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Circular summation of theta functions in Ramanujan's Lost Notebook

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Abstract

In this paper, we prove Ramanujan's circular summation formulas previously studied by S.S. Rangachari, S.H. Son, K. Ono, S. Ahlgren and K.S. Chua using properties of elliptic and theta functions. We also derive identities similar to Ramanujan's summation formula and connect these identities to Jacobi's and Dixon's elliptic functions. At the end of the paper, we discuss the connection of our results with the recent thesis of E. Conrad.

Keywords: Circular summation; Elliptic; Theta; Ramanujan; Jacobi; Dixon

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1. Introduction

The Jacobi theta function $\theta_3(z \mid \tau)$ is defined as

$$\theta_3(z \mid \tau) = \sum_{m=-\infty}^{\infty} q^{m^2} e^{2miz}, \qquad q = e^{\pi i \tau}, \quad \text{with } \operatorname{Im}\tau > 0. \tag{1.1}$$

On p. 54 of his Lost Notebook [12], S. Ramanujan recorded the following statement (translated here in terms of $\theta_3(z \mid \tau)$):

Theorem 1.1. *For any positive integer $n \geqslant 2$,*

$$\sum_{k=0}^{n-1} q^{k^2} e^{2kiz} \theta_3^n(z + k\pi\tau \mid n\tau) = \theta_3(z \mid \tau) F_n(\tau). \tag{1.2}$$

When $n \geqslant 3$,

$$F_n(\tau) = 1 + 2nq^{n-1} + \cdots. \tag{1.3}$$

The first proof of Theorem 1.1 was given by Rangachari [13]. The statement was later proved by Son [15] using an entirely different method.

In [13], Rangachari also discussed Ramanujan's explicit expressions of F_n for $n = 2, 3, 4, 5$ and 7. When n is a prime, Rangachari established Ramanujan's F_n by first showing that if p is a prime, then

$$F_p(\tau) = \Theta_{A_{p-1}^*}(p\tau), \tag{1.4}$$

where A_{p-1}^* is the dual of the root lattice,

$$A_{p-1} = \left\{ \mathbf{x} \in \mathbf{Z}^p \;\middle|\; \sum_{i=1}^{p} x_i = 0 \right\},$$

and

$$\Theta_L(\tau) = \sum_{\mathbf{x} \in L} q^{\mathbf{x}\cdot\mathbf{x}}.$$

The primality condition in Rangachari's result (1.4) was subsequently removed by Chua [4].

After the work of Rangachari and Son, Ono [11], Ahlgren [1] and Chua [4,5] devoted several papers to the evaluations of $F_n(\tau)$ for other integers n not found in Ramanujan's work. These authors quoted Ramanujan's assertion but did not realize that Rangachari's proof of (1.3) is incorrect.

In this paper, we will give a proof of (1.2) and possibly the first proof of (1.3). We will also establish results similar to Theorem 1.1 with $\theta_3(z \mid \tau)$ replaced by

$$\theta_1(z \mid \tau) = -iq^{1/4} \sum_{m=-\infty}^{\infty} (-1)^m q^{m(m+1)} e^{(2m+1)iz},$$

$$\theta_2(z \mid \tau) = q^{1/4} \sum_{m=-\infty}^{\infty} q^{m(m+1)} e^{(2m+1)iz},$$

or

$$\theta_4(z \mid \tau) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m^2} e^{2miz}.$$

At the end of the paper, we define analogues of Dixon's functions and discuss their connection with recent results of Conrad [6].

2. Proof of Theorem 1.1

In this section, we will establish (1.2), followed by (1.3).

Proof of Theorem 1.1. From (1.1), we find that

$$\theta_3(z+\pi \mid \tau) = \theta_3(z \mid \tau) \quad \text{and} \quad \theta_3(z+\pi\tau \mid \tau) = q^{-1}e^{-2iz}\theta_3(z \mid \tau). \tag{2.1}$$

Let $f(z)$ be the left-hand side of (1.2). Then

$$\begin{aligned} f(z+\pi\tau) &= \sum_{k=0}^{n-1} q^{k^2} e^{2ki(z+\pi\tau)} \theta_3^n\big(z+(k+1)\pi\tau \mid n\tau\big) \\ &= q^{-1}e^{-2iz} \sum_{k=0}^{n-1} q^{(k+1)^2} e^{2(k+1)iz} \theta_3^n\big(z+(k+1)\pi\tau \mid n\tau\big). \end{aligned} \tag{2.2}$$

Replacing $k+1$ to k, we find that

$$\sum_{k=0}^{n-1} q^{(k+1)^2} e^{2(k+1)iz} \theta_3^n\big(z+(k+1)\pi\tau \mid n\tau\big) = \sum_{k=1}^{n} q^{k^2} e^{2kiz} \theta_3^n(z+k\pi\tau \mid n\tau). \tag{2.3}$$

