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ELLIPTIC FUNCTIONS AND THE QUINTUPLE, HIRSCHHORN AND WINQUIST PRODUCT IDENTITIES

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In this article, we derive the quintuple, Hirschhorn and Winquist product identities using the theory of elliptic functions. Our method can be used to establish generalizations of these identities due to the second author.

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1. Introduction

The Rogers–Ramanujan continued fraction is defined by

$$R(q) = \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots,$$

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with real $|q| < 1$ and $q^{1/5}$ being the real fifth root of q. The continued fraction $R(q)$ can be expressed as [11, p. 42]

$$R(q) = q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)} \tag{1.1}$$

where

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2}b^{k(k-1)/2},$$

with $|a| < 1$ and $|b| < 1$.

One of the most important identities associated with $R(q)$ in the form (1.1) is [11, p. 45]

$$\frac{1}{R(q)} - R(q) - 1 = q^{-1/5}\frac{f(-q^{1/5})}{f(-q^5)} \tag{1.2}$$

with

$$f(-q) = \prod_{k=1}^{\infty}(1-q^k).$$

An application of (1.2) is the explicit evaluation of $R(q)$ at $q = e^{-2\pi\sqrt{n}}$, where n is a positive rational number. For example, when $n = 1$, the right-hand side of (1.2) has value $\sqrt{5}$ by the famous transformation formula satisfied by $f(-q)$ [2, p. 43, Entry 27(iii)]. Solving for $R(e^{-2\pi})$ using (1.2), we derive Ramanujan's famous continued fraction

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2}.$$

One of the first few proofs of (1.1) appeared in [11, p. 45]. It was established using a variant of the quintuple product identity

$$f(B^3q, q^5/B^3) - B^2 f(q/B^3, B^3q^5) = f(-q^2)\frac{f(-B^2,-q^2/B^2)}{f(Bq,q/B)}. \tag{1.3}$$

For a short history of (1.3), we refer the reader to [2, p. 83].

There are other applications of (1.3). For example, by multiplying two identities arising from the quintuple product identity, Kang [6] established the Winquist identity [13]

$$\begin{aligned}&\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{3m^2+3n^2+3m+n}\\ &\qquad \times(a^{-3m}b^{-3n} - a^{-3m}b^{3n+1} - a^{-3n+1}b^{-3m-1} + a^{3n+2}b^{-3m-1})\\ &\quad = (q^2;q^2)_\infty^2(a, a^{-1}q^2, b, b^{-1}q^2, ab, a^{-1}b^{-1}q^2, ab^{-1}, a^{-1}bq^2; q^2)_\infty, \end{aligned} \tag{1.4}$$

where

$$(a_1, a_2, \ldots, a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty,$$

with

$$(a; q)_\infty = \prod_{k \geq 1} (1 - aq^{k-1}).$$

Another recent application of (1.3) appears in the proofs of new identities that arise from the study of cranks [3]. An example of these identities is [3, Theorem 4.2]

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k(k+1)/2} \frac{q^k - 1}{1 + q^{3k} + q^{6k}} = q(q; q)_\infty \frac{f(-q^3, -q^{24}) f(-q^{12}, -q^{15})}{(q^{27}; q^{27})_\infty}.$$

In Sec. 2, we give proofs of (1.3) and one of its generalizations due to Liu using the theory of elliptic functions. In Sec. 3, we use the same method to establish Hirschhorn's product identity

$$\begin{aligned}
&(q^2, q^2, x, q^2/x; q^2)_\infty (q^2x^2, q^2/x^2, x^2, q^4/x^2; q^4)_\infty \\
&= \sum_{m=-\infty}^{\infty} (-1)^m q^{5m^2+m} \left(\sum_{k=-\infty}^{\infty} (-1)^k q^{5k^2+3k} x^{5k+3} + \sum_{k=-\infty}^{\infty} (-1)^k q^{5k^2-3k} x^{5k} \right) \\
&\quad - \sum_{m=-\infty}^{\infty} (-1)^m q^{5m^2+3m} \\
&\quad \times \left(\sum_{k=-\infty}^{\infty} (-1)^k q^{5k^2+k} x^{5k+2} + \sum_{k=-\infty}^{\infty} (-1)^k q^{5k^2-k} x^{5k+1} \right).
\end{aligned} \tag{1.5}$$

