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The 26th power of Dedekind's η-function ☆

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Abstract

We prove a simple and explicit formula, which expresses the 26th power of Dedekind's η-function as a double series. The proof relies on properties of Ramanujan's Eisenstein series P, Q and R, and parameters from the theory of elliptic functions.

The formula reveals a number of properties of the product $\prod_{j=1}^{\infty}(1-q^j)^{26}$, for example its lacunarity, the action of the Hecke operator, and sufficient conditions for a coefficient to be zero.

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1. Introduction

Dedekind's η-function is defined by

$$\eta(z) = q^{1/24}\prod_{j=1}^{\infty}\left(1-q^j\right), \quad \text{where } q = e^{2\pi i z},\ \operatorname{Im}(z) > 0.$$

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For certain values of d, the expansion of $\eta^d(z)$ in powers of q has a particularly simple form. For example, the cases $d=1$ and $d=3$ are due to L. Euler and C.G.J. Jacobi, respectively:

$$\eta(24z)=\sum_{j=-\infty}^{\infty}(-1)^j q^{(6j+1)^2},$$

$$\eta^3(8z)=\sum_{j=-\infty}^{\infty}(4j+1)q^{(4j+1)^2}.$$

For $d=8$ we have

$$\eta^8(12z)=\frac{1}{2}\sum_{i=-\infty}^{\infty}(2i)^2 q^{3(2i)^2}\sum_{j=-\infty}^{\infty}(6j-2)q^{(6j-2)^2}$$
$$+\frac{1}{2}\sum_{i=-\infty}^{\infty}(2i+1)^2 q^{3(2i+1)^2}\sum_{j=-\infty}^{\infty}(6j+1)q^{(6j+1)^2},$$

and there are analogous formulas for $d=10$ and 14 in terms of double sums. The case $d=8$ was first considered by F. Klein and R. Fricke [7, p. 373]. The formula for $d=10$ was discovered by L. Winquist [17], who used it to give an elementary proof of S. Ramanujan's congruence

$$p(11m+6)\equiv 0 \pmod{11},$$

where $p(n)$ is the number of partitions of n. F.J. Dyson [5, p. 637] reports that the case $d=14$ was discovered by A.O.L. Atkin, and furthermore that Atkin had a formula for $d=26$. Apart from a special case cited by Dyson [5, p. 651], no details of Atkin's work on $d=26$ have been published.

Dyson found that there are elegant multiple series expansions for $d=3,8,10,14,15,21,24,26,28,35,36,\ldots$. At about the same time, I.G. Macdonald [10] discovered there is an elegant multiple series expansion for any value of d which is the dimension of a finite dimensional simple Lie algebra. Macdonald's results include all of the numbers on Dyson's list, except $d=26$. In [8] and [9], V.E. Leininger and S.C. Milne utilized [11] and multiple basic hypergeometric series techniques to derive new non-trivial explicit multiple series expansions for additional infinite families of values of d not in [10], but not for $d=26$. They also simplified Macdonald's results corresponding to affine root systems of type A_ℓ.

The purpose of this article is to prove a formula for $\eta^{26}(z)$ in terms of a double series. A special case of our formula is as follows. Suppose $12n+13$ is prime. Let $a+ib$ and $c+id$ be the unique Gaussian integers which satisfy the conditions:

$$a^2+b^2=24n+26,\quad a,b\equiv 1 \pmod 6,\ a<b,$$
$$c^2+3d^2=36n+39,\quad d\equiv 1 \pmod 6,\ c>0.$$

Then the coefficient of q^n in $\prod_{j=1}^{\infty}(1-q^j)^{26}$ is

$$\frac{1}{2^6 3^4 11^2 13}\left(\frac{(-1)^{(a+b-2)/6}}{2^6}\operatorname{Re}\bigl((a+ib)^{12}\bigr)+\frac{(-1)^{(c+d-1)/6}}{3^6}\operatorname{Re}\bigl((c+id\sqrt{3})^{12}\bigr)\right).$$

This is different from the formula of Atkin quoted by Dyson. As an example, when $n = 2$ we have $a+ib = -5+7i$ and $c+id = 6-5i$, so the coefficient of q^2 in $\prod_{j=1}^{\infty}(1-q^j)^{26}$ is therefore

$$\begin{aligned}&\frac{1}{2^6 3^4 11^2 13}\left(\frac{1}{64}\operatorname{Re}\bigl((-5+7i)^{12}\bigr)+\frac{1}{729}\operatorname{Re}\bigl(\bigl(6-5i\sqrt{3}\bigr)^{12}\bigr)\right)\\&=\frac{1}{2^6 3^4 11^2 13}(1025046359+1413128809)\\&=299.\end{aligned}$$