Using the second identity in (2.1), we find that

$$q^{n^2} e^{2inz} \theta_3^n(z+n\pi\tau \mid n\tau) = \theta_3^n(z \mid n\tau). \tag{2.4}$$

Combining (2.2)–(2.4), we deduce that

$$f(z+\pi\tau) = q^{-1}e^{-2iz} f(z). \tag{2.5}$$

Using the first identity, we find that

$$f(z+\pi) = f(z). \tag{2.6}$$

From (2.1), (2.5) and (2.6), we deduce that $f(z)/\theta_3(z \mid \tau)$ is an elliptic function with periods π and $\pi\tau$. It is well known that $\theta_3(z \mid \tau)$ has only a simple zero at $z = (\pi+\pi\tau)/2$ in the period parallelogram. Hence $f(z)/\theta_3(z \mid \tau)$ is a constant, say $F_n(\tau)$, since it is an elliptic function with only one simple pole in a period parallelogram. This concludes the proof of (1.2). Rangachari proved (1.2) by first proving it for odd n and then for even n. This is not necessary as shown in the above proof. However, when $\theta_3(z \mid \tau)$ is replaced by $\theta_1(z \mid \tau)$, we have to consider the corresponding identities according to the parity of n. For more details, see the proof of Theorem 4.1.

We now prove (1.3) to complete the proof of Theorem 1.1. We will first show that

$$F_n(\tau)=\sum_{k=0}^{n-1}\sum_{\substack{m_1,m_2,\dots,m_n\in\mathbf{Z}\\ m_1+m_2+\cdots+m_n=k}} q^{n(m_1^2+m_2^2+\cdots+m_n^2)-k^2}. \tag{2.7}$$

We replace $e^{2\pi iz}$ by x in the series representation for $\theta_3(z\mid\tau)$ and rewrite (1.2) as

$$\sum_{k=0}^{n-1}\left(q^{k^2}x^k\left\{\sum_{m=-\infty}^{\infty}q^{nm^2-2km}x^{-m}\right\}^n\right)=F_n(\tau)\sum_{m=-\infty}^{\infty}q^{m^2}x^m. \tag{2.8}$$

Substituting the expansion

$$\left\{\sum_{m=-\infty}^{\infty}q^{nm^2-2km}x^{-m}\right\}^n$$
$$=\sum_{m_1,m_2,\dots,m_n=-\infty}^{\infty}q^{n(m_1^2+m_2^2+\cdots+m_n^2)-2k(m_1+m_2+\cdots+m_n)}x^{-m_1-m_2-\cdots-m_n}$$

into (2.8) and then equating the constants, we arrive at (2.7). Now we use (2.7) to prove (1.3).

By the Cauchy–Schwarz inequality we have

$$n\left(m_1^2+m_2^2+\cdots+m_n^2\right)\geqslant(m_1+m_2+\cdots+m_n)^2=k^2.$$

Thus $F_n(\tau)$ is a power series in q. To prove (1.3), we need to study the number of the solutions of the following diophantine equations:

$$\begin{cases} n(m_1^2+m_2^2+\cdots+m_n^2)-k^2=t,\\ m_1+m_2+\cdots+m_n=k. \end{cases} \tag{2.9}$$

Let $N(t)$ denote the number of the solutions of the above equations. Then we have

$$F_n(\tau)=N(0)+N(1)q+\cdots+N(n-1)q^{n-1}+\cdots.$$

It is obvious that for any integer m, $m^2\geqslant m$. Thus we have

$$m_1^2+m_2^2+\cdots+m_n^2\geqslant m_1+m_2+\cdots+m_n.$$

Combining this with (2.9), we find

$$t\geqslant k(n-k),\quad \text{where } 0\leqslant k\leqslant n-1. \tag{2.10}$$

When $t=0$, this inequality holds only when $k=0$. Then (2.9) becomes

$$\begin{cases} m_1^2+m_2^2+\cdots+m_n^2=0,\\ m_1+m_2+\cdots+m_n=0. \end{cases}$$

The only solution of this equation is $m_1=m_2=\cdots=m_n=0$; and thus we have $N(0)=1$.

When $1\leqslant k\leqslant n-1$, we find from (2.10) that

$$t\geqslant k(n-k)\geqslant n-1. \tag{2.11}$$

Hence, $N(t)=0$ for $1\leqslant t\leqslant n-2$. The equality in (2.11) holds if and only if $k=1$ or $k=n-1$.

When $k=1$, (2.9) becomes

$$\begin{cases} m_1^2+m_2^2+\cdots+m_n^2=1, \\ m_1+m_2+\cdots+m_n=1. \end{cases}$$

The solutions of the above equation is $(1,0,\ldots,0),(0,1,\ldots,0),\ldots,(0,0,\ldots,1)$; and the number of the solutions is n.

