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RAMANUJAN'S EISENSTEIN SERIES AND POWERS OF DEDEKIND'S ETA-FUNCTION

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Abstract

In this article, we use the theory of elliptic functions to construct theta function identities which are equivalent to Macdonald's identities for $\mathbf{A}_2, \mathbf{B}_2$ and $\mathbf{G}_2$. Using these identities, we express, for $d = 8, 10$ or 14, certain theta functions in the form $\eta^d(\tau)F(P,Q,R)$, where $\eta(\tau)$ is Dedekind's eta-function, and $F(P,Q,R)$ is a polynomial in Ramanujan's Eisenstein series P, Q and R. We also derive identities in the case when $d = 26$. These lead to a new expression for $\eta^{26}(\tau)$. This work generalizes the results for $d = 1$ and $d = 3$ which were given by Ramanujan on page 369 of 'The Lost Notebook'.

1. *Introduction*

Let $\mathrm{Im}(\tau) > 0$ and put $q = \exp(2\pi i\tau)$. Dedekind's eta-function is defined by

$$\eta(\tau) = q^{1/24}\prod_{k=1}^{\infty}(1-q^k),$$

and Ramanujan's Eisenstein series are

$$P = P(q) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k},$$

$$Q = Q(q) = 1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}$$

and

$$R = R(q) = 1 - 504\sum_{k=1}^{\infty}\frac{k^5q^k}{1-q^k}.$$

On page 369 of The Lost Notebook [**28**], Ramanujan gave the following results.

THEOREM 1.1 (Ramanujan). *Let*

$$S_1(m) = \sum_{\alpha\equiv 1 \pmod 6}(-1)^{(\alpha-1)/6}\alpha^m q^{\alpha^2/24},$$

$$S_3(m) = \sum_{\alpha\equiv 1 \pmod 4}\alpha^m q^{\alpha^2/8}.$$

Then

$$S_1(0) = \eta(\tau),$$
$$S_1(2) = \eta(\tau)P,$$
$$S_1(4) = \eta(\tau)(3P^2 - 2Q),$$
$$S_1(6) = \eta(\tau)(15P^3 - 30PQ + 16R),$$

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and in general

$$S_1(2m) = \eta(\tau) \sum_{i+2j+3k=m} a_{ijk} P^i Q^j R^k,$$

where a_{ijk} are integers and i, j and k are non-negative integers. Also

$$\begin{aligned} S_3(1) &= \eta^3(\tau), \\ S_3(3) &= \eta^3(\tau)P, \\ S_3(5) &= \eta^3(\tau)\frac{(5P^2 - 2Q)}{3}, \\ S_3(7) &= \eta^3(\tau)\frac{(35P^3 - 42PQ + 16R)}{9}, \end{aligned}$$

and in general

$$S_3(2m+1) = \eta^3(\tau) \sum_{i+2j+3k=m} b_{ijk} P^i Q^j R^k,$$

where b_{ijk} are rational numbers and i, j and k are non-negative integers.

The results for $S_1(0)$ and $S_3(1)$ are well-known consequences of the Jacobi triple product identity [**1**, p. 500]. Ramanujan also listed the values of $S_1(8)$, $S_1(10)$, $S_3(9)$ and $S_3(11)$. He indicated that these results may be proved by induction, using differentiation and the Ramanujan differential equations [**26**, equation (30)]

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \quad q\frac{dQ}{dq} = \frac{PQ - R}{3}, \quad q\frac{dR}{dq} = \frac{PR - Q^2}{2}.$$

Theorem 1.1 has been studied by Venkatachaliengar [**36**, pp. 31–32] (where both S_1 and S_3 are studied), Berndt and Yee [**5**] (where S_1 is studied) and Berndt, Chan, Liu and Yesilyurt [**6**] (where S_3 is studied). For a different approach to these identities, see Ramanujan [**27**, Chapter 16, Entry 35(i)] (for S_3), Berndt [**4**, p. 61] (for S_3) and Liu [**22**] (for S_1).

The first purpose of this article is to prove analogous results corresponding to the 2nd, 4th, 6th, 8th, 10th, 14th and 26th powers of $\eta(\tau)$, these being the even powers of $\eta(\tau)$ that are lacunary [**33**, Theorem 1]. For example, the result for the 14th power is as follows. For non-negative integers m, n, ℓ, let

$$\begin{aligned} S_{14}(m,n,\ell) = \sum_{\substack{\alpha\equiv 2 \pmod 6 \\ \beta\equiv 1 \pmod 4}} & (-1)^{(\alpha-2)/6} \left(\beta(\alpha^2-\beta^2)\right)^m \left(\alpha(\alpha^2-9\beta^2)\right)^n \\ & \times \left(\alpha^2+3\beta^2\right)^\ell q^{(\alpha^2+3\beta^2)/12}. \end{aligned}$$

Then

$$S_{14}(2m+1, 2n+1, \ell) = \eta^{14}(\tau) \sum_{i+2j+3k=3m+3n+\ell} c_{ijk} P^i Q^j R^k, \quad m,n,\ell \geqslant 0, \tag{1.1}$$

where c_{ijk} are rational numbers and i, j and k are non-negative integers. The first few instances of (1.1) are

$$\begin{aligned} S_{14}(1,1,0) &= -30\eta^{14}(\tau), \\ S_{14}(1,1,1) &= -210\eta^{14}(\tau)P, \\ S_{14}(1,1,2) &= -210\eta^{14}(\tau)(8P^2 - Q), \\ S_{14}(3,1,0) &= -5\eta^{14}(\tau)(56P^3 - 21PQ + 19R), \\ S_{14}(1,3,0) &= -15\eta^{14}(\tau)(504P^3 - 189PQ - 115R). \end{aligned}$$

An equation equivalent to the one for $S_{14}(1,1,0)$ was stated without proof by Winquist [**38**]. Since

$$\beta(\alpha^2-\beta^2)\alpha(\alpha^2-9\beta^2)=\alpha^5\beta-10\alpha^3\beta^3+9\alpha\beta^5=\frac{1}{6\sqrt{3}}\operatorname{Im}\left((\alpha+i\beta\sqrt{3})^6\right),$$

the result for $S_{14}(1,1,0)$ may be written as

$$\sum_{\substack{\alpha\equiv 2 \pmod 6\\ \beta\equiv 1 \pmod 4}}(-1)^{(\alpha-2)/6}\operatorname{Im}\left((\alpha+i\beta\sqrt{3})^6\right)q^{(\alpha^2+3\beta^2)/12}=-180\sqrt{3}\eta^{14}(\tau).$$

The second purpose of this article is to prove results of the type

$$\sum_{\substack{\alpha\equiv 2 \pmod 6\\ \beta\equiv 1 \pmod 4}}(-1)^{(\alpha-2)/6}\operatorname{Im}\left((\alpha+i\beta\sqrt{3})^{6n}\right)q^{(\alpha^2+3\beta^2)/12}=\sqrt{3}\eta^{14}(\tau)\sum_{2j+3k=3(n-1)}d_{jk}Q^jR^k, \tag{1.2}$$

where d_{jk} are rational numbers and j and k are non-negative integers. We shall state analogues of this result for the 2nd, 4th, 6th, 8th, 10th and 26th powers of $\eta(\tau)$ and give a detailed proof for the 10th power.

This work is organized as follows.

Notation and properties of theta functions are established in Section 2.

Sections 3, 4 and 5 are devoted to the 8th, 10th and 14th powers of $\eta(\tau)$, respectively. Each section begins with a multivariate theta function identity which is then used to prove the analogues of (1.1) for the 8th, 10th or 14th power of $\eta(\tau)$.

Section 6 is concerned with the analogues of (1.1) for 2nd, 4th and 6th powers of $\eta(\tau)$. These follow from Ramanujan's Theorem 1.1.

In Section 7, we prove results analogous to (1.2) for the 2nd, 4th, 6th, 8th, 10th and 14th powers of $\eta(\tau)$. Since Ramanujan's Eisenstein series P does not occur in these results, the modular transformation for multiple theta series given by Schoeneberg [**32**] can be used to prove them.