Identity (1.5) is previously known as the Farkas–Kra septuple identity, named after H. M. Farkas and I. Kra, who discovered it in 1999 [4]. However, in [8], Liu indicated that the identity was discovered much earlier by Hirschhorn [5, (3.1)]. As such, we shall refer to (1.5) as the Hirschhorn identity.

In Sec. 4, we prove one of Liu's identities and establish the Winquist identity (1.4).

2. A Proof of the Quintuple Identity

Let $q = e^{\pi i \tau}$ and $\operatorname{Im}(\tau) > 0$. Define

$$\theta_1(u|\tau) = -iq^{1/4} \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k+1)} e^{(2k+1)iu}$$

and

$$\theta_4(u|\tau) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2kiu}.$$

Using the arguments in [2, p. 82] and the Jacobi triple product formula [12, pp. 469–472]

$$\theta_1(u|\tau) = 2q^{1/4} \sin u (q^2;q^2)_\infty (q^2 e^{2iu};q^2)_\infty (q^2 e^{-2iu};q^2)_\infty, \tag{2.1}$$

we can rewrite (1.3) as (see the last paragraph of [2, p. 82])

$$(q^2;q^2)_\infty \frac{\theta_1(2u|\tau)}{\theta_1(u|\tau)} = e^{iu}\theta_4\left(3u + \frac{\pi\tau}{2}|3\tau\right) + e^{-iu}\theta_4\left(3u - \frac{\pi\tau}{2}|3\tau\right). \tag{2.2}$$

The quintuple formula in the form (2.2) appeared in Liu's paper [7, Theorem 4]. An alternative but equivalent form of (2.2) can be found in [1, (1.6)], where the identity was proved using a well-known identity satisfied by the Weierstrass σ-function [1, (3.1)].

We now proceed to give a proof of (2.2) different from those given in [7] and [1, (1.6)]. Using the transformation formulas

$$\theta_1(u+\pi|\tau) = -\theta_1(u|\tau), \quad \theta_1(u+\pi\tau|\tau) = -q^{-1}e^{-2iu}\theta_1(u|\tau),$$
$$\theta_4(u+\pi|\tau) = \theta_4(u|\tau) \quad \text{and} \quad \theta_4(u+\pi\tau|\tau) = -q^{-1}e^{-2iu}\theta_4(u|\tau),$$

we find that the functions

$$G(u) := \frac{\theta_1(2u|\tau)}{\theta_1(u|\tau)}, \quad g(u) := e^{iu}\theta_4\left(3u + \frac{\pi\tau}{2}|3\tau\right), \quad g(-u) := e^{-iu}\theta_4\left(3u - \frac{\pi\tau}{2}|3\tau\right)$$

satisfy the relations

$$f(u+\pi) = -f(u) \quad \text{and} \quad f(u+\pi\tau) = -q^{-3}e^{-6iu}f(u). \tag{2.3}$$

Therefore, the functions

$$f_1(u) := g(u)/G(u) \quad \text{and} \quad f_2(u) := g(-u)/G(u)$$

are both elliptic functions with periods π and $\pi\tau$. The function $\theta_1(2u|\tau)$ has zeros at $0, \pi/2, \pi\tau/2$, and $(\pi+\pi\tau)/2$ but $f_1(u)$ and $f_2(u)$ are analytic at $u=0$ since $u=0$ is also a zero of $\theta_1(u|\tau)$. Hence, the functions $f_1(u)$ and $f_2(u)$ have simple poles at $\pi/2, \pi\tau/2, (\pi+\pi\tau)/2$, which are the remaining zeros of $\theta_1(2u|\tau)$ in a period parallelogram. In general, linear combinations of these two functions would not give us an expression independent of u. However, since the function $g(u)$ satisfies (2.3), we find that

$$g\left(\frac{\pi}{2}\right) + g\left(-\frac{\pi}{2}\right) = 0 \quad \text{and} \quad g\left(\frac{\pi\tau}{2}\right) + g\left(-\frac{\pi\tau}{2}\right) = 0. \tag{2.4}$$