When $k=n-1$, (2.9) becomes

$$\begin{cases} m_1^2+m_2^2+\cdots+m_n^2=n-1, \\ m_1+m_2+\cdots+m_n=n-1. \end{cases} \tag{2.12}$$

The solutions of the above equation is $(0,1,\ldots,1),(1,0,\ldots,1),\ldots,(1,1,\ldots,0)$, since

$$m_1(m_1-1)+\cdots+m_n(m_n-1)=0,$$

and $m_i^2\geqslant m_i$ implies that $m_i(m_i-1)=0$. This implies that $m_i=0$ or 1. Therefore, the number of the solutions in this case is n and combining with the case $k=1$ we conclude that $N(n-1)=2n$. This completes the proof of (1.3). □

Remark. A different expression for $F_n(\tau)$ which follows from [7, p. 115] can be found in Chua [4, Corollary 2.2].

3. A related identity

In this section we will prove the following identity which may be regarded as an equivalent form of the circular summation formula (1.2).

Theorem 3.1. *We have*

$$\sum_{k=0}^{n-1}\theta_3^n\left(z+\frac{k\pi}{n}\;\Big|\;\tau\right)=G_n(\tau)\theta_3(nz\mid n\tau), \tag{3.1}$$

where

$$G_n(\tau)=\sqrt{n}(-i\tau)^{\frac{1-n}{2}}F_n\left(-\frac{1}{n\tau}\right). \tag{3.2}$$

Proof. We recall the Jacobi imaginary transformation formula [16, p. 475]

$$\theta_3\left(\frac{z}{\tau}\;\Big|\;-\frac{1}{\tau}\right)=\sqrt{-i\tau}\,e^{iz^2/\pi\tau}\theta_3(z\mid\tau). \tag{3.3}$$

Replacing τ by $-1/n\tau$ and then z by z/τ in (1.2), we have

$$F_n\left(-\frac{1}{n\tau}\right)\theta_3\left(\frac{z}{\tau}\;\Big|\;-\frac{1}{n\tau}\right)=\sum_{k=0}^{n-1}e^{-\frac{\pi ik^2}{n\tau}+\frac{2ikz}{\tau}}\theta_3^n\left(\frac{z}{\tau}-\frac{k\pi}{n\tau}\;\Big|\;-\frac{1}{\tau}\right). \tag{3.4}$$

Using (3.3) in (3.4), we find that

$$F_n\left(-\frac{1}{n\tau}\right)\sqrt{-in\tau}\, e^{\frac{inz^2}{\pi\tau}}\theta_3(nz \mid n\tau) = (-i\tau)^{\frac{n}{2}} e^{\frac{inz^2}{\pi\tau}} \sum_{k=0}^{n-1}\theta_3^n\left(z - \frac{k\pi}{n} \;\middle|\; \tau\right).$$

Comparing the above two equations yields

$$\sum_{k=0}^{n-1}\theta_3^n\left(z - \frac{k\pi}{n} \;\middle|\; \tau\right) = \sqrt{n}(-i\tau)^{\frac{1-n}{2}} F_n\left(-\frac{1}{n\tau}\right)\theta_3(nz \mid n\tau).$$

Replacing z by $-z$ and noting that $\theta_3(z \mid \tau)$ is an even function of z, we arrive at (3.1). We complete the proof of Theorem 3.1. □

We now compute a representation for $G_n(\tau)$. Replace $e^{2\pi iz}$ by x in the series expansion of $\theta_3(z \mid \tau)$ and rewrite Theorem 3.1 as

$$\sum_{k=0}^{n-1}\left\{\sum_{m=-\infty}^{\infty} q^{m^2}\omega_n^{km}x^m\right\}^n = G_n(\tau)\sum_{m=-\infty}^{\infty} q^{nm^2}x^{nm}, \tag{3.5}$$

where $\omega_n = \exp\frac{2\pi i}{n}$. Since

$$\left\{\sum_{m=-\infty}^{\infty} q^{m^2}\omega_n^{km}\right\}^n = \sum_{m_1,m_2,\dots,m_n=-\infty}^{\infty} q^{m_1^2+m_2^2+\cdots+m_n^2}\omega_n^{k(m_1+m_2+\cdots+m_n)}x^{m_1+m_2+\cdots+m_n},$$

we deduce from (3.5) that

$$G_n(\tau) = n\sum_{\substack{m_1+m_2+\cdots+m_n=0\\ m_1,m_2,\dots,m_n=-\infty}}^{\infty} q^{m_1^2+m_2^2+\cdots+m_n^2}. \tag{3.6}$$

