}^2\mathbf{Y}^4 \\ &\quad - 25665\mathbf{X}^4\mathbf{Y}^2 + 62080\mathbf{X}^2\mathbf{Y}^5 + 62080\mathbf{X}^5\mathbf{Y}^2 + 92860\mathbf{X}^3\mathbf{Y}^3 - 59520\mathbf{X}^3\mathbf{Y}^4 \\ &\quad - 59520\mathbf{X}^4\mathbf{Y}^3 - 1167360\mathbf{X}^3\mathbf{Y}^5 - 1167360\mathbf{X}^5\mathbf{Y}^3 + 7864320\mathbf{X}^4\mathbf{Y}^5 \\ &\quad + 7864320\mathbf{X}^5\mathbf{Y}^4 - 16777216\mathbf{X}^5\mathbf{Y}^5 - \mathbf{X}\mathbf{Y} \\ &\quad + 815\mathbf{X}^2\mathbf{Y}^2 + 3338240\mathbf{X}^4\mathbf{Y}^4 = 0. \end{aligned} \quad (4.20)$$

When  $q = e^{-\pi/\sqrt{15}}$ , then from (4.16) and (4.17), we find that

$$\mathbf{X}(e^{-\pi/\sqrt{15}}) = \mathbf{X}(e^{-\pi\sqrt{5/3}}) = \mathbf{Y}(e^{-\pi/\sqrt{15}}).$$

This shows that  $\mathbf{X}(e^{-\pi/\sqrt{15}})$  is a root of the polynomial  $\Phi(T, T)$  where  $\Phi$  is given by (4.20). This polynomial factors as

$$-T^2(T-1)(64T-1)(64T^2-11T+1)(64T^2+1)(1+8T)^2,$$

<sup>8</sup>This can be verified using Ramanujan's modular equation of degree 5 in the theory of signature 3 [2].

and a quick check shows that

$$\mathbf{X}(e^{-\pi/\sqrt{15}}) = \frac{1}{64}. \quad (4.21)$$

Next, from (4.20), we can compute  $\frac{d\mathbf{X}}{dY}$  and substitute the result into (4.19). Differentiating the result with respect to  $\mathbf{X}$  and using (4.21) and (4.18), we conclude that

$$\left. \frac{d\mathbf{M}_N}{d\mathbf{X}} \right|_{q=e^{-\pi/\sqrt{15}}} = -128. \quad (4.22)$$

With these values, we conclude that

$$a_n = \frac{3}{8\sqrt{5}} \quad \text{and} \quad b_n = \frac{3\sqrt{5}}{8}.$$

Substituting these values into Theorem 2.1 yields (1.3).

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