Hence $f_1(u) + f_2(u)$ is an elliptic function with at most one pole and we conclude that the function must be independent of u. Hence,

$$A(q)\frac{\theta_1(2u|\tau)}{\theta_1(u|\tau)} = e^{iu}\theta_4\left(3u + \frac{\pi\tau}{2}|3\tau\right) + e^{-iu}\theta_4\left(3u - \frac{\pi\tau}{2}|3\tau\right), \tag{2.5}$$

for some $A(q)$ independent of u.

To determine $A(q)$, we substitute $u = 0$ into (2.5) and deduce that

$$A(q) = \theta_4\left(\frac{\pi\tau}{2}|3\tau\right) = (q^2; q^2)_\infty,$$

where the last equality follows from

$$\theta_4(u|\tau) = (q^2; q^2)_\infty(qe^{2iu}; q^2)_\infty(qe^{-2iu}; q^2)_\infty. \tag{2.6}$$

Identity (2.6) is a variant of (2.1) [12, p. 469].

We may replace $g(u)$ in our proof by any entire function $h(u)$ satisfying (2.3). The function $h(u)$ would have zeros at $u = \pi/2$ and $\pi\tau/2$ (see (2.4)) and by exactly the same argument as above, we deduce that

$$h(u) + h(-u) = h(0)\frac{\theta_1(2u|\tau)}{\theta_1(u|\tau)}. \tag{2.7}$$

The above generalization of (2.2) is due to Liu [10, Theorem 2] and first proved using the residue theorem.

There are several ways of proving (1.3). One of the most popular methods is to use the properties of the product on the left-hand side of (1.3) to determine the power series of the right-hand side of (1.3). However, such a method does not allow us to deduce the generalization (2.7). There are many applications of (2.7). For more details, see [8].

3. The Hirschhorn Product Identity

To prove (1.5), we first set $x = e^{2iu}$. We can then rewrite (1.5) as

$$\begin{aligned} &-q^{-1/2}\theta_1(u|\tau)\theta_1(2u|\tau) \\ &\quad = \sum_{m=-\infty}^{\infty} (-1)^m q^{5m^2+m}\left(e^{-3iu}\theta_4\left(5u - \frac{3\pi\tau}{2}|5\tau\right) + e^{3iu}\theta_4\left(5u + \frac{3\pi\tau}{2}|5\tau\right)\right) \\ &\qquad - \sum_{m=-\infty}^{\infty} (-1)^m q^{5m^2+3m}\left(e^{iu}\theta_4\left(5u + \frac{\pi\tau}{2}|5\tau\right) + e^{-iu}\theta_4\left(5u - \frac{\pi\tau}{2}|5\tau\right)\right). \end{aligned} \tag{3.1}$$

To prove (3.1), we observe that the functions

$$H(u) = \theta_1(u|\tau)\theta_1(2u|\tau),$$
$$h_1(u) = e^{-3iu}\theta_4\left(5u - \frac{3\pi\tau}{2}|5\tau\right), \quad h_1(-u) = e^{3iu}\theta_4\left(5u + \frac{3\pi\tau}{2}|5\tau\right),$$
$$h_2(u) = e^{iu}\theta_4\left(5u + \frac{\pi\tau}{2}|5\tau\right) \quad \text{and} \quad h_2(-u) = e^{-iu}\theta_4\left(5u - \frac{\pi\tau}{2}|5\tau\right)$$

satisfy

$$f(u+\pi) = -f(u) \quad \text{and} \quad f(u+\pi\tau) = -q^{-5}e^{-10iu}f(u). \tag{3.2}$$

Hence the functions $h_3(u)/H(u)$ and $h_4(u)/H(u)$, where

$$h_3(u) := h_1(u) + h_1(-u) \quad \text{and} \quad h_4(u) := h_2(u) + h_2(-u),$$

are even elliptic functions with periods π and $\pi\tau$ having a double pole at $u = 0$ and simple poles at $\pi/2, \pi\tau/2$ and $(\pi + \pi\tau)/2$.