Remark. Chua [5] observed that when $n = p$, where $p > 3$ is an odd prime, the function

$$\frac{F_p(\tau)\eta(\tau)}{\eta^p(p\tau)}$$

is a modular function invariant under $\Gamma_0(p)$. He then computed $F_p(\tau)$ for $p = 5, 7$ and 13. We note here that it follows from the transformation formula (3.2), the function

$$\frac{G_p(\tau)\eta(\tau)}{\eta^p(p\tau)}$$

is also a modular function invariant under $\Gamma_0(p)$. For example, when $p = 5$, we have the identity

$$5\sum_{m_1,m_2,m_3,m_4=-\infty}^{\infty} q^{m_1^2+m_2^2+m_3^2+m_4^2+(m_1+m_2+m_3+m_4)^2}$$

$$= 5q^2\left(\prod_{k\geqslant 1}\frac{(1-q^{10k})^5}{(1-q^{2k})}\right)\left(\frac{1}{q^2}\prod_{k\geqslant 1}\frac{(1-q^{2k})^6}{(1-q^{10k})^6}+25\right).$$

For a general p, where the genus of $\Gamma_0(p)$ is not zero, we can still compute G_p using results from [3]. For example, for $p=11$,

$$F_{11}(\tau)=q^{10}\left(\prod_{k\geqslant 1}\frac{(1-q^{22k})^{11}}{(1-q^{2k})}\right)\left(P_1P_2-7P_1^2+9P_2-37P_1+3\right)$$

and

$$G_{11}(\tau)=q^{10}\left(\prod_{k\geqslant 1}\frac{(1-q^{22k})^{11}}{(1-q^{2k})}\right)\left(11P_1P_2+1133P_1^2+15829P_2 + 18953P_1+119823\right),$$

where

$$P_1=F_{2,1,0,2,2} \quad\text{and}\quad P_2=F_{3,2,0,2,4},$$

$$F_{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5}=\sum_{j=1}^{5}\prod_{k=1}^{5}W_{11,2^jk}^{\alpha_k},$$

and

$$W_{i,j}=q^{2(6j^2/l-j)}\prod_{n=1}^{\infty}\frac{(1-q^{2(l(n-1)+4j)})(1-q^{2(ln-4j)})}{(1-q^{2(l(n-1)+2j)})(1-q^{2(ln-2j)})},\quad 1\leqslant j\leqslant\frac{l-1}{2}.$$

Our representation for F_{11} is different from that of Ono [11].

4. Two related identities

It is clear that Theorem 3.1 is obtained from Theorem 1.1 via the transformation that sends τ to $-1/(n\tau)$. In this section, we use different transformations to derive further analogues of Theorem 1.1:

Theorem 4.1.

$$\sum_{k=0}^{n-1}(-1)^kq^{k^2}e^{2kiz}\theta_1^n(z+k\pi\tau\mid n\tau)=\begin{cases}F_n(\tau)\theta_4(z\mid\tau), & n \text{ is even},\\ F_n(\tau)\theta_1(z\mid\tau), & n \text{ is odd}.\end{cases}\tag{4.1}$$

Theorem 4.2. *We have*

$$\sum_{k=0}^{n-1}\theta_1^n\left(z+\frac{k\pi}{n}\,\middle|\,\tau\right)=\begin{cases}i^{-n}G_n(\tau)\theta_2(nz\mid n\tau), & n \text{ is even},\\ i^{1-n}G_n(\tau)\theta_1(nz\mid n\tau), & n \text{ is odd}.\end{cases}$$

We will only prove Theorem 4.1. The proof of Theorem 4.2 is similar and we omit its proof.

Proof. Replacing z by $z+\frac{\pi+n\pi\tau}{2}$ in Theorem 1.1, we have

$$\sum_{k=0}^{n-1}(-1)^k q^{k^2+nk} e^{2kiz}\theta_3^n\left(z+k\pi\tau+\frac{\pi+n\pi\tau}{2}\,\Big|\, n\tau\right)$$
$$=\theta_3\left(z+\frac{\pi+n\pi\tau}{2}\,\Big|\,\tau\right)F_n(\tau). \tag{4.2}$$

From the definitions of $\theta_1(z\mid\tau)$ and $\theta_3(z\mid\tau)$ we can readily find that

$$\theta_3\left(z+\frac{\pi+\pi\tau}{2}\,\Big|\,\tau\right)=iq^{-1/4}e^{-iz}\theta_1(z\mid\tau).$$

Replacing τ by $n\tau$ and then z by $z+k\pi\tau$, we have

$$\theta_3\left(z+k\pi\tau+\frac{\pi+n\pi\tau}{2}\,\Big|\,n\tau\right)=iq^{-n/4-k}e^{-iz}\theta_1(z+k\pi\tau\mid n\tau).$$