In Section 8, we give a simple proof of a series expansion for $\eta^{26}(\tau)$, as well as analogues of (1.1) and (1.2) for the 26th power of $\eta(\tau)$ which are new. The proofs rely on two different analogues of (1.2) for $\eta^2(\tau)$.

Finally, in Section 9, we make some remarks about lacunary series and the Hecke operator, and a new formula for $\eta^{24}(\tau)$ is presented.

2. *Preliminaries*

In the classical theory of theta functions [**37**], the notation $q=\exp(\pi i\tau)$ is used, whereas in the theory of modular forms $q=\exp(2\pi i\tau)$. Because we will use both theories, we let $t=2\tau$ and define

$$q=\exp(\pi it)=\exp(2\pi i\tau).$$

We will use t when working with theta functions and τ for modular forms and Dedekind's eta-function.

The Jacobi theta functions [**1**, p. 509; **37**, Chapter 21], are defined by

$$\theta_1(z|t)=2\sum_{k=0}^{\infty}(-1)^kq^{(k+1/2)^2}\sin(2k+1)z,$$

$$\theta_2(z|t)=2\sum_{k=0}^{\infty}q^{(k+1/2)^2}\cos(2k+1)z,$$

$$\theta_3(z|t)=1+2\sum_{k=1}^{\infty}q^{k^2}\cos 2kz$$

and

$$\theta_4(z|t) = 1 + 2\sum_{k=1}^{\infty}(-1)^k q^{k^2}\cos 2kz.$$

Let

$$\begin{aligned}
G_2(z|t) &= 2 \sum_{\alpha\equiv 1 \pmod 6} q^{\alpha^2/12}\sin(\alpha z),\\
G_3(z|t) &= 2 \sum_{\alpha\equiv 4 \pmod 6} q^{\alpha^2/12}\sin(\alpha z),\\
H(z|t) &= G_2(4z|4t) - G_3(4z|4t)\\
&= 2 \sum_{\alpha\equiv 2 \pmod 6} (-1)^{(\alpha-2)/6} q^{\alpha^2/12}\sin(2\alpha z)
\end{aligned}$$

and

$$T(z|t) = \theta_1(2z|t).$$

These functions satisfy the transformation properties

$$\begin{aligned}
&\theta_1(z+\pi|t) = -\theta_1(z|t), \quad \theta_1(z+\pi t|t) = -q^{-1}e^{-2iz}\theta_1(z|t),\\
&\theta_2(z+\pi|t) = -\theta_2(z|t), \quad \theta_2(z+\pi t|t) = q^{-1}e^{-2iz}\theta_2(z|t),\\
&\theta_3(z+\pi|t) = \theta_3(z|t), \quad \theta_3(z+\pi t|t) = q^{-1}e^{-2iz}\theta_3(z|t),\\
&\theta_4(z+\pi|t) = \theta_4(z|t), \quad \theta_4(z+\pi t|t) = -q^{-1}e^{-2iz}\theta_4(z|t),\\
&G_2(z+\pi|t) = -G_2(z|t), \quad G_2(z+\pi t|t) = q^{-3}e^{-6iz}G_2(z|t),\\
&G_3(z+\pi|t) = G_3(z|t), \quad G_3(z+\pi t|t) = q^{-3}e^{-6iz}G_3(z|t),\\
&H\left(z+\frac{\pi}{2}\middle|t\right) = H(z|t), \quad H\left(z+\frac{\pi t}{2}\middle|t\right) = -q^{-3}e^{-12iz}H(z|t),\\
&T\left(z+\frac{\pi}{2}\middle|t\right) = -T(z|t), \quad T\left(z+\frac{\pi t}{2}\middle|t\right) = -q^{-1}e^{-4iz}T(z|t).
\end{aligned}$$

By the Jacobi triple product identity [**1**, p. 497],

$$\theta_1(z|t) = 2q^{1/4}\sin z\prod_{k=1}^{\infty}(1-q^{2k}e^{2iz})(1-q^{2k}e^{-2iz})(1-q^{2k}).$$

Therefore, $\theta_1(z|t)$ has simple zeros at $z = \pi m + \pi t n$, m, $n \in \mathbb{Z}$, and no other zeros.

We will also need the results

$$\theta_2(z|t)G_2(z|t) = \eta(2\tau)\theta_1(2z|t), \tag{2.1}$$

$$\theta_3(z|t)G_3(z|t) = -\eta(2\tau)\theta_1(2z|t). \tag{2.2}$$

These are equivalent to the quintuple product identity. For example, see [**34**, Proposition 2.1], where these and two other similar equations are given. Equations (2.1) and (2.2), together with the Jacobi triple product identity, imply that $G_2(z|t)$ has simple zeros when $z = \pi m/2 + \pi t n/2$, where m and n are integers and $(m,n) \not\equiv (1,0) \pmod 2$, and no other zeros. Similarly, $G_3(z|t)$ has simple zeros when $z = \pi m/2 + \pi t n/2$, where m and n are integers and $(m,n) \not\equiv (1,1) \pmod 2$, and no other zeros. Equations (2.1) and (2.2) also imply that

$$\theta_2(z|t)G_2(z|t) + \theta_3(z|t)G_3(z|t) = 0.$$

The following lemma is of fundamental importance and will be used several times in the proofs in the subsequent sections. Let $f^{(\ell)}(z|t)$ denote the ℓth derivative of $f(z|t)$ with respect to z.

Lemma 2.1.

$$\theta_1^{(2\ell_1+1)}\left(0\middle|\frac{t}{2}\right)\theta_1^{(2\ell_2+1)}\left(0\middle|\frac{t}{2}\right)\cdots\theta_1^{(2\ell_m+1)}\left(0\middle|\frac{t}{2}\right) = (\eta(\tau))^{3m}\sum_{i+2j+3k=\ell_1+\ell_2+\cdots+\ell_m} a_{ijk}P^iQ^jR^k$$

for some rational numbers a_{ijk}, where i, j and k are non-negative integers.

Proof. Let us first consider the case $m = 1$. From the definition of θ_1, we have

$$\theta_1^{(2\ell+1)}(z|t) = 2(-1)^\ell\sum_{k=0}^{\infty}(-1)^k(2k+1)^{2\ell+1}q^{(k+(1/2))^2}\cos(2k+1)z.$$

Therefore

$$\begin{aligned}\theta_1^{(2\ell+1)}\left(0\middle|\frac{t}{2}\right) &= 2(-1)^\ell\sum_{k=0}^{\infty}(-1)^k(2k+1)^{2\ell+1}q^{(k+(1/2))^2/2}\\ &= 2(-1)^\ell\sum_{k=-\infty}^{\infty}(4k+1)^{2\ell+1}q^{(4k+1)^2/8}\\ &= 2(-1)^\ell S_3(2\ell+1)\\ &= \eta^3(\tau)\sum_{i+2j+3k=\ell}a_{ijk}P^iQ^jR^k\end{aligned}$$

by Theorem 1.1. The general case $m \geqslant 1$ now follows by multiplying m copies of this result together. □

Finally, we define the standard notation for products:

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-xq^k)$$

and

$$(x_1, x_2, \ldots, x_m; q)_\infty = (x_1;q)_\infty(x_2;q)_\infty\cdots(x_m;q)_\infty.$$

3. *The eighth power of $\eta(\tau)$*

The main tool used in this section is the following.

Theorem 3.1.

$$G_2(x|t)\theta_2(y|t) + G_3(x|t)\theta_3(y|t) = \frac{1}{\eta(\tau)}\theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right).$$

Proof. Let

$$M_8(x,y|t) = G_2(x|t)\theta_2(y|t) + G_3(x|t)\theta_3(y|t)$$

and

$$N_8(x,y|t) = \theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right).$$

Then the formulae listed in Section 2 imply that M_8 and N_8 satisfy the transformation properties

$$\begin{aligned} f(x+2\pi, y|t) &= f(x,y|t), \quad f(x+\pi t, y|t) = q^{-3}e^{-6ix}f(x,y|t),\\ f(x, y+2\pi|t) &= f(x,y|t), \quad f(x, y+\pi t|t) = q^{-1}e^{-2iy}f(x,y|t). \end{aligned}$$

Fix y and consider M_8 and N_8 as functions of x; N_8 has simple zeros at $x = \pi m + \pi t n/2$, $\pm y + 2\pi m + \pi t n$, $m, n \in \mathbb{Z}$, and no other zeros. By the results in Section 2, we see that M_8 also has zeros at these points, and possibly at other points too. Therefore, $M_8(x,y|t)/N_8(x,y|t)$ is an elliptic function of x with no poles and thus is a constant independent of x.