Our proof is based on the observation that $\eta^2(z)$ may be expressed as a product of two theta functions in two different ways:

$$\eta^2(z)=\left(\sum_{j=-\infty}^{\infty}(-1)^j q^{(6j+1)^2/24}\right)^2=\left(\sum_{j=-\infty}^{\infty}(-1)^j q^{(6j+1)^2/12}\right)\left(\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2}\right).$$

Atkin's proof [1] uses properties of $\eta^{10}(z)E_4^2(z)$ and $\eta^{14}(z)E_6(z)$, where E_4 and E_6 are Eisenstein series of weights 4 and 6, respectively. Atkin's notes [1] indicate that he discovered his formula for $\eta^{26}(z)$ in 1965, and in 1966 he found another formula, different from the one quoted by Dyson [5, p. 651]. For a published proof of a formula for $\eta^{26}(z)$, see the paper by J.-P. Serre [16].

2. Statement of results

Let m and n be real numbers and define

$$\begin{aligned}f(m,n)&=\sum_{j=0}^{6}\binom{12}{2j}(-1)^j m^j n^{6-j}\\&=m^6-66m^5n+495m^4n^2-924m^3n^3+495m^2n^4-66mn^5+n^6.\end{aligned}$$

Observe that

$$f\bigl(m^2,n^2\bigr)=\operatorname{Re}\bigl((m+in)^{12}\bigr).$$

Let

$$P=1-24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j},\qquad Q=1+240\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j},$$
$$R=1-504\sum_{j=1}^{\infty}\frac{j^5q^j}{1-q^j}.$$

We will prove the following identities:

Lemma 1.

$$\eta^2(z)\big(3999Q^3 - 4000R^2\big) = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j} f\left(\frac{(6i+1)^2}{2}, \frac{(6j+1)^2}{2}\right) q^{[(6i+1)^2+(6j+1)^2]/24}.$$

Lemma 2.

$$\eta^2(z)\big(5439Q^3 - 5438R^2\big) = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j} f\big(12i^2, (6j+1)^2\big) q^{i^2+(6j+1)^2/12}.$$

If we add these results and use the fact that ([13], [15, pp. 140–144])

$$Q^3 - R^2 = 1728\eta^{24}(z),$$

we obtain our main result:

Theorem 3.

$$16308864\,\eta^{26}(z) = \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j} f\left(\frac{(6i+1)^2}{2}, \frac{(6j+1)^2}{2}\right) q^{[(6i+1)^2+(6j+1)^2]/24} + \sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j} f\big(12i^2, (6j+1)^2\big) q^{i^2+(6j+1)^2/12}.$$

By comparing coefficients on both sides we readily obtain:

Corollary 4. *Let*

$$\eta^r(z) = q^{r/24}\sum_{n=0}^{\infty} p_r(n) q^n,$$

where the coefficients $p_r(n)$ *are defined by*

$$\prod_{m=1}^{\infty}\big(1-q^m\big)^r = \sum_{n=0}^{\infty} p_r(n) q^n.$$

Then

$$16308864 p_{26}(n) = \sum_{\substack{\alpha^2+\beta^2=24n+26\\ \alpha,\beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-2)/6} f\left(\frac{\alpha^2}{2}, \frac{\beta^2}{2}\right) + \sum_{\substack{\gamma^2+3\delta^2=36n+39\\ \gamma\equiv 0 \pmod 6,\, \delta\equiv 1 \pmod 6}} (-1)^{(\gamma+\delta-1)/6} f\left(\frac{\gamma^2}{3}, \delta^2\right).$$

3. Proof of Lemma 1

Let

$$V_{2\ell} = \frac{\sum_{j=-\infty}^{\infty} (-1)^j (6j+1)^{2\ell} q^{(6j+1)^2/24}}{\sum_{j=-\infty}^{\infty} (-1)^j q^{(6j+1)^2/24}}.$$