It follows that

$$\theta_3^n\left(z+k\pi\tau+\frac{\pi+n\pi\tau}{2}\,\Big|\,n\tau\right)=i^n q^{-n^2/4-kn}e^{-niz}\theta_1^n(z+k\pi\tau\mid n\tau).$$

Combining this with (4.2), we have

$$i^n q^{-n^2/4}e^{-niz}\sum_{k=0}^{n-1}(-1)^k q^{k^2}e^{2kiz}\theta_1^n(z+k\pi\tau\mid n\tau)=\theta_3\left(z+\frac{\pi+n\pi\tau}{2}\,\Big|\,\tau\right)F_n(\tau).$$

Using the identity $\theta_3(z+\frac{\pi}{2}\mid\tau)=\theta_4(z\mid\tau)$ in the right-hand side of the above equation, we have

$$i^n q^{-n^2/4}e^{-niz}\sum_{k=0}^{n-1}(-1)^k q^{k^2}e^{2kiz}\theta_1^n(z+k\pi\tau\mid n\tau)=\theta_4\left(z+\frac{n\pi\tau}{2}\,\Big|\,\tau\right)F_n(\tau). \tag{4.3}$$

From the definition of $\theta_4(z\mid\tau)$ we find that

$$\theta_4(z+\pi\tau\mid\tau)=-q^{-1}e^{-2iz}\theta_4(z\mid\tau).$$

Iterating the above equation, we find that

$$\theta_4(z+l\pi\tau\mid\tau)=(-1)^l q^{-l^2}e^{-2liz}\theta_4(z\mid\tau).$$

When n is even, we set $l=n/2$ and obtain

$$\theta_4\left(z+\frac{n\pi\tau}{2}]\,\Big|\,\tau\right)=i^n q^{-n^2/4}e^{-niz}\theta_4(z\mid\tau). \tag{4.4}$$

When n is odd, we have

$$\begin{aligned}\theta_4\left(z+\frac{n\pi\tau}{2}\,\Big|\,\tau\right)&=\theta_4\left(z+\frac{\pi\tau}{2}+\frac{(n-1)\pi\tau}{2}\,\Big|\,\tau\right)\\&=i^{n-1}q^{-(n-1)^2/4-(n-1)/2}e^{-(n-1)iz}\theta_4\left(z+\frac{\pi\tau}{2}\,\Big|\,\tau\right).\end{aligned}\tag{4.5}$$

Using the identity

$$\theta_4\left(z+\frac{\pi\tau}{2}\,\Big|\,\tau\right)=iq^{-1/4}e^{-iz}\theta_1(z\mid\tau)$$

in (4.5), we find that

$$\theta_4\left(z+\frac{n\pi\tau}{2}\,\Big|\,\tau\right)=i^nq^{-n^2/4}e^{niz}\theta_1(z\mid\tau).\tag{4.6}$$

Combining (4.3), (4.4) and (4.6), we obtain (4.1). This completes the proof of Theorem 4.1. □

In the next two sections, we will specialize our theorems by setting $n=2$ and $n=3$.

5. Some identities when $n=2$

Theorem 5.1. *We have*

$$\theta_2^2(z\mid 2\tau)+\theta_3^2(z\mid 2\tau)=\theta_3(0\mid\tau)\theta_3(z\mid\tau),\tag{5.1}$$

$$\theta_1^2(z\mid 2\tau)+\theta_4^2(z\mid 2\tau)=\theta_3(0\mid\tau)\theta_4(z\mid\tau)\quad\textit{and}\tag{5.2}$$

$$\theta_3^4(z\mid\tau)-\theta_2^4(z\mid\tau)=\theta_4^3(0\mid\tau)\theta_4(2z\mid\tau).\tag{5.3}$$

Proof. When $n=2$, Theorem 1.1 reduces to

$$\theta_3^2(z\mid 2\tau)+qe^{2iz}\theta_3^2(z+\pi\tau\mid 2\tau)=F_2(\tau)\theta_3(z\mid\tau).\tag{5.4}$$

Using the identity $\theta_3(z+\frac{\pi\tau}{2}\mid\tau)=q^{-1/4}e^{-iz}\theta_2(z\mid\tau)$, we find

$$\theta_3(z+\pi\tau\mid 2\tau)=q^{-1/2}e^{-iz}\theta_2(z\mid 2\tau).\tag{5.5}$$

Combining (5.4) and (5.5), we have

$$\theta_2^2(z\mid 2\tau)+\theta_3^2(z\mid 2\tau)=F_2(\tau)\theta_3(z\mid\tau).\tag{5.6}$$

Similarly when $n=2$, from Theorem 4.1 we can find

$$\theta_1^2(z\mid 2\tau)+\theta_4^2(z\mid 2\tau)=F_2(\tau)\theta_4(z\mid\tau).\tag{5.7}$$