Now fix x and consider M_8 and N_8 as functions of y; N_8 has simple zeros at $y = \pm x + 2\pi m + \pi t n$ and no other zeros. It is easy to check that M_8 also has zeros at these points, and possibly at other points too. Therefore, M_8/N_8 is an elliptic function of y with no poles and thus is a constant independent of y.

It follows that

$$\frac{M_8(x,y|t)}{N_8(x,y|t)} = C(q)$$

for some $C(q)$ independent of x and y. To calculate $C(q)$, let $x = \pi/2$ and $y = \pi$. Since $G_3(\pi/2|t) = 0$, we have

$$\begin{aligned} M_8\left(\frac{\pi}{2}, \pi \middle| t\right) &= G_2\left(\frac{\pi}{2}\middle| t\right)\theta_2(\pi|t)\\ &= -2\sum_{k=-\infty}^{\infty}(-1)^k q^{(6k+1)^2/12}\sum_{j=-\infty}^{\infty} q^{(j+(1/2))^2}\\ &= -4\eta(2\tau)\, q^{1/4}(-q^2,-q^2,q^2;q^2)_\infty\\ &= -4\eta^2(4\tau). \end{aligned}$$

On the other hand

$$\begin{aligned} N_8\left(\frac{\pi}{2}, \pi \middle| t\right) &= \theta_1\left(-\frac{\pi}{4}\middle|\frac{t}{2}\right)\theta_1\left(\frac{\pi}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{3\pi}{4}\middle|\frac{t}{2}\right)\\ &= -\left(2q^{1/8}\right)^3 \sin\frac{\pi}{4}\sin\frac{\pi}{2}\sin\frac{3\pi}{4}(iq,-iq,q;q)_\infty^2(-q,-q,q;q)_\infty\\ &= -4\eta(\tau)\eta^2(4\tau), \end{aligned}$$

after simplifying. Therefore

$$C(q) = \frac{M_8(\pi/2,\pi|t)}{N_8(\pi/2,\pi|t)} = \frac{1}{\eta(\tau)}.$$

This completes the proof of Theorem 3.1. □

THEOREM 3.2. *Let m and n be non-negative integers and define*

$$S_8(m,n) = \sum_{\substack{\alpha\equiv 1 \pmod 3\\ \alpha+\beta\equiv 0 \pmod 2}} \alpha^m\beta^n q^{(\alpha^2+3\beta^2)/12}.$$

Then $S_8(1,0) = 0$ and

$$S_8(2m+1,2n) = \eta^8(\tau)\sum_{i+2j+3k=m+n-1} a_{ijk}P^iQ^jR^k, \tag{3.1}$$

provided $m+n \geqslant 1$. Here, a_{ijk} are rational numbers and i, j and k are non-negative integers.

Proof. Apply $\partial^{2m+2n+1}/(\partial x^{2m+1}\partial y^{2n})$ to the identity in Theorem 3.1 and let $x = y = 0$. The left-hand side is

$$\begin{aligned}
&G_2^{(2m+1)}(0|t)\,\theta_2^{(2n)}(0|\tau) + G_3^{(2m+1)}(0|t)\,\theta_3^{(2n)}(0|\tau)\\
&= 2(-1)^{m+n}\sum_{\alpha\equiv 1 \pmod 6}\alpha^{2m+1}q^{\alpha^2/12}\sum_{\beta\equiv 1 \pmod 2}\beta^{2n}q^{\beta^2/4}\\
&\quad + 2(-1)^{m+n}\sum_{\alpha\equiv 4 \pmod 6}\alpha^{2m+1}q^{\alpha^2/12}\sum_{\beta\equiv 0 \pmod 2}\beta^{2n}q^{\beta^2/4}\\
&= 2(-1)^{m+n}\sum_{\substack{\alpha\equiv 1 \pmod 3\\ \alpha+\beta\equiv 0 \pmod 2}}\alpha^{2m+1}\beta^{2n}q^{(\alpha^2+3\beta^2)/12}. \qquad (3.2)
\end{aligned}$$

Since $\theta_1(z|t)$ is an odd function, the right-hand side is a linear combination of terms of the form

$$\frac{1}{\eta(\tau)}\theta_1^{(2\ell_1+1)}\left(0\middle|\frac{t}{2}\right)\theta_1^{(2\ell_2+1)}\left(0\middle|\frac{t}{2}\right)\theta_1^{(2\ell_3+1)}\left(0\middle|\frac{t}{2}\right),$$

where $(2\ell_1+1)+(2\ell_2+1)+(2\ell_3+1) = 2m+2n+1$. By Lemma 2.1, the right-hand side is therefore of the form

$$\eta^8(\tau)\sum_{i+2j+3k=m+n-1}a_{ijk}P^iQ^jR^k. \qquad (3.3)$$

If we combine (3.2) and (3.3), then we complete the proof of the theorem for the case $m+n \geqslant 1$. The result for $S_8(1,0)$ is obtained similarly. □

The following identities are consequences of Theorem 3.2:

$$\begin{aligned}
S_8(1,0) &= 0,\\
S_8(3,0) &= -6\eta^8(\tau),\\
S_8(5,0) &= -30\eta^8(\tau)P,\\
S_8(7,0) &= -\frac{63}{2}\eta^8(\tau)(5P^2-Q),\\
S_8(7,2) &= 2\eta^8(\tau)R,\\
S_8(5,4) &= \eta^8(\tau)(5P^3-3PQ).
\end{aligned}$$

We also have

$$\begin{aligned}
&S_8(3,0) : S_8(1,2) = -3 : 1,\\
&S_8(5,0) : S_8(3,2) : S_8(1,4) = -15 : 1 : 1,\\
&S_8(7,0) : S_8(5,2) : S_8(3,4) : S_8(1,6) = -63 : 1 : 1 : 1,\\
&\begin{pmatrix} S_8(9,0)\\ S_8(3,6)\\ S_8(1,8)\end{pmatrix} = \begin{pmatrix} -66 & -189\\ 1/3 & 2/3\\ 2/9 & 7/9\end{pmatrix}\begin{pmatrix} S_8(7,2)\\ S_8(5,4)\end{pmatrix}.
\end{aligned}$$

An identity equivalent to $S_8(1,2) = 2\eta^8(\tau)$ was stated without proof by Winquist [**38**]. The formula for $\eta^8(\tau)$ given by Klein and Fricke [**19**, p. 373] can be shown to be equivalent to

$S_8(3,0)+27S_8(1,2)=48\eta^8(\tau)$. Schoeneberg [**31**, equation (11)] gave the attractive form

$$\eta^8(\tau)=\frac{1}{6}\sum_{\mu\in\mathbb{Z}[\exp(2\pi i/3)]}\chi(\mu)\mu^3\exp\left(\frac{2\pi i\tau|\mu|^2}{3}\right),$$

where

$$\chi(\mu)=\begin{cases}1 & \text{if } \mu\equiv 1 \pmod{\sqrt{-3}},\\ -1 & \text{if } \mu\equiv -1 \pmod{\sqrt{-3}}.\end{cases}$$

(The sum over the terms satisfying $\mu\equiv 0 \,(\mathrm{mod}\ \sqrt{-3})$ is zero.) Schoeneberg's formula can be deduced from the formulae for $S_8(3,0)$ and $S_8(1,2)$.

Theorem 3.1 is equivalent to Macdonald's identity for $\mathbf{A}_2$ (see [**10**; **11**, Theorem 2.1; **23**], or [**35**, p. 146]) in the form

$$(u,qu^{-1},v,qv^{-1},uv,qu^{-1}v^{-1},q,q;q)_\infty=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}q^{3m^2-3mn+3n^2+m+n}h_{m,n}(u,v),$$

where $u=e^{i(x+y)}, v=e^{i(x-y)}$ and

$$h_{m,n}(u,v)=uv\Big\{(u^{-3m-1}v^{-3n-1}-u^{3m+1}v^{3n+1})+(u^{3n-3m}v^{3n+1}-u^{3m-3n}v^{-3n-1}) + (u^{3n+1}v^{3n-3m}-u^{-3n-1}v^{3m-3n})\Big\}.$$

4. *The tenth power of* $\eta(\tau)$

The main tool used in this section is the following.