Now, since $h_1(u)$ and $h_2(u)$ satisfy (3.2), we deduce that $h_3(u)$ and $h_4(u)$ vanish at $u = \pi/2$ and $\pi\tau/2$.

Next, we hope to find two expressions α and β such that $\alpha h_3(u) + \beta h_4(u)$ have a zero of order at least two at $u = 0$. Since $h_3(u)$ and $h_4(u)$ are even, it suffices to remove the constant term in $\alpha h_3(u) + \beta h_4(u)$. An obvious choice is to set $\alpha = h_4(0)$ and $\beta = -h_3(0)$. Hence, $h_5(u) = h_4(0)h_3(u) - h_3(0)h_4(u)$ has a double zero at $u = 0$ and vanishes at $u = \pi/2$ and $\pi\tau/2$. As a result, the function $h_5(u)/H(u)$ is an elliptic function that can have at most one pole and hence, must be independent of u. Therefore, we must have

$$\begin{aligned} &B(q)\theta_1(u|\tau)\theta_1(2u|\tau) \\ &= \sum_{m=-\infty}^{\infty} (-1)^m q^{5m^2+m}\left(e^{-3iu}\theta_4\left(5u - \frac{3\pi\tau}{2}|5\tau\right) + e^{3iu}\theta_4\left(5u + \frac{3\pi\tau}{2}|5\tau\right)\right) \\ &\quad - \sum_{m=-\infty}^{\infty} (-1)^m q^{5m^2+3m}\left(e^{iu}\theta_4\left(5u + \frac{\pi\tau}{2}|5\tau\right) + e^{-iu}\theta_4\left(5u - \frac{\pi\tau}{2}|5\tau\right)\right), \end{aligned} \tag{3.3}$$

for some function $B(q)$.

To determine the constant $B(q)$, we first observe that

$$\begin{aligned} &e^{-3iu}\theta_4\left(5u - \frac{3\pi\tau}{2}|5\tau\right) + e^{3iu}\theta_4\left(5u + 3\frac{\pi\tau}{2}|5\tau\right) \\ &= 2\sum_{k=-\infty}^{\infty} (-1)^k q^{5k^2+3k}\cos(10k+3)u, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &e^{iu}\theta_4\left(5u + \frac{\pi\tau}{2}|5\tau\right) + e^{-iu}\theta_4\left(5u - \frac{\pi\tau}{2}|5\tau\right) \\ &= 2\sum_{k=-\infty}^{\infty} (-1)^k q^{5k^2+k}\cos(10k+1)u. \end{aligned} \tag{3.5}$$

Substituting $u = \pi/5$ into (3.4) and (3.5) and using (2.6), we find that the right-hand side of (3.3) is

$$\begin{aligned} &2\left(\cos\left(\frac{3\pi}{5}\right) - \cos\left(\frac{\pi}{5}\right)\right)\sum_{m,k\in\mathbf{Z}}(-1)^{m+k}q^{5m^2+m+5k^2+3k} \\ &= -4\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)\theta_4\left(\frac{\pi\tau}{2}|5\tau\right)\theta_4\left(\frac{3\pi\tau}{2}|5\tau\right) \\ &= -4\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)(q^{10}, q^6, q^4, q^{10}, q^2, q^8; q^{10})_\infty \\ &= -4\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)(q^2; q^2)_\infty(q^{10}; q^{10})_\infty. \end{aligned} \tag{3.6}$$