Setting $z=0$ in the above equation, we find that

$$F_2(\tau)=\frac{\theta_4^2(0\mid 2\tau)}{\theta_4(0\mid\tau)}.$$

From the infinite product representations for $\theta_3(z\mid\tau)$ and $\theta_4(z\mid\tau)$, namely,

$$\theta_3(z\mid\tau)=\prod_{n=1}^{\infty}\left(1-q^{2n}\right)\left(1+e^{2iz}q^{2n-1}\right)\left(1+e^{-2iz}q^{2n-1}\right),$$

and

$$\theta_4(z \mid \tau) = \prod_{n=1}^{\infty} (1-q^{2n})(1-e^{2iz}q^{2n-1})(1-e^{-2iz}q^{2n-1})$$

we find that

$$\theta_3(z \mid \tau)\theta_4(z \mid \tau) = \theta_4(0 \mid 2\tau)\theta_4(2z \mid 2\tau). \tag{5.8}$$

Setting $z = 0$, we find that

$$F_2(\tau) = \frac{\theta_4^2(0 \mid 2\tau)}{\theta_4(0 \mid \tau)} = \theta_3(0 \mid \tau).$$

Substituting this into (5.6) and (5.7), we obtain (5.1) and (5.2), respectively.

Replacing q by $-q$ in (5.1), we have

$$\theta_3^2(z \mid 2\tau) - \theta_2^2(z \mid 2\tau) = \theta_4(0 \mid \tau)\theta_4(z \mid \tau).$$

Multiplying this with (5.1) and then using (5.8), we find that

$$\theta_3^4(z \mid 2\tau) - \theta_2^4(z \mid 2\tau) = \theta_4(0 \mid 2\tau)\theta_4(2z \mid 2\tau).$$

Replacing q^2 by q, we arrive at (5.3). We complete the proof of Theorem 5.1. □

Remark. If we set $z = 0$ in (5.3), we recover Jacobi's famous identity

$$\theta_1^4(0 \mid \tau) - \theta_2^4(0 \mid \tau) = \theta_4^4(0 \mid \tau).$$

Identity (5.3) can also be found in [16, p. 488, Example 4].

6. Some identities when $n = 3$

In this section we will prove Theorem 6.1 followed by Theorem 6.2.

Theorem 6.1. *We have*

$$\begin{aligned}\theta_3^3(z \mid 3\tau) + qe^{2iz}\theta_3^3(z+\pi\tau \mid 3\tau) + qe^{-2iz}\theta_3^3(z-\pi\tau \mid 3\tau) &= a(\tau)\theta_3(z \mid \tau),\\ \theta_1^3(z \mid 3\tau) - qe^{2iz}\theta_1^3(z+\pi\tau \mid 3\tau) - qe^{-2iz}\theta_1^3(z-\pi\tau \mid 3\tau) &= a(\tau)\theta_1(z \mid \tau),\end{aligned} \tag{6.1}$$

where $a(\tau)$ is the Ramanujan function defined as

$$a(\tau) = 1 + 6\sum_{m=0}^{\infty}\left(\frac{q^{2(3m+1)}}{1-q^{2(3m+1)}} - \frac{q^{2(3m+2)}}{1-q^{2(3m+2)}}\right). \tag{6.2}$$

Proof. Taking $n = 3$ in Theorem 1.1, we have

$$\theta_3^3(z \mid 3\tau) + qe^{2iz}\theta_3^3(z+\pi\tau \mid 3\tau) + q^4e^{4iz}\theta_3^3(z+2\pi\tau \mid 3\tau) = F_3(\tau)\theta_3(z \mid \tau). \tag{6.3}$$

Replacing τ by 3τ in the second identity in (2.1), we have

$$\theta_3(z+3\pi\tau \mid 3\tau) = q^{-3}e^{-2iz}\theta_3(z \mid 3\tau).$$

Replacing z by $z-\pi\tau$, we find that

$$\theta_3(z+2\pi\tau \mid 3\tau)=q^{-1}e^{-2iz}\theta_3(z-\pi\tau \mid 3\tau). \tag{6.4}$$

Substituting (6.4) into (6.3), we arrive at

$$\theta_3^3(z \mid 3\tau)+qe^{2iz}\theta_3^3(z+\pi\tau \mid 3\tau)+qe^{-2iz}\theta_3^3(z-\pi\tau \mid 3\tau)=F_3(\tau)\theta_3(z \mid \tau). \tag{6.5}$$

In the same way when $n=3$, Theorem 4.1 reduces to

$$\theta_1^3(z \mid 3\tau)-qe^{2iz}\theta_1^3(z+\pi\tau \mid 3\tau)-qe^{-2iz}\theta_1^3(z-\pi\tau \mid 3\tau)=F_3(\tau)\theta_1(z \mid \tau). \tag{6.6}$$