THEOREM 4.1.

$$G_3(x|t)G_2(y|t)-G_2(x|t)G_3(y|t)=\frac{1}{\eta^2(\tau)}\theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(y\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right).$$

Proof. Apply the technique used in the proof of Theorem 3.1. Let

$$M_{10}(x,y|t):=G_3(x|t)G_2(y|t)-G_2(x|t)G_3(y|t)$$

and

$$N_{10}(x,y|t):=\theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(y\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right).$$

Then M_{10} and N_{10} satisfy the transformation formulae

$$\begin{aligned}f(x+2\pi,y|t)&=f(x,y|t), & f(x+\pi t,y|t)&=q^{-3}e^{-6ix}f(x,y|t),\\ f(x,y+2\pi|t)&=f(x,y|t), & f(x,y+\pi t|t)&=q^{-3}e^{-6iy}f(x,y|t).\end{aligned}$$

Let y be fixed. Then N_{10} has simple zeros at $x=\pi m+\pi tn/2,\ \pm y+2\pi m+\pi tn,\ m,\ n\in\mathbb{Z}$, and no other zeros. The results in Section 2 imply that M_{10} also has zeros at the same points as N_{10}, and possibly at other points too. Thus $M_{10}(x,y|t)/N_{10}(x,y|t)$ is an elliptic function of x with no poles and, therefore, is a constant which is independent of x.

By the symmetry in x and y, we find that $M_{10}(x,y|t)/N_{10}(x,y|t)$ is also independent of y and, therefore, depends only on q. Let us denote the constant by $D(q)$. To determine its value, let $x = \pi/2$ and $y = \pi/6$. Since $G_3(\pi/2|t) = 0$ we have

$$\begin{aligned}
M_{10}\left(\frac{\pi}{2}, \frac{\pi}{6}\middle| t\right) &= -G_2\left(\frac{\pi}{2}\middle| t\right) G_3\left(\frac{\pi}{6}\middle| t\right) \\
&= -4 \sum_{j=-\infty}^{\infty} q^{(6j+1)^2/12} \sin(3j + \tfrac{1}{2})\pi \sum_{k=-\infty}^{\infty} q^{(6k-2)^2/12} \sin(k - \tfrac{1}{3})\pi \\
&= 2\sqrt{3} \left(q^{1/12} \sum_{j=-\infty}^{\infty} (-1)^j q^{3j^2+j} \right) \left(q^{1/3} \sum_{k=-\infty}^{\infty} (-1)^k q^{3k^2-2k} \right) \\
&= 2\sqrt{3}\, \eta(2\tau) q^{1/3} (q, q^5, q^6; q^6)_\infty \\
&= 2\sqrt{3}\, \frac{\eta(\tau)\eta^2(6\tau)}{\eta(3\tau)}.
\end{aligned}$$

On the other hand, writing $\gamma = \exp(i\pi/3)$, we have

$$\begin{aligned}
N_{10}\left(\frac{\pi}{2}, \frac{\pi}{6}\middle| t\right) &= \theta_1\left(\frac{\pi}{6}\middle|\frac{t}{2}\right)^2 \theta_1\left(\frac{\pi}{3}\middle|\frac{t}{2}\right) \theta_1\left(\frac{\pi}{2}\middle|\frac{t}{2}\right) \\
&= \left(2q^{1/8}\right)^4 \sin^2\frac{\pi}{6} \sin\frac{\pi}{3} \sin\frac{\pi}{2} (\gamma q, \gamma^5 q, q; q)_\infty^2 (\gamma^2 q, \gamma^4 q, q; q)_\infty (\gamma^3 q, \gamma^3 q, q; q)_\infty \\
&= 2\sqrt{3}\, \frac{\eta^3(\tau)\eta^2(6\tau)}{\eta(3\tau)}
\end{aligned}$$

after simplifying the infinite products. So,

$$D(q) = \frac{M_{10}(\pi/3, (\pi/6)|t)}{N_{10}(\pi/3, (\pi/6)|t)} = \frac{1}{\eta^2(\tau)}. \qquad \square$$

THEOREM 4.2. *Let*

$$S_{10}(m, n) = \sum_{\substack{\alpha \equiv 1 \pmod 6 \\ \beta \equiv 4 \pmod 6}} (\alpha^m \beta^n - \alpha^n \beta^m) q^{(\alpha^2+\beta^2)/12}.$$

Then

$$S_{10}(2m+1, 2n+1) = \eta^{10}(\tau) \sum_{i+2j+3k=m+n-1} a_{ijk} P^i Q^j R^k, \tag{4.1}$$

where a_{ijk} are rational numbers and i, j and k are non-negative integers.

Proof. Apply $\partial^{2m+2n+2}/(\partial x^{2m+1} \partial y^{2n+1})$ to both sides of Theorem 4.1; then let $x = y = 0$. We omit the details as they are similar to those in the proof of Theorem 3.2. $\square$

The first few examples of Theorem 4.2 are

$$\begin{aligned}
S_{10}(3,1) &= 6\eta^{10}(\tau), \\
S_{10}(5,1) &= 30\eta^{10}(\tau)P, \\
S_{10}(7,1) &= \frac{63}{2}\eta^{10}(\tau)(5P^2 - Q), \\
S_{10}(5,3) &= \frac{3}{2}\eta^{10}(\tau)(15P^2 + Q), \\
S_{10}(9,1) &= 3\eta^{10}(\tau)(315P^3 - 189PQ + 44R), \\
S_{10}(7,3) &= \frac{3}{2}\eta^{10}(\tau)(105P^3 - 21PQ - 4R).
\end{aligned}$$

Theorem 4.1 is equivalent to Winquist's identity [**38**, Theorem 1.1]: put $a = e^{i(x+y)}$, $b = e^{i(x-y)}$ in Theorem 4.1 to get [**38**, Theorem 1.1]. Observe that the left-hand side of

Theorem 4.1 is a difference of two terms, and each term is a product of two series that can be summed by the quintuple product identity. This was first noticed by Kang [**18**]. More information on Winquist's identity can be found in [**6**, **7**, **9**, **14**, **17**, **20**, **21**].

5. *The fourteenth power of* $\eta(\tau)$

The main tool used in this section is the following.

THEOREM 5.1.

$$H(x|t)T(y|t) + H\left(\frac{x-y}{2}\middle|t\right)T\left(\frac{3x+y}{2}\middle|t\right) + H\left(\frac{x+y}{2}\middle|t\right)T\left(\frac{-3x+y}{2}\middle|t\right)$$
$$= \frac{1}{\eta^4(\tau)}\theta_1\left(x\middle|\frac{t}{2}\right)\theta_1\left(y\middle|\frac{t}{2}\right)\theta_1\left(\frac{x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{x-y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{3x+y}{2}\middle|\frac{t}{2}\right)\theta_1\left(\frac{-3x+y}{2}\middle|\frac{t}{2}\right).$$

Proof. Apply the elliptic function method used in the previous two sections. By the results in Section 2, it may be checked that both sides satisfy the transformation formulae

$$f(x+2\pi, y|t) = f(x,y|t), \quad f(x+\pi t, y|t) = q^{-12}e^{-24ix}f(x,y|t),$$
$$f(x, y+2\pi|t) = f(x,y|t), \quad f(x, y+\pi t|t) = q^{-4}e^{-8iy}f(x,y|t).$$

It is straightforward to check that for a fixed value of x or y, the left-hand side is zero whenever the right-hand side is zero. Finally, the constant may be evaluated by letting $x = -\pi/8$, $y = 7\pi/8$. ☐

Because the left-hand side of Theorem 5.1 is more complicated than the left-hand sides of Theorems 3.1 and 4.1, some extra analysis is needed before differentiating. We will need the following.