On the other hand, by (2.1) and the identity

$$(1-x)(1-e^{2\pi i/5}x)(1-e^{-2\pi i/5}x)(1-e^{4\pi i/5}x)(1-e^{-4\pi i/5}x) = (1-x^5),$$

we find that

$$\begin{aligned} &\theta_1(\pi/5|\tau)\theta_1(2\pi/5|\tau) \\ &= 4q^{1/2}\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)(q^2, q^2, q^2e^{2\pi i/5}, q^2e^{-2\pi i/5}, q^2e^{4\pi i/5}, q^2e^{-4\pi i/5}; q^2)_\infty \\ &= 4q^{1/2}\sin\left(\frac{2\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)(q^2; q^2)_\infty(q^{10}; q^{10})_\infty. \end{aligned} \tag{3.7}$$

Combining (3.3), (3.6) and (3.7), we conclude that

$$B(q) = -q^{-1/2}$$

and this completes the proof of (3.1).

The Hirschhorn identity in the form (3.1) is slightly different from that given in [8, (1.4)], where the right-hand side is expressed in terms of θ_1 instead of θ_4.

We may also replace $h_1(u)$ and $h_2(u)$ by any two entire even functions $g_1(u)$ and $g_2(u)$ satisfying (3.2) and deduce that

$$C(q)\theta_1(u|\tau)\theta_1(2u|\tau) = g_2(0)(g_1(u) + g_1(-u)) - g_1(0)(g_2(u) + g_2(-u)),$$

for some $C(q)$ independent of u. The above generalization, which is due to Liu [8, Theorem 1] and first proved using the residue theorem, can be proved in exactly the same way as in the proof of (3.1).

In the proof of (3.1), we need several identities to determine $B(q)$. In general, the determination of $C(q)$ in closed form is usually very challenging. We end this section by stating an identity which is a consequence of (3.1). By comparing the coefficients of u^2 on both sides of (3.1) using (3.4) and (3.5), we deduce Hirschhorn's identity [5] for $(q^2; q^2)_\infty^6$:

$$\begin{aligned} 8(q^2; q^2)_\infty^6 = \sum_{m,n\in\mathbf{Z}}(-1)^{m+n}\Big(&q^{5m^2+m+5n^2+3n}(10n+3)^2 \\ &- q^{5m^2+3m+5n^2+n}(10n+1)^2\Big). \end{aligned} \tag{3.8}$$

It is not surprising that this identity gives rise to one of Ramanujan's famous partition congruence (the key is to solve for pairs (m,n) for which $(5m^2+3m+5n^2+n)/2+2\equiv 0 \pmod 7$)

$$p(7n+5)\equiv 0 \pmod 7.$$

4. A New Proof of the Winquist identity

In this section, we first show that the Winquist identity (1.4) follows from [9, Theorems 4 and 7][a]

$$\begin{aligned}(x,q^2/x,y,q^2/y,y/x,q^2x/y,xy,q^2/xy,q^2,q^2,q^2,q^2;q^2)_\infty\\ =(q^2;q^2)^2_\infty\sum_{n=-\infty}^{\infty}(-1)^n q^{3n(n-1)}y^{3n}\sum_{m=-\infty}^{\infty}q^{m(m-1)/3}x^m\\ -yx^{-1}(q^2;q^2)^2_\infty\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/3}y^n\sum_{m=-\infty}^{\infty}(-1)^m q^{3m(m-1)}x^{3m}.\end{aligned}\tag{4.1}$$

We first observe that

$$\begin{aligned}\sum_{m=-\infty}^{\infty}(-1)^m q^{m(m-1)/3}u^m\\ =\sum_{k=-\infty}^{\infty}\left((-1)^{3k}q^{k(3k+1)}(u^{-3k}+u^{3k+1})+(-1)^{3k+2}q^{(3k+2)(3k+1)/3}u^{3k+2}\right).\end{aligned}$$