Using the relation

$$V_{2\ell+2} = P V_{2\ell} + 24q \frac{dV_{2\ell}}{dq}$$

and the Ramanujan differential equations for P, Q and R, Ramanujan [14, p. 369] showed that

$$\begin{aligned}
V_0 &= 1,\\
V_2 &= P,\\
V_4 &= 3P^2 - 2Q,\\
V_6 &= 15P^3 - 30PQ + 16R,\\
V_8 &= 105P^4 - 420P^2Q + 448PR - 132Q^2,\\
V_{10} &= 945P^5 - 6300P^3Q + 10080P^2R - 5940PQ^2 + 1216QR,\\
V_{12} &= 10395P^6 - 103950P^4Q + 221760P^3R\\
&\quad - 196020P^2Q^2 + 80256PQR - 2712Q^3 - 9728R^2.
\end{aligned} \tag{1}$$

Observe that

$$\begin{aligned}
&V_0V_{12} - 66V_2V_{10} + 495V_4V_8 - 924V_6^2 + 495V_8V_4 - 66V_{10}V_2 + V_{12}V_0\\
&\quad = 64\big(3999Q^3 - 4000R^2\big).
\end{aligned}$$

If we multiply this by $\eta^2(z)/64$ we complete the proof of Lemma 1. □

4. Proof of Lemma 2

The key to proving Lemma 2 is to write

$$\begin{aligned}\eta^2(z) &= q^{1/12}\prod_{j=1}^{\infty}\big(1-q^j\big)^2\\ &= q^{1/12}\prod_{j=1}^{\infty}\big(1-q^{2j}\big)\prod_{j=1}^{\infty}\frac{(1-q^j)^2}{(1-q^{2j})}\\ &= \left(\sum_{j=-\infty}^{\infty}(-1)^j q^{(6j+1)^2/12}\right)\left(\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2}\right). \end{aligned} \tag{2}$$

Let

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2}, \qquad \psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2},$$

and define

$$z = \varphi(q)^2, \qquad x = 16q\frac{\psi(q^2)^4}{\varphi(q)^4}.$$

Let

$$\begin{aligned}\overline{V}_{2\ell} &= \frac{\sum_{j=-\infty}^{\infty}(-1)^j(6j+1)^{2\ell}q^{(6j+1)^2/12}}{\sum_{j=-\infty}^{\infty}(-1)^j q^{(6j+1)^2/12}} = V_{2\ell}\big(q^2\big),\\ W_{2\ell} &= \frac{\sum_{j=-\infty}^{\infty}(-1)^j(12j^2)^{\ell}q^{j^2}}{\sum_{j=-\infty}^{\infty}(-1)^j q^{j^2}}.\end{aligned}$$

We will express $\overline{V}_{2\ell}$ and $W_{2\ell}$ in terms of P, z and x.

From [3, pp. 126–127] we have

$$P\big(q^2\big) = \frac{1}{2}\big(P + z^2(1+x)\big), \qquad Q\big(q^2\big) = z^4\big(1-x+x^2\big),$$
$$R\big(q^2\big) = z^6(1+x)\left(1-\frac{x}{2}\right)(1-2x).$$

Using these in (1) we obtain:

$$\begin{aligned}\overline{V}_0 &= 1,\\ 2\overline{V}_2 &= P + z^2(1+x),\\ 4\overline{V}_4 &= 3P^2 + 6Pz^2(1+x) - z^4\big(5-14x+5x^2\big),\end{aligned}$$

$$\begin{aligned}
8\overline{V}_6 &= 15P^3 + 45P^2z^2(1+x) - 15z^4P\big(5-14x+5x^2\big)\\
&\quad + z^6(1+x)\big(23x^2-170x+23\big),\\
16\overline{V}_8 &= 105P^4 + 420P^3z^2(1+x) - 210P^2z^4\big(5-14x+5x^2\big)\\
&\quad + 28Pz^6(1+x)\big(23-170x+23x^2\big)\\
&\quad - z^8\big(103-1172x+16458x^2-1172x^3+103x^4\big),\\
32\overline{V}_{10} &= 945P^5 + 4725P^4z^2(1+x) - 3150P^3z^4\big(5-14x+5x^2\big)\\
&\quad + 630P^2z^6(1+x)\big(23-170x+23x^2\big)\\
&\quad - 45Pz^8\big(103-1172x+16458x^2-1172x^3+103x^4\big)\\
&\quad + z^{10}(1+x)\big(257-7852x-346266x^2-7852x^3+257x^4\big),\\
64\overline{V}_{12} &= 10395P^6 + 62370P^5z^2(1+x) - 51975P^4z^4\big(5-14x+5x^2\big)\\
&\quad + 13860P^3z^6(1+x)\big(23-170x+23x^2\big)\\
&\quad - 1485P^2z^8\big(103-1172x+16458x^2-1172x^3+103x^4\big)\\
&\quad + 66Pz^{10}(1+x)\big(23-170x+23x^2\big)\big(257-7852x-346266x^2-7852x^3+257x^4\big)\\
&\quad + z^{12}\big(4387+12282x-10840467x^2-17010388x^3-10840467x^4\\
&\quad + 12282x^5+4387x^6\big).
\end{aligned}$$