LEMMA 5.2. *Let* $D_x = \partial/\partial x$ *and* $D_y = \partial/\partial y$. *Let* $f(z)$ *and* $g(z)$ *be analytic functions. Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix} \right\}.$$

Then

$$\begin{aligned} &D_xD_y(D_x^2 - D_y^2)(D_x^2 - 9D_y^2)\left(f(ax+by)g(cx+dy)\right) \\ &\quad = f^{(5)}(ax+by)g'(cx+dy) - 10f'''(ax+by)g'''(cx+dy) \\ &\qquad +9f'(ax+by)g^{(5)}(cx+dy). \end{aligned} \tag{5.1}$$

More generally, for non-negative integers m, n *and* ℓ, *define an operator* $D_{x,y}(m,n,\ell)$ *and coefficients* $c_{i,j}(m,n,\ell)$ *by*

$$\begin{aligned} D_{x,y}(m,n,\ell) &= \left(D_y(D_x^2 - D_y^2)\right)^m \left(D_x(D_x^2 - 9D_y^2)\right)^n \left(D_x^2 + 3D_y^2\right)^\ell \\ &= \sum_{i+j=3m+3n+2\ell} c_{i,j}(m,n,\ell)D_x^iD_y^j. \end{aligned}$$

Then

$$\begin{aligned} &D_{x,y}(2m+1, 2n+1, \ell)\left(f(ax+by)g(cx+dy)\right) \\ &\quad = \sum_{i+j=6(m+n+1)+2\ell} c_{i,j}(2m+1, 2n+1, \ell)\left(f^{(i)}(ax+by)g^{(j)}(cx+dy)\right). \end{aligned} \tag{5.2}$$

Proof. The result is trivial if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In either of the other cases, calculations using the chain rule imply that

$$\begin{aligned} &D_y(D_x^2 - D_y^2)\left(f(ax+by)g(cx+dy)\right) \\ &\quad = -f''(ax+by)g'(cx+dy) + f(ax+by)g'''(cx+dy), \qquad (5.3) \\ &D_x(D_x^2 - 9D_y^2)\left(f(ax+by)g(cx+dy)\right) \\ &\quad = -f'''(ax+by)g(cx+dy) + 9f'(ax+by)g''(cx+dy), \qquad (5.4) \\ &(D_x^2 + 3D_y^2)\left(f(ax+by)g(cx+dy)\right) \\ &\quad = f''(ax+by)g(cx+dy) + 3f(ax+by)g''(cx+dy). \qquad (5.5) \end{aligned}$$

If we combine (5.3) and (5.4), then we obtain (5.1), which is the case $m = n = \ell = 0$ of (5.2). The general result (5.2) now follows by induction on m, n and ℓ, using (5.3)–(5.5). □

THEOREM 5.3. *Let*

$$S_{14}(m,n,\ell) = \sum_{\substack{\alpha \equiv 2 \pmod 6 \\ \beta \equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6} \left(\beta(\alpha^2 - \beta^2)\right)^m \left(\alpha(\alpha^2 - 9\beta^2)\right)^n \left(\alpha^2 + 3\beta^2\right)^\ell q^{(\alpha^2+3\beta^2)/12}.$$

Then

$$S_{14}(2m+1, 2n+1, \ell) = \eta^{14}(\tau) \sum_{i+2j+3k=3m+3n+\ell} a_{ijk} P^i Q^j R^k, \qquad (5.6)$$

where a_{ijk} are rational numbers and i, j and k are non-negative integers.

Proof. Apply the operator $D_{x,y}(2m+1, 2n+1, \ell)$ to the identity in Theorem 5.1, then let $x = y = 0$. For the left-hand side use Lemma 5.2, and for the right-hand side use Lemma 2.1. □

Since

$$(\alpha^2 + 3\beta^2)^3 = 27\beta^2(\alpha^2 - \beta^2)^2 + \alpha^2(\alpha^2 - 9\beta^2)^2,$$

it follows that

$$S_{14}(2m+1, 2n+1, \ell+3) = 27S_{14}(2m+3, 2n+1, \ell) + S_{14}(2m+1, 2n+3, \ell).$$

Therefore, without loss of generality, we may assume that $0 \leqslant \ell \leqslant 2$.

The first few examples of Theorem 5.3 were given in Section 1. Theorem 5.1 is equivalent to Macdonald's identity for $\mathbf{G}_2$ (see [**11**, equation (1.8)]) written in the form

$$\begin{aligned} &(u, qu^{-1}, uv, qu^{-1}v^{-1}, u^2v, qu^{-2}v^{-1}, u^3v, qu^{-3}v^{-1}, v, qv^{-1}, u^3v^2, qu^{-3}v^{-2}, q, q; q)_\infty \\ &\quad = \sum_m \sum_n q^{12m^2 - 12mn + 4n^2 - m - n} H_{m,n}(u, v), \end{aligned}$$

where $u = e^{2ix}$, $v = e^{i(y-3x)}$ and

$$\begin{aligned} H_{m,n}(u,v) = u^5v^3 \Big\{ &(u^{12m-5}v^{4n-3} + u^{-12m+5}v^{-4n+3}) \\ &- (u^{12n-12m-4}v^{4n-3} + u^{12m-12n+4}v^{-4n+3}) \\ &+ (u^{12n-12m-4}v^{8n-12m-1} + u^{12m-12n+4}v^{12m-8n+1}) \\ &- (u^{12n-24m+1}v^{8n-12m-1} + u^{24m-12n-1}v^{12m-8n+1}) \\ &+ (u^{12n-24m+1}v^{4n-12m+2} + u^{24m-12n-1}v^{12m-4n-2}) \\ &- (u^{-12m+5}v^{4n-12m+2} + u^{12m-5}y^{12m-4n-2}) \Big\}. \end{aligned}$$

6. *Second, fourth and sixth powers of $\eta(\tau)$*

Analogous results for $\eta^2(\tau)$, $\eta^4(\tau)$ and $\eta^6(\tau)$ can be obtained trivially by multiplying Ramanujan's results for S_1 and S_3. Specifically, let

$$S_2(m,n) = S_1(m)S_1(n),$$
$$S_4(m,n) = S_1(m)S_3(n),$$
$$S_6(m,n) = S_3(m)S_3(n).$$

Then

$$S_2(2m,2n) = \eta^2(\tau) \sum_{i+2j+3k=m+n} a_{ijk} P^i Q^j R^k, \tag{6.1}$$

$$S_4(2m,2n+1) = \eta^4(\tau) \sum_{i+2j+3k=m+n} a_{ijk} P^i Q^j R^k, \tag{6.2}$$

$$S_6(2m+1,2n+1) = \eta^6(\tau) \sum_{i+2j+3k=m+n} a_{ijk} P^i Q^j R^k. \tag{6.3}$$

In each case, a_{ijk} are rational numbers and i, j and k are non-negative integers.

Another form for $\eta^6(\tau)$ was given by Schoeneberg [**31**, equation (8)]:

$$\eta^6(\tau) = \frac{1}{2} \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} \mathrm{Re}(a+2ib)^2 q^{(a^2+4b^2)/4}.$$

This formula can be shown to be equivalent to the identity for $S_6(1,1)$ by direct series manipulations.

Results of a different type for $\eta^6(\tau)$ may be obtained using a series given by Hirschhorn [**16**]. Let

$$S_6^*(m,n) = \sum_{\substack{\alpha\equiv 1 \pmod{10} \\ \beta\equiv 3 \pmod{10}}} (-1)^{(\alpha+\beta-4)/10} (\alpha^m\beta^n - \alpha^n\beta^m) q^{(\alpha^2+\beta^2)/40}.$$

Hirschhorn's result is

$$S_6^*(0,2) = 8\eta^6(\tau).$$

Using the techniques in this paper, it can be shown that if $m+n \geqslant 1$, then

$$S_6^*(2m,2n) = \eta^6(\tau) \sum_{i+2j+3k=m+n-1} a_{ijk} P^i Q^j R^k,$$

where a_{ijk} are rational numbers and i, j and k are non-negative integers.

7. *Identities obtained using Schoeneberg's theta functions*

In this section, we prove (1.2) and analogous results for 2nd, 4th, 6th, 8th and 10th powers of $\eta(\tau)$. Most of the results in this section are new. A few special cases can be found in Ramanujan's The Lost Notebook, for example [**28**, p. 249]. Some of Ramanujan's identities have recently been examined by Rangachari [**29**, **30**], using Hecke's theta functions [**15**].