Hence, if we write the first and second term in the right-hand side of (4.1) as T_1 and T_2 respectively, we find that

$$\begin{aligned}T_1=(q^2;q^2)^2_\infty\Bigg(\sum_{n,k\in\mathbf{Z}}(-1)^{n+k}q^{3n^2+3n+3k^2+k}(y^{-3n}x^{-3k}-y^{-3n}x^{3k+1})\\ +(-1)^{n+k}q^{3n^2+3n+3k^2+3k+2/3}y^{-3n}x^{3k+2}\Bigg)\end{aligned}$$

and

$$\begin{aligned}T_2=(q^2;q^2)^2_\infty\Bigg(\sum_{n,k\in\mathbf{Z}}(-1)^{n+k}q^{3n^2+3n+3k^2+k}(y^{-3k+1}x^{-3n-1}-y^{3k+2}x^{-3n-1})\\ +(-1)^{n+k}q^{3n^2+3n+3k^2+3k+2/3}y^{3k+3}x^{-3n-1}\Bigg)\end{aligned}$$

with $x=b$ and $y=a$ we find that T_1-T_2 is precisely the left-hand side of (1.4).

[a]This observation is also made independently by Hirschhorn.

The advantage of writing Winquist's identity (1.4) in the form (4.1) is that the latter is a direct consequence (see the proof of [9, Theorem 7]) of (2.1) and the following identity [9, Theorem 4]):

$$\begin{aligned} &q^{1/6}(q^2;q^2)_\infty^2(\theta_1(u|\tau/3)\theta_1(3v|3\tau)-\theta_1(v|\tau/3)\theta_1(3u|3\tau)) \\ &\quad=\theta_1(u|\tau)\theta_1(v|\tau)\theta_1(u-v|\tau)\theta(u+v|\tau). \end{aligned} \tag{4.2}$$

Identity (4.2) is proved in [9] using the residue theorem. We now establish (4.2) in the same way as the identities proved in the previous sections.

The functions (viewed as a function of u)

$$\theta_1(u|\tau/3), \theta_1(3u|3\tau), \theta_1(u|\tau)\theta_1(u-v|\tau)\theta_1(u+v|\tau)$$

satisfy the functional equations

$$f(u+\pi)=-f(u) \quad \text{and} \quad f(u+\pi\tau)=-q^{-3}e^{-6iu}f(u).$$

Hence,

$$F_1(u)=\frac{\theta_1(u|\tau)\theta_1(u+v|\tau)\theta_1(u-v|\tau)}{\theta_1(u|\tau/3)} \quad \text{and} \quad F_2(u)=\frac{\theta_1(3u|3\tau)}{\theta_1(u|\tau/3)}$$

are elliptic functions with only two simple poles at $u=\pi\tau/3$ and $-\pi\tau/3$. Note that F_1/F_2 is not independent of u since the zeros of F_1 and F_2 are different. Therefore, we conclude that there must exist $C_1(v)=C_1(v,q)$, $C_2(v)=C_2(v,q)$ such that the elliptic function $C_1(v)F_1(u)+C_2(v)F_2(u)$ has only one pole. It follows that this elliptic function must be independent of u and hence,

$$C_1(v)\theta_1(u|\tau/3)+C_2(v)\theta_1(3u|3\tau)=C_3(v)\theta_1(u|\tau)\theta_1(u+v|\tau)\theta_1(u-v|\tau), \tag{4.3}$$

for some function $C_3(v)=C_3(v,q)$. Let $u=v$ in (4.3). Then we find that

$$C_1(v)\theta(v|\tau/3)+C_2(v)\theta(3v|3\tau)=0.$$

Hence

$$C_1(v)=-C_2(v)\frac{\theta_1(3v|3\tau)}{\theta_1(v|\tau/3)}.$$

We can therefore rewrite (4.3) as

$$\begin{aligned} &-C_2(v)\theta(3v|3\tau)\theta_1(u|\tau/3)+C_2(v)\theta_1(v|\tau/3)\theta_1(3u|3\tau) \\ &\quad=C_3(v)\theta_1(v|\tau/3)\theta_1(u|\tau)\theta_1(u+v|\tau)\theta_1(u-v|\tau). \end{aligned} \tag{4.4}$$