Now we will express $W_{2\ell}$ in terms of P, z and x. First, observe that $W_0 = 1$. Next, from [3, pp. 120–129] we have:

$$\begin{gathered}
\varphi(-q) = z^{1/2}(1-x)^{1/4},\\
q\frac{dx}{dq} = z^2x(1-x),\\
12q\frac{dz}{dq} = Pz + z^3(5x-1),\\
Q = z^4\big(1+14x+x^2\big),\\
R = z^6(1+x)\big(1-34x+x^2\big).
\end{gathered}$$

Using these we obtain

$$\begin{aligned}
W_2 &= 12q\frac{d}{dq}\log\sqrt{\varphi(-q)}\\
&= 3q\frac{d}{dq}\log z(1-x)^{1/2}\\
&= \frac{3}{z}q\frac{dz}{dq} - \frac{3}{2(1-x)}q\frac{dx}{dq}\\
&= \frac{1}{2}\big(P - z^2 - z^2x\big).
\end{aligned}$$

If we apply $q\frac{d}{dq}$ to the equation defining $W_{2\ell}$ and simplify, we obtain the differential recurrence relation

$$W_{2\ell+2} = W_2 W_{2\ell} + 12q\frac{d}{dq}W_{2\ell}.$$

Additional values of $W_{2\ell}$ can be computed using the differential recurrence relation together with Ramanujan's differential equations for P, Q and R [13, Eq. (30), p. 142]. We obtain:

$$\begin{aligned}
\frac{4}{3}W_4 &= P^2 - 2Pz^2(1+x) + z^4\bigl(1-22x+x^2\bigr),\\
\frac{8}{3}W_6 &= 5P^3 - 15P^2z^2(1+x) + 15Pz^4\bigl(1-22x+x^2\bigr) - z^6(1+x)\bigl(5+226x+5x^2\bigr),\\
\frac{16}{3}W_8 &= 35P^4 - 140P^3z^2(1+x) + 210P^2z^4\bigl(1-22x+x^2\bigr)\\
&\quad - 28Pz^6(1+x)\bigl(5+226x+5x^2\bigr)\\
&\quad + z^8\bigl(35-2596x-6990x^2-2596x^3+35x^4\bigr),\\
\frac{32}{27}W_{10} &= 35P^5 - 175P^4z^2(1+x) + 350P^3z^4\bigl(1-22x+x^2\bigr)\\
&\quad - 70P^2z^6(1+x)\bigl(5+226x+5x^2\bigr)\\
&\quad + 5Pz^8\bigl(35-2596x-6990x^2-2596x^3+35x^4\bigr)\\
&\quad - z^{10}(1+x)\bigl(35-196x+18546x^2-196x^3+35x^4\bigr),\\
\frac{64}{27}W_{12} &= 385P^6 - 2310P^5z^2(1+x) + 5775P^4z^4\bigl(1-22x+x^2\bigr)\\
&\quad - 1540P^3z^6(1+x)\bigl(5+226x+5x^2\bigr)\\
&\quad + 165P^2z^8\bigl(35-2596x-6990x^2-2596x^3+35x^4\bigr)\\
&\quad - 66Pz^{10}(1+x)\bigl(35-196x^3+18546x^2-196x^3+35x^4\bigr)\\
&\quad + z^{12}\bigl(385+18078x-50385x^2-841180x^3-50385x^4+18078x^5+385x^6\bigr).
\end{aligned}$$

Observe that

$$\begin{aligned}
&\overline{V}_0W_{12} - 66\overline{V}_2W_{10} + 495\overline{V}_4W_8 - 924\overline{V}_6W_6 + 495\overline{V}_8W_4 - 66\overline{V}_{10}W_2 + \overline{V}_{12}W_0\\
&\quad = z^{12}\bigl(1+587346x-2348625x^2+3526652x^3-2348625x^4+587346x^5+x^6\bigr)\\
&\quad = 5439z^{12}\bigl(1+14x+x^2\bigr)^3 - 5438z^{12}(1+x)^2\bigl(1-34x+x^2\bigr)^2\\
&\quad = 5439Q^3 - 5438R^2.
\end{aligned}$$

If we multiply this by $\eta^2(z)$ and use (2) we complete the proof of Lemma 2.