The results we shall prove are as follows.

THEOREM 7.1. *Let*

$$C_2(n|\tau) = \sum_{\substack{\alpha\equiv 1 \pmod 6 \\ \beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-2)/6} (\alpha+i\beta)^n q^{(\alpha^2+\beta^2)/24}.$$

Then $C_2(4n|\tau)/\eta^2(\tau)$ is a modular form of weight $4n$ on $\mathrm{SL}_2(\mathbb{Z})$.

THEOREM 7.2. *Let*

$$C_2^*(n|\tau) = \sum_{\substack{\alpha\equiv 0 \pmod 6 \\ \beta\equiv 1 \pmod 6}} (-1)^{(\alpha+\beta-1)/6}(\alpha+i\beta\sqrt{3})^n q^{(\alpha^2+3\beta^2)/36}.$$

Then $C_2^*(6n|\tau)/\eta^2(\tau)$ *is a modular form of weight* $6n$ *on* $\mathrm{SL}_2(\mathbb{Z})$.

THEOREM 7.3. *Let*

$$C_4(n|\tau) = \sum_{\substack{\alpha\equiv 1 \pmod 6 \\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-1)/6}\,\mathrm{Im}\left((\alpha+i\beta\sqrt{3})^n\right) q^{(\alpha^2+3\beta^2)/24}.$$

Then $C_4(2n+1|\tau)/\eta^4(\tau)$ *is a modular form of weight* $2n$ *on* $\mathrm{SL}_2(\mathbb{Z})$.

THEOREM 7.4. *Let*

$$C_6(n|\tau) = \sum_{\substack{\alpha\equiv 1 \pmod 4 \\ \beta\equiv 1 \pmod 4}} (\alpha+i\beta)^n q^{(\alpha^2+\beta^2)/8}.$$

Then $C_6(4n+2|\tau)/\eta^6(\tau)$ *is a modular form of weight* $4n$ *on* $\mathrm{SL}_2(\mathbb{Z})$.

THEOREM 7.5. *Let*

$$C_8(n|\tau) = \sum_{\substack{\alpha\equiv 1 \pmod 3 \\ \alpha+\beta\equiv 0 \pmod 2}} (\alpha+i\beta\sqrt{3})^n q^{(\alpha^2+3\beta^2)/12}.$$

Then $C_8(6n+3|\tau)/\eta^8(\tau)$ *is a modular form of weight* $6n$ *on* $\mathrm{SL}_2(\mathbb{Z})$.

THEOREM 7.6. *Let*

$$C_{10}(n|\tau) = \sum_{\substack{\alpha\equiv 1 \pmod 6 \\ \beta\equiv 4 \pmod 6}} \mathrm{Im}\left((\alpha+i\beta)^n\right) q^{(\alpha^2+\beta^2)/12}.$$

Then $C_{10}(4n+4|\tau)/\eta^{10}(\tau)$ *is a modular form of weight* $4n$ *on* $\mathrm{SL}_2(\mathbb{Z})$.

THEOREM 7.7. *Let*

$$C_{14}(n|\tau) = \sum_{\substack{\alpha\equiv 2 \pmod 6 \\ \beta\equiv 1 \pmod 4}} (-1)^{(\alpha-2)/6}\,\mathrm{Im}\left((\alpha+i\beta\sqrt{3})^n\right) q^{(\alpha^2+3\beta^2)/12}.$$

Then $C_{14}(6n+6|\tau)/\eta^{14}(\tau)$ *is a modular form of weight* $6n$ *on* $\mathrm{SL}_2(\mathbb{Z})$.

In order to prove Theorems 7.1–7.7, we first recall some properties of a class of theta functions studied by Schoeneberg [**32**].

Let f be an even positive integer and $\mathbf{A} = (a_{\mu,\nu})$ be a *symmetric* $f \times f$ matrix such that
(1) $a_{\mu,\nu} \in \mathbb{Z}$;
(2) $a_{\mu,\mu}$ is even; and
(3) $\mathbf{x}^t\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^f$ such that $\mathbf{x} \neq \mathbf{0}$.
Let N be the smallest positive integer such that $N\mathbf{A}^{-1}$ also satisfies conditions (1)–(3). Let

$$P_k^{\mathbf{A}}(\mathbf{x}) := \sum_{\mathbf{y}} c_{\mathbf{y}}(\mathbf{y}^t\mathbf{A}\mathbf{x})^k,$$

where the sum is over finitely many $\mathbf{y} \in \mathbb{C}^f$ with the property $\mathbf{y}^t\mathbf{A}\mathbf{y} = 0$, and $c_{\mathbf{y}}$ are arbitrary complex numbers.

When $\mathbf{Ah} \equiv \mathbf{0} \,(\mathrm{mod}\ N)$ and $\mathrm{Im}\,\tau > 0$, we define

$$\vartheta_{\mathbf{A},\mathbf{h},P_k^A}(\tau) = \sum_{\substack{\mathbf{n}\in\mathbb{Z}^f \\ \mathbf{n}\equiv\mathbf{h} \pmod N}} P_k^{\mathbf{A}}(\mathbf{n}) e^{((2\pi i\tau)/N)(1/2)((\mathbf{n}^t\mathbf{A}\mathbf{n})/N)}.$$

The result which we need is the following [**32**, Theorem 2, p. 210].

THEOREM 7.8. *The function $\vartheta_{\mathbf{A},\mathbf{h},P_k^{\mathbf{A}}}$ satisfies the following transformation formulae*

$$\vartheta_{\mathbf{A},\mathbf{h},P_k^{\mathbf{A}}}(\tau+1) = e^{((2\pi i)/N)(1/2)((\mathbf{h}^t\mathbf{A}\mathbf{h})/N)}\vartheta_{\mathbf{A},\mathbf{h},P_k^{\mathbf{A}}}(\tau)$$

and

$$\vartheta_{\mathbf{A},\mathbf{h},P_k^{\mathbf{A}}}\left(-\frac{1}{\tau}\right) = \frac{(-i)^{(f/2)+2k}\tau^{(f/2)+k}}{\sqrt{|\det \mathbf{A}|}} \sum_{\substack{\mathbf{g} \pmod N \\ \mathbf{A}\mathbf{g}\equiv\mathbf{0} \pmod N}} e^{((2\pi i)/N)((\mathbf{g}^t\mathbf{A}\mathbf{h})/N)}\vartheta_{\mathbf{A},\mathbf{g},P_k^{\mathbf{A}}}(\tau).$$

We will also need the following.

LEMMA 7.9. *Let*

$$\varphi_{r,s}(n;\tau) = \sum_{\substack{\alpha\equiv r \pmod{12} \\ \beta\equiv s \pmod{12}}} (\alpha - i\beta)^n e^{((2\pi i\tau)/12)(1/2)((6(\alpha^2+\beta^2))/12)}.$$

Then

$$\varphi_{r,s}(4n;\tau+1) = e^{6\pi i(r^2+s^2)/12^2}\varphi_{r,s}(4n;\tau) \tag{7.1}$$

and

$$\varphi_{r,s}\left(4n;-\frac{1}{\tau}\right) = \frac{(-i)\tau^{4n+1}}{6} \sum_{\substack{(u,v) \pmod{12} \\ (6u,6v)\equiv(0,0) \pmod{12}}} e^{\pi i(ru+sv)/12}\varphi_{u,v}(4n;\tau). \tag{7.2}$$

Proof. These follow from Theorem 7.8 on taking

$$\mathbf{A} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} r \\ s \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} i \\ 1 \end{pmatrix},$$

$N = 12$, $k = 4n$, and $f = 2$. □

We are now ready to prove Theorems 7.1–7.7. We shall give a detailed proof of Theorem 7.6. The details for the other theorems are similar.