We recall the infinite product representation for $\theta_1(z \mid \tau)$, namely,

$$\begin{aligned}\theta_1(z \mid \tau)&=2q^{1/4}(\sin z)\prod_{n=1}^{\infty}\bigl(1-q^{2n}\bigr)\bigl(1-e^{2iz}q^{2n}\bigr)\bigl(1-e^{-2iz}q^{2n}\bigr)\\&=iq^{1/4}e^{-iz}\prod_{n=1}^{\infty}\bigl(1-q^{2n}\bigr)\bigl(1-e^{2iz}q^{2n-2}\bigr)\bigl(1-e^{-2iz}q^{2n}\bigr),\end{aligned} \tag{6.7}$$

from which we can infer

$$\theta_1'(0 \mid \tau)=2q^{1/4}\prod_{n=1}^{\infty}\bigl(1-q^{2n}\bigr)^3 \quad \text{and} \quad \theta_1(\pi\tau \mid 3\tau)=iq^{-1/4}\prod_{n=1}^{\infty}\bigl(1-q^{2n}\bigr). \tag{6.8}$$

Using the technique of logarithmic differentiation on (6.7), we obtain

$$\frac{\theta_1'}{\theta_1}(z \mid \tau)=-i-2i\sum_{n=0}^{\infty}\frac{q^{2n}e^{2iz}}{1-q^{2n}e^{2iz}}+2i\sum_{n=1}^{\infty}\frac{q^{2n}e^{-2iz}}{1-q^{2n}e^{-2iz}}.$$

Comparing this with (6.2), we infer that

$$a(\tau)=-2+3i\frac{\theta_1'}{\theta_1}(\pi\tau \mid 3\tau). \tag{6.9}$$

Next we turn to determine $F_3(\tau)$. We differentiate both sides of (6.6) and then set $z=0$ to get

$$2qi\theta_1^3(\pi\tau \mid 3\tau)\left\{-2+3i\frac{\theta_1'}{\theta_1}(\pi\tau \mid 3\tau)\right\}=F_3(\tau)\theta_1'(0 \mid \tau).$$

Using (6.8) and (6.9) in the above equation, we conclude that $F_3(\tau)=a(\tau)$. Substituting this into (6.5) and (6.6), we complete the proof of Theorem 6.1. □

Theorem 6.2. *We have*

$$\begin{aligned}&\theta_3^3(z \mid \tau)+\theta_3^3\left(z+\frac{\pi}{3} \,\middle|\, \tau\right)+\theta_3^3\left(z-\frac{\pi}{3} \,\middle|\, \tau\right)=3a(\tau)\theta_3(3z \mid 3\tau),\\&\theta_1^3\left(z+\frac{\pi}{3} \,\middle|\, \tau\right)+\theta_1^3\left(z-\frac{\pi}{3} \,\middle|\, \tau\right)-\theta_1^3(z \mid \tau)=3a(\tau)\theta_1(3z \mid 3\tau).\end{aligned} \tag{6.10}$$

Proof. Taking $n=3$ in Theorem 3.1, we have

$$\theta_3^3(z \mid \tau)+\theta_3^3\left(z+\frac{\pi}{3} \,\middle|\, \tau\right)+\theta_3^3\left(z+\frac{2\pi}{3} \,\middle|\, \tau\right)=G_3(\tau)\theta_3(3z \mid 3\tau). \tag{6.11}$$

Using the identity $\theta_3(z+\pi \mid \tau)=\theta_3(z \mid \tau)$, we find that $\theta_3(z+\frac{2\pi}{3} \mid \tau)=\theta_3(z-\frac{\pi}{3} \mid \tau)$. Thus we have

$$\theta_3^3(z \mid \tau)+\theta_3^3\left(z+\frac{\pi}{3} \,\middle|\, \tau\right)+\theta_3^3\left(z-\frac{\pi}{3} \,\middle|\, \tau\right)=G_3(\tau)\theta_3(3z \mid 3\tau). \tag{6.12}$$

Similarly, from Theorem 4.2 we deduce that

$$\theta_1^3\left(z+\frac{\pi}{3} \,\middle|\, \tau\right)+\theta_1^3\left(z-\frac{\pi}{3} \,\middle|\, \tau\right)-\theta_1^3(z \mid \tau)=G_3(\tau)\theta_1(3z \mid 3\tau). \tag{6.13}$$

Using the infinite product representation for $\theta_1(z \mid \tau)$, we can find that

$$\theta_1\left(\frac{\pi}{3} \,\middle|\, \tau\right)=\sqrt{3}\,q^{1/4}\prod_{n=1}^{\infty}\left(1-q^{6n}\right). \tag{6.14}$$