5. Consequences

5.1. Lacunarity

A series $q^{\nu}\sum_{n=0}^{\infty} a_n q^n$ is called lacunary if almost all of the coefficients a_n are zero, i.e., $\lim_{N\to\infty}\{M(N)/N\}=0$, where $M(N)$ is the number of $n\leqslant N$ with $a_n\neq 0$. Serre [16] showed that the only even values of d for which $\eta^d(z)$ is lacunary are $d=2,4,6,8,10,14$ or 26. It is still unknown if there are any odd values of d, besides $d=1$ and 3, for which $\eta^d(z)$ is lacunary.

The series representation given in Theorem 3, together with a theorem of Landau [2, p. 244, Theorem 10.5], imply that $\eta^{26}(z)$ is lacunary.

5.2. The Hecke operator

If $p\equiv 11 \pmod{12}$, then Corollary 4 together with the elementary method used in [4] imply

$$p_{26}\left(pn+\frac{13}{12}(p^2-1)\right)=p^{12}p_{26}\left(\frac{n}{p}\right).$$

This provides an elementary derivation of the case $r=26$ of the following theorem of Newman [12]:

Theorem 5. *Suppose that r is one of the numbers 2, 4, 6, 8, 10, 14, 26. Let p be a prime >3 such that $r(p+1)\equiv 0 \pmod{24}$. Let $\Delta=(p^2-1)/24$, and define $p_r(x)=0$ if x is not a non-negative integer. Then*

$$p_r(pn+r\Delta)=(-p)^{r/2-1}p_r\left(\frac{n}{p}\right).$$

Furthermore there are no other values of r for which the theorem is true.

Elementary proofs for the cases $r=2$, 4, 6, 8, 10 and 14 of Newman's theorem were given in [4].

5.3. Values of n for which $p_{26}(n)=0$

The explicit formula in Corollary 4 may be used to prove two conditions given by Serre [16, p. 213] which imply $p_{26}(n)=0$.

Proposition 6. *Suppose that the factorization of $12n+13$ into distinct primes contains at least one prime congruent to -1 modulo 4 raised to an odd power, and also at least one prime congruent to -1 modulo 3 raised to an odd power. Then $p_{26}(n)=0$.*

Proof. Let $12n+13=\prod_p p^{\lambda_p}$ be the factorization of $12n+13$ into primes. Let $r_2(m)$ and $s(m)$ denote the number of solutions in integers of $x^2+y^2=m$ and $x^2+3y^2=m$, respectively. Two classical results [6] are, for any positive integer m,

$$r_2(m) = 4\big(d_{1,4}(m) - d_{3,4}(m)\big), \tag{3}$$

$$s(m) = 2\big(d_{1,3}(m) - d_{2,3}(m)\big) + 4\big(d_{4,12}(m) - d_{8,12}(m)\big), \tag{4}$$

where $d_{j,k}(m)$ denotes the number of divisors of m which are congruent to j modulo k. It follows that

$$r_2(24n+26) = 4 \prod_{p\equiv 1 \pmod 4} (\lambda_p + 1) \prod_{p\equiv -1 \pmod 4} \frac{1+(-1)^{\lambda_p}}{2}, \tag{5}$$

$$s(36n+39) = 2 \prod_{p\equiv 1 \pmod 3} (\lambda_p + 1) \prod_{p\equiv -1 \pmod 3} \frac{1+(-1)^{\lambda_p}}{2}. \tag{6}$$

Consequently, $r_2(24n+26) = s(36n+39) = 0$. Therefore the sums in Corollary 4 are empty and it follows that $p_{26}(n) = 0$. □

Proposition 7. *Suppose* $12n+13$ *is a square and all the prime factors of* $12n+13$ *are congruent to* -1 *modulo* 12. *Then* $p_{26}(n) = 0$.