Since (4.4) holds for any u and v, we can interchange u and v and deduce that

$$\begin{aligned} &-C_2(u)\theta(3u|3\tau)\theta_1(v|\tau/3)+C_2(u)\theta_1(u|\tau/3)\theta_1(3v|3\tau) \\ &\quad=C_3(u)\theta_1(u|\tau/3)\theta_1(v|\tau)\theta_1(u+v|\tau)\theta_1(v-u|\tau). \end{aligned} \tag{4.5}$$

Dividing (4.4) by (4.5) and simplifying, we find that

$$\frac{C_2(v)\theta_1(v|\tau)}{C_3(v)}=\frac{C_2(u)\theta_1(u|\tau)}{C_3(u)}.$$

Hence, the expression $C_2(x)\theta_1(x)/C_3(x)$ must be independent of x and we must have

$$C_2(v) = \frac{C_3(v)}{\theta_1(v|\tau)} C(q)$$

for some function $C(q)$. We conclude that

$$\begin{aligned} &C(q)(-\theta_1(u|\tau/3)\theta_1(3v|3\tau) + \theta_1(v|\tau/3)\theta_1(3u|3\tau)) \\ &\quad = \theta_1(u|\tau)\theta_1(v|\tau)\theta_1(u-v|\tau)\theta(u+v|\tau). \end{aligned} \tag{4.6}$$

By comparing the coefficients of v on both sides of (4.6), we find that

$$C(q)\left(-3\theta_1'(0|3\tau)\theta_1(u|\tau/3) + \theta_1'(0|\tau/3)\theta_1(3u|3\tau)\right) = \theta_1'(0|\tau)\theta_1^3(u|\tau). \tag{4.7}$$

Let $u = \pi/3$ in (4.7). Using the identities (see [12, pp. 469–472] for the proofs of these identities)

$$\begin{aligned} \theta_1'(0|\tau) &= 2q^{1/4}(q^2;q^2)_\infty^3, \\ \theta_1(\pi|3\tau) &= 0 \end{aligned}$$

and

$$\theta_1(\pi/3|\tau) = \sqrt{3}q^{1/4}(q^6;q^6)_\infty,$$

we conclude that

$$C(q) = -q^{1/6}(q^2;q^2)_\infty^2$$

and this completes the proof of (4.2).

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References

[1] W. N. Bailey, An expression for $\vartheta_1(nz)/\vartheta_1(z)$, *Proc. Amer. Math. Soc.* **4**(4) 569–572.

[2] B. C. Berndt, *Ramanujan's Notebooks*, *Part III* (Springer-Verlag, New York, 1991).

[3] B. C. Berndt, H. H. Chan, S. H. Chan and W. C. Liaw, Cranks and dissections in Ramanujan's lost notebook, *J. Combin. Theory, Ser. A* **109** (2005) 91–120.

[4] H. M. Farkas and I. Kra, On the quintuple product identity, *Proc. Amer. Math. Soc.* **88** (1999) 771–778.

[5] M. D. Hirschhorn, A simple proof of an identity of Ramanujan, *J. Austral. Math. Soc. Ser. A* **34** (1983) 31–35.

[6] S.-Y. Kang, A new proof of Winquist's identity, *J. Combin. Theory, Ser. A* **78** (1997) 313–318.

[7] Z.-G. Liu, Residue Theorem and theta function identities, *The Ramanujan J.* **5** (2001) 129–151.

[8] Z.-G. Liu, Some theta function identities associated with the modular equations of degree 5, *Integers: Electronic J. Combin. Number Theory* **1** (2001) (A#03).

[9] Z.-G. Liu, A theta function identity and its implications, *Trans. Amer. Math. Soc.* **357** (2005) 825–835.

[10] Z.-G. Liu, A three term theta function identity and its applications, *Adv. in Math.*, to appear.

[11] G. N. Watson, Theorems stated by Ramanujan: Theorems on continued fractions, *J. Lond. Math. Soc.* **4** (1929) 39–48.

[12] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th edn. (Cambridge University Press, Cambridge, 1966).

[13] L. Winquist, An elementary proof of $p(11m+6) \equiv 0 \pmod{11}$, *J. Combin. Theory*, **6** (1969) 56–59.