It is well known that the trigonometric series expansion for the logarithmic derivative of $\theta_1(z \mid \tau)$ is

$$\frac{\theta_1'}{\theta_1}(z \mid \tau)=\cot z+4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\sin 2nz. \tag{6.15}$$

Thus, we have

$$\frac{\theta_1'}{\theta_1}\left(\frac{\pi}{3} \,\middle|\, \tau\right)=\frac{1}{\sqrt{3}}a(\tau). \tag{6.16}$$

Differentiating both sides of (6.13) with respect to z and then setting $z=0$, we find

$$2\theta_1^3\left(\frac{\pi}{3} \,\middle|\, \tau\right)\times\frac{\theta_1'}{\theta_1}\left(\frac{\pi}{3} \,\middle|\, \tau\right)=G_3(\tau)\theta_1'(0 \mid 3\tau). \tag{6.17}$$

Using (6.14) and (6.16) in the above equation, we find $G_3(\tau)=a(\tau)$. Substituting this into (6.13) and (6.14), we complete the proof of Theorem 6.2. □

Remark. Identities (6.1) and (6.10) have been recently proved by Liu [9].

7. Analogues of the Dixon functions

It is well known [16, p. 492] from the theory of Jacobian elliptic functions and theta functions that

$$\operatorname{sn}(u \mid \tau)=\frac{\theta_3(0 \mid \tau)}{\theta_2(0 \mid \tau)}\frac{\theta_1(u/\theta_3^2(0) \mid \tau)}{\theta_4(u/\theta_3^2(0) \mid \tau)}.$$

We can therefore think of

$$\operatorname{sn}\big(u\big(\theta_3^2(0)\big) \mid 2\tau\big)=Ce^{iu}\frac{\theta_1(u \mid 2\tau)}{\theta_1(u-\pi\tau \mid 2\tau)}$$

for some constant C. Inspired by this, one could define the analogue of $\mathrm{sn}(u)$ as

$$\mathcal{S}(u) = q^{-1/3} e^{2iu/3} \frac{\theta_1(u \mid 3\tau)}{\theta_1(u - \pi\tau \mid 3\tau)}.$$

We also define

$$\mathcal{C}(u) = -e^{4iu/3} \frac{\theta_1(u + \pi\tau \mid 3\tau)}{\theta_1(u - \pi\tau \mid 3\tau)}.$$

From (6.1), we conclude that

$$\mathcal{S}^3(u) + \mathcal{C}^3(u) - 1 = -a(\tau) \prod_{n=1}^{\infty} \frac{(1 - q^{2n})}{(1 - q^{6n})^3} \mathcal{S}(u)\mathcal{C}(u),$$

where we have used the relation [9, (5.1)]

$$\theta_1\left(u \,\Big|\, \frac{\tau}{3}\right) = \prod_{n=1}^{\infty} \frac{(1 - q^{n/3})}{(1 - q^n)^3} \theta_1(u \mid \tau)\theta_1\left(u + \frac{\pi\tau}{3} \,\Big|\, \tau\right)\theta_1\left(u - \frac{\pi\tau}{3} \,\Big|\, \tau\right).$$

Hence, we deduce that

$$\mathcal{S}^3(u) + \mathcal{C}^3(u) - 1 = -3\frac{a(\tau)}{c(\tau)} \mathcal{S}(u)\mathcal{C}(u),$$

where we have used the product representation of $c(\tau)$ [2, p. 109, (5.5)], $c(\tau)$ being one of the Borwein's function

$$c(\tau) = \sum_{m,n=-\infty}^{\infty} q^{2((m+1/3)^2 + (m+1/3)(n+1/3) + (n+1/3)^2)} = 3q^{2/3} \prod_{n=1}^{\infty} \frac{(1 - q^{6n})^3}{(1 - q^{2n})}.$$

Remarks.

(1) The functions $\mathcal{S}(u)$ and $\mathcal{C}(u)$ satisfy a relation similar to that of Dixon's functions $\mathrm{sm}(u)$ and $\mathrm{cm}(u)$ (see [8] for their definitions), and hence they can be viewed as analogues of Dixon's functions. This connects Ramanujan's circular summation formula to the theory of elliptic functions developed by Dixon.
(2) Dixon's functions were studied recently by Conrad [6] and Ohyama [10]. Corresponding results satisfied by $\mathrm{sm}(u)$ and $\mathrm{cm}(u)$ for $\mathcal{S}(u)$ and $\mathcal{C}(u)$ can probably be derived using the same methods illustrated in those papers.
(3) The function $\mathcal{C}(u)$ also appears in L.C. Shen's paper [14, p. 130] in his study of Ramanujan's elliptic functions to the cubic base.

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