Proof of Theorem 7.6. From the first example following Theorem 4.2 and the definition of $C_{10}(4|\tau)$, it follows that

$$C_{10}(4|\tau) = 24\eta^{10}(\tau). \tag{7.3}$$

Next, observe that

$$
\begin{aligned}
&C_{10}(4n|\tau)\\
&= \frac{1}{2i}\left(\sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 4 \pmod 6}} (\alpha+i\beta)^{4n} q^{(\alpha^2+\beta^2)/12} - \sum_{\substack{\alpha\equiv 1 \pmod 6\\ \beta\equiv 4 \pmod 6}} (\alpha-i\beta)^{4n} q^{(\alpha^2+\beta^2)/12}\right)\\
&= \frac{1}{2^{4n+1}i}\left(\sum_{\substack{\alpha\equiv 8 \pmod{12}\\ \beta\equiv 2 \pmod{12}}} (\alpha-i\beta)^{4n} q^{3(\alpha^2+\beta^2)/12^2} - \sum_{\substack{\alpha\equiv 2 \pmod{12}\\ \beta\equiv 8 \pmod{12}}} (\alpha-i\beta)^{4n} q^{3(\alpha^2+\beta^2)/12^2}\right)\\
&= \frac{1}{2^{4n+1}i}\left(\varphi_{8,2}(4n;\tau)-\varphi_{2,8}(4n;\tau)\right).
\end{aligned} \tag{7.4}
$$

Equation (7.1) implies that

$$\varphi_{8,2}(4n;\tau+1)-\varphi_{8,2}(4n;\tau+1) = e^{5\pi i/6}\left(\varphi_{8,2}(4n;\tau)-\varphi_{2,8}(4n;\tau)\right). \tag{7.5}$$

Equation (7.2) gives

$$\varphi_{8,2}\left(4n;-\frac{1}{\tau}\right)-\varphi_{2,8}\left(4n;-\frac{1}{\tau}\right) = -\frac{i\tau^{4n+1}}{6}\sum_{j=1}^{6}\sum_{k=1}^{6}\left(e^{\pi i(4j+k)/3}-e^{\pi i(j+4k)/3}\right)\varphi_{2j,2k}(4n;\tau).$$

If we use the relation $\varphi_{r,s}(4n;\tau)=\varphi_{12-r,12-s}(4n;\tau)$ and simplify, then we find that

$$
\begin{aligned}
\varphi_{8,2}\left(4n;-\frac{1}{\tau}\right)-\varphi_{2,8}\left(4n;-\frac{1}{\tau}\right) = -\frac{i\tau^{4n+1}}{6}\Big(&4(\varphi_{2,4}-\varphi_{4,2})(4n;\tau)+2(\varphi_{8,2}-\varphi_{2,8})(4n;\tau)\\
&+2(\varphi_{12,2}-\varphi_{2,12})(4n;\tau)+2(\varphi_{4,6}-\varphi_{6,4})(4n;\tau)\\
&+2(\varphi_{6,12}-\varphi_{12,6})(4n;\tau)\Big).
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
\varphi_{2,12}(4n;\tau) &= \varphi_{12,2}(4n;\tau),\\
\varphi_{4,6}(4n;\tau) &= \varphi_{6,4}(4n;\tau),\\
\varphi_{6,12}(4n;\tau) &= \varphi_{12,6}(4n;\tau),\\
\varphi_{2,4}(4n;\tau) &= \varphi_{8,2}(4n;\tau),\\
\varphi_{4,2}(4n;\tau) &= \varphi_{2,8}(4n;\tau).
\end{aligned}
$$

Therefore,

$$\varphi_{8,2}\left(4n;-\frac{1}{\tau}\right)-\varphi_{2,8}\left(4n;-\frac{1}{\tau}\right) = -i\tau^{4n+1}\left(\varphi_{8,2}(4n;\tau)-\varphi_{2,8}(4n;\tau)\right). \tag{7.6}$$

Equations (7.3)–(7.6) imply that the function

$$F(\tau) := \frac{C_{10}(4n|\tau)}{\eta^{10}(\tau)}$$

satisfies the transformation properties

$$F(\tau+1)=F(\tau),\quad F\left(-\frac{1}{\tau}\right)=\tau^{4n-4}F(\tau).$$

That is, $F(\tau)$ is a modular form of weight $4n-4$ on $\mathrm{SL}_2(\mathbb{Z})$. This completes the proof of Theorem 7.6. □

8. *The twenty-sixth power of $\eta(\tau)$*

The analogue of (1.2) for the 26th power of $\eta(\tau)$ is as follows.

THEOREM 8.1. *For $n \geqslant 1$, the function*

$$\frac{1}{\eta^{26}(\tau)}\left(\frac{C_2^*(12n|\tau)}{3^{6n}} - (-1)^n\frac{C_2(12n|\tau)}{2^{6n}}\right)$$

is a modular form of weight $12n-12$ on $\mathrm{SL}_2(\mathbb{Z})$.

Proof. Calculations using Theorems 7.1 and 7.2 imply that the first few terms in the q-expansions are

$$\begin{aligned} C_2(12n|\tau) &= (-64)^n q^{1/12}\left(1-\left((2+3i)^{12n}+(2-3i)^{12n}\right)q\right.\\ &\quad \left.+\left(5^{12n}-(4+3i)^{12n}-(4-3i)^{12n}\right)q^2+\cdots\right),\\ C_2^*(12n|\tau) &= (729)^n q^{1/12}\left(1-\left((1+2i\sqrt{3})^{12n}+(1-2i\sqrt{3})^{12n}\right)q-5^{12n}q^2+\cdots\right). \end{aligned}$$

The q^2 terms in the two expansions are different because $\mathrm{Re}(((4+3i)/5)^{12n}) \neq 1$ for any integer n [**25**, Corollary 3.12]. Therefore, $C_2(12n|\tau)$ and $C_2^*(12n|\tau)$ are linearly independent. It follows that

$$\frac{1}{\eta^{2}(\tau)}\left(\frac{C_2^*(12n|\tau)}{3^{6n}} - (-1)^n\frac{C_2(12n|\tau)}{2^{6n}}\right)$$

is a cusp form of weight $12n$ on $\mathrm{SL}_2(\mathbb{Z})$ and so must be of the form $\eta^{24}(\tau)F$, where F is a modular form of weight $12n-12$. This completes the proof. □

COROLLARY 8.2.

$$\eta^{26}(\tau) = \frac{1}{16308864}\left(\frac{C_2(12|\tau)}{64} + \frac{C_2^*(12|\tau)}{729}\right).$$

Proof. Take $n=1$ in Theorem 8.1 and observe that

$$(2+3i)^{12}+(2-3i)^{12}-(1+2i\sqrt{3})^{12}-(1-2i\sqrt{3})^{12} = 16\,308\,864.$$ □

Corollary 8.2 was discovered and proved in [**8**]. An equivalent form of this identity had been discovered in 1966 by Atkin [**2**] (unpublished), and the first published proof was given in 1985 by Serre [**33**]. The proof we have given here is different from those in the literature.

Here is the analogue of (1.1) for $\eta^{26}(\tau)$.

COROLLARY 8.3. *Let n and ℓ be integers satisfying $n \geqslant 1$, $\ell \geqslant 0$, and define*

$$S_{26}(n,\ell) = \left(q\frac{d}{dq}\right)^{\ell}\left(\frac{C_2^*(12n|\tau)}{3^{6n}} - (-1)^n\frac{C_2(12n|\tau)}{2^{6n}}\right).$$

Then

$$S_{26}(n,\ell) = \eta^{26}(\tau)\sum_{i+2j+3k=6(n-1)+\ell} a_{ijk}P^iQ^jR^k, \tag{8.1}$$

where a_{ijk} are rational numbers and i, j and k are non-negative integers.

Proof. This follows immediately from Theorem 8.1 and the Ramanujan differential equations. □

9. *Concluding remarks*

9.1. *Lacunarity and the Hecke operator*

By a theorem of Landau [**3**, Theorem 10.5, p. 244], all of the series $S_2(2m, 2n)$, $S_4(2m, 2n+1)$, $S_6(2m+1, 2n+1)$, $S_8(2m+1, 2n)$, $S_{10}(2m+1, 2n+1)$, $S_{14}(2m+1, 2n+1, \ell)$ and $S_{26}(n, \ell)$ are lacunary. Hence the corresponding expressions on the right-hand sides of (3.1), (4.1), (5.6), (6.1)–(6.3) and (8.1) are lacunary.