Proof. Write $12n+13 = x^2 = \prod_p p^{\lambda_p}$, where the product is over primes $p \equiv -1 \pmod{12}$, and all the exponents λ_p are even. Without loss of generality we can assume $x \equiv 1 \pmod{12}$. Then (5) implies $r_2(26n+24) = 4$, and in fact the representations are $24n+26 = (\pm x)^2 + (\pm x)^2$. Similarly, (6) implies $s(36n+39) = 2$, and the representations are $36n+39 = (0)^2 + 3(\pm x)^2$. Corollary 4 implies

$$16308864 p_{26}(n) = f\left(\frac{x^2}{2}, \frac{x^2}{2}\right) + f\big(0, x^2\big) = x^{12} - x^{12} = 0. \qquad \square$$

5.4. The case when $12n+13$ *is prime*

For this section, let us consider the case when $12n+13$ is prime, in which case we write $p = 12n+13$. Corollary 4 implies

$$16308864 p_{26}(n) = \sum_{\substack{\alpha^2+\beta^2=2p \\ \alpha,\beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-2)/6} f\left(\frac{\alpha^2}{2}, \frac{\beta^2}{2}\right)$$
$$+ \sum_{\substack{\gamma^2+3\delta^2=3p \\ \gamma\equiv 0 \pmod 6,\ \delta\equiv 1 \pmod 6}} (-1)^{(\gamma+\delta-1)/6} f\left(\frac{\gamma^2}{3}, \delta^2\right).$$

Equation (3) implies that $\alpha^2 + \beta^2 = 2p$ has exactly eight solutions in integers, say $(\alpha, \beta) = (\pm a, \pm b), (\pm b, \pm a)$, where $a, b \equiv 1 \pmod 6$ and $a < b$. Two of these eight solutions satisfy the conditions $\alpha, \beta \equiv 1 \pmod 6$, namely $(\alpha, \beta) = (a, b)$ and $(\alpha, \beta) = (b, a)$. Therefore

$$\sum_{\substack{\alpha^2+\beta^2=2p \\ \alpha,\beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-2)/6} f\left(\frac{\alpha^2}{2}, \frac{\beta^2}{2}\right) = 2(-1)^{(a+b-2)/6} f\left(\frac{a^2}{2}, \frac{b^2}{2}\right).$$

Similarly, Eq. (4) implies that $\gamma^2 + 3\delta^2 = 3p$ has exactly four solutions in integers, say $(\gamma, \delta) = (\pm c, \pm d)$, where $c > 0$ and $d \equiv 1 \pmod 6$. Two of these four solutions satisfy $\gamma \equiv 0 \pmod 6$, $\delta \equiv 1 \pmod 6$, namely $(\gamma, \delta) = (c, d)$ and $(\gamma, \delta) = (-c, d)$. Therefore

$$\sum_{\substack{\gamma^2+3\delta^2=3p \\ \gamma\equiv 0 \ (\mathrm{mod}\ 6),\, \delta\equiv 1 \ (\mathrm{mod}\ 6)}} (-1)^{(\gamma+\delta-1)/6} f\left(\frac{\gamma^2}{3}, \delta^2\right) = 2(-1)^{(c+d-1)/6} f\left(\frac{c^2}{3}, d^2\right).$$

Therefore we have proved a result which is equivalent to the formula mentioned in the introduction:

Theorem 8. *Let $p = 12n + 13$ be prime. Let (a, b) be the unique solution in integers of*

$$a^2 + b^2 = 2p, \quad a, b \equiv 1 \pmod 6,\ a < b,$$

and let (c, d) be the unique solution in integers of

$$c^2 + 3d^2 = 3p, \quad d \equiv 1 \pmod 6,\ c > 0.$$

Then

$$8154432 p_{26}(n) = (-1)^{(a+b-2)/6} f\left(\frac{a^2}{2}, \frac{b^2}{2}\right) + (-1)^{(c+d-1)/6} f\left(\frac{c^2}{3}, d^2\right). \tag{7}$$

5.5. Remark

We note that

$$f\left(m^2, n^2\right) = \prod_{j=1}^{6} \left(m^2 - a_j^2 n^2\right)$$

where

$$a_1 = \sqrt{2} + 1, \qquad a_2 = \sqrt{2} - 1,$$
$$a_3 = \sqrt{6} + \sqrt{3} + \sqrt{2} + 2, \qquad a_4 = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2,$$
$$a_5 = \sqrt{6} - \sqrt{3} - \sqrt{2} + 2, \qquad a_6 = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2,$$

and

$$a_1 a_2 = a_3 a_4 = a_5 a_6 = 1.$$

As a result, (7) may be written as a sum of two expressions, each of which factors into linear factors involving a and b, and c and d, respectively.

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