Let us write

$$S_{14}(2m+1, 2n+1, \ell) = Aq^{7/12} \sum_{k=0}^{\infty} a(k) q^k,$$

where A is a numerical constant selected to make $a(0) = 1$. Then the technique used in [**12**] implies that if $p \equiv 5 \pmod 6$ is prime, then

$$a\left(pk + \frac{7}{12}(p^2 - 1)\right) = (-1)^{(p+1)/6} p^{6(m+n+1)+2\ell} a\left(\frac{k}{p}\right).$$

Similar results for S_2, S_4, S_6, S_8, S_{10} and S_{26} may also be written down. These results generalize a theorem of Newman [**24**].

9.2. *Ramanujan's τ function*

Ramanujan's function $\tau(n)$ is defined by

$$q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n.$$

If we multiply the results for $S_{10}(3, 1)$ and $S_{14}(1, 1, 0)$, then we obtain

$$\eta^{24}(\tau) = -\frac{1}{180} \sum_{\substack{\alpha \equiv 1 \pmod 6,\ \beta \equiv 4 \pmod 6 \\ \gamma \equiv 2 \pmod 6,\ \delta \equiv 1 \pmod 4}} (-1)^{(\gamma-2)/6} \alpha\beta(\alpha^2 - \beta^2)\gamma\delta(\gamma^2 - \delta^2)(\gamma^2 - 9\delta^2) \times q^{(\alpha^2+\beta^2+\gamma^2+3\delta^2)/12}.$$

If we extract the coefficient of q^n on both sides, then we obtain

$$\tau(n) = -\frac{1}{4320\sqrt{3}} \sum (-1)^{(\gamma-2)/6} \operatorname{Im}\left((\alpha + i\beta)^4\right) \operatorname{Im}\left((\gamma + i\delta\sqrt{3})^6\right),$$

where the summation is over integers satisfying

$$\alpha^2 + \beta^2 + \gamma^2 + 3\delta^2 = 12n,$$
$$\alpha \equiv 1 \pmod 6, \quad \beta \equiv 4 \pmod 6, \quad \gamma \equiv 2 \pmod 6, \quad \delta \equiv 1 \pmod 4.$$

This is different from the representation given by Dyson [**13**, p. 636].

References

1. G. Andrews, R. Askey and R. Roy, *Special functions* (Cambridge University Press, Cambridge, 1999).
2. A. O. L. Atkin, Email to S. Cooper, September 29, 2004.
3. P. T. Bateman and H. G. Diamond, *Lectures in analytic number theory* (World Scientific, Singapore, 2004).
4. B. C. Berndt, *Ramanujan's notebooks*, Part III (Springer, New York, 1991).
5. B. C. Berndt and A. J. Yee, 'A page on Eisenstein series in Ramanujan's lost notebook', *Glasgow Math. J.* 45 (2003) 123–129.
6. B. C. Berndt, S. H. Chan, Z.-G. Liu and H. Yesilyurt, 'A new identity for $(q;q)_\infty^{10}$ with an application to Ramanujan's partition congruence modulo 11', *Quart. J. Math.* 55 (2004) 13–30.

7. L. Carlitz and M. V. Subbarao, 'On a combinatorial identity of Winquist and its generalization', *Duke Math. J.* 39 (1972) 165–172.
8. H. H. Chan, S. Cooper and P. C. Toh, 'The 26th power of Dedekind's η-function', *Adv. Math.* 207 (2006) 532–543.
9. H. H. Chan, Z.-G. Liu and S. T. Ng, 'Elliptic functions and the quintuple, Hirschhorn and Winquist product identities', *Int. J. Number Theory* 1 (2005) 33–43.
10. S. Cooper, 'A new proof of the Macdonald identities for A_{n-1}', *J. Australian Math. Soc.* A 62 (1997) 345–360.
11. S. Cooper, 'The Macdonald identities for G_2 and some extensions', *New Zealand J. Math.* 26 (1997) 161–182.
12. S. Cooper, M. Hirschhorn and R. Lewis, 'Powers of Euler's product and related identities', *Ramanujan J.* 4 (2000) 137–155.
13. F. J. Dyson, 'Missed opportunities', *Bull. Amer. Math. Soc.* 78 (1972) 635–652.
14. P. Hammond, R. Lewis and Z.-G. Liu,'Hirschhorn's identities', *Bull. Australian Math. Soc.* 60 (1999) 73–80.
15. E. Hecke, 'Zur Theorie der elliptischen Modulfunktionen', *Math. Ann.* 97 (1926) 210–242.
16. M. D. Hirschhorn, 'A simple proof of an identity of Ramanujan', *J. Australian Math. Soc.* A 34 (1983) 31–35.
17. M. D. Hirschhorn, 'A generalisation of Winquist's identity and a conjecture of Ramanujan', *J. Indian Math. Soc.* 51 (1987) 49–55.
18. S.-Y. Kang, 'A new proof of Winquist's identity', *J. Combin. Theory* A 78 (1997) 313–318.
19. F. Klein and R. Fricke, *Vorlesungen über die Theorie der elliptische Modulfunktionen,* vol. 2 (Teubner, Leipzig, 1892).
20. S. Kongsiriwong and Z.-G. Liu, 'Uniform proofs of q-series-product identities', *Results Math.* 44 (2003) 312–339.
21. Z.-G. Liu, 'A theta function identity and its implications', *Trans. Amer. Math. Soc.* 357 (2005) 825–835.
22. Z.-G. Liu, 'A three-term theta function identity and its applications', *Adv. Math.* 195 (2005) 1–23.
23. I. Macdonald, 'Affine root systems and Dedekind's η-function', *Invent. Math.* 15 (1972) 91–143.
24. M. Newman, 'An identity for the coefficients of certain modular forms', *J. London Math. Soc.* 30 (1955) 488–493.
25. I. Niven, 'Irrational numbers', *The Carus mathematical monographs*, vol. 11 (MAA, Distributed by Wiley, New York, 1956).
26. S. Ramanujan, 'On certain arithmetical functions', *Trans. Cambridge Philos. Soc.* 22 (1916) 159–184.
27. S. Ramanujan, *Notebooks* (2 volumes) (Tata Institute of Fundamental Research, Bombay, 1957).
28. S. Ramanujan, *The lost notebook and other unpublished papers* (Narosa, New Delhi, 1988).
29. S. S. Rangachari, 'Ramanujan and Dirichlet series with Euler products', *Proc. Indian Acad. Sci. Math. Sci.* 91 (1982) 1–15.
30. S. S. Rangachari, 'Euler products, modular identities and elliptic integrals in Ramanujan's manuscripts. II', *Ramanujan revisited*, Urbana-Champaign, IL, 1987 (Academic Press, Boston, MA, 1988) 347–357.
31. B. Schoeneberg, 'Über den Zusammenhang der Eisensteinschen Reihen und Thetareihen mit der Diskriminante der elliptischen Funktionen', *Math. Ann.* 126 (1953) 177–184.
32. B. Schoeneberg, 'Elliptic modular functions: an introduction.' Translated from the German by J. R. Smart and E. A. Schwandt. Die Grundlehren der mathematischen Wissenschaften, Band 203. Springer-Verlag, New York-Heidelberg, 1974.
33. J.-P. Serre, 'Sur la lacunarité des puissances de η', *Glasgow Math. J.* 27 (1985) 203–221.
34. L.-C. Shen, 'On the products of three theta functions', *Ramanujan J.* 3 (1999) 343–357.
35. D. Stanton, 'An elementary approach to the Macdonald identities', *q-series and partitions* (ed. D. Stanton; Springer, New York, 1989) 139–149.
36. K. Venkatachaliengar, 'Development of elliptic functions according to Ramanujan', Technical Report 2 (Madurai Kamaraj University, Department of Mathematics, Madurai, 1988).
37. E. T. Whittaker and G. N. Watson, *A course of modern analysis*, 4th edn (Cambridge Univerisity Press, Cambridge, 1966).
38. L. Winquist, 'An elementary proof of $p(11m+6) \equiv 0 \pmod{11}$', *J. Combin. Theory* 6 (1969) 56–59.

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