RAMANUJAN'S CUBIC CONTINUED FRACTION REVISITED

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Abstract. In this article, we derive a sequence of numbers which converge to $1/\pi$. We will also derive a new series for $1/\pi$. These new results are motivated by the study of Ramanujan's cubic continued fraction.

1. Introduction

Let $q = e^{2\pi i\tau}$ and

$$G(q) = \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{\cdots}{}.$$

In 1996, inspired by page 366 of Ramanujan's Lost Notebook [9], H.H. Chan [4] derived several new results satisfied by $G(q)$. For example, he showed that

$$G^3(q) = G(q^3)\frac{1-G(q^3)+G^2(q^3)}{1+2G(q^3)+4G^2(q^3)}. \tag{1.1}$$

From (1.1), Chan constructed an algorithm for computing e^{π}. This iteration prompted F.G. Garvan to ask if there were any iteration to π which can be derived from the study of $G(q)$. In this paper, we will show that such an iteration exists. We will also derive the following series for $1/\pi$:

$$\frac{2\sqrt{3}(3+2\sqrt{2})}{9\pi} = \sum_{k=0}^{\infty} C_k \left(k+1-\frac{2}{3}\sqrt{2}\right)\left(-1+\frac{3}{4}\sqrt{2}\right)^k \tag{1.2}$$

where

$$C_k = \sum_{m=0}^{k}\left\{\sum_{j=0}^{m}\binom{m}{j}^3 \sum_{i=0}^{k-m}\binom{k-m}{i}^3\right\}. \tag{1.3}$$

The proof of (1.2) involves the identity

$$G^3(e^{-2\pi/\sqrt{6}}) = -1+\frac{3}{4}\sqrt{2}.$$

2. A triplication formula for $G(q)$ and a new iteration to $1/\pi$

In [1], C. Adiga, T. Kim, M.S.M. Naika and H.S. Madhusudhan gave a new proof of (1.1) by first proving the identity

$$1-3\frac{G(q^3)}{1+G(q^3)}=\left(1-9\frac{G^3(q)}{1+G^3(q)}\right)^{1/3}. \tag{2.1}$$

This identity allows one to write $G(q^3)$ in terms of $G(q)$, namely,

$$G(q^3)=\frac{1-H(q)}{2+H(q)}, \tag{2.2}$$

with

$$H(q)=\left(\frac{1-8G^3(q)}{1+G^3(q)}\right)^{1/3}.$$

The above triplication formula for $G(q)$ is analogous to the Borweins-Ramanujan triplication formula for the cubic singular modulus defined by

$$\frac{1}{\alpha(q)}=1+\frac{1}{27}\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12}, \tag{2.3}$$

where $q=e^{2\pi i\tau}$ and

$$\eta(\tau)=q^{1/24}\prod_{k=1}^{\infty}(1-q^k).$$

In the case for $\alpha(q)$, the triplication formula is given by

$$\alpha(q^3)=\left(\frac{1-\sqrt[3]{1-\alpha(q)}}{1+2\sqrt[3]{1-\alpha(q)}}\right)^3. \tag{2.4}$$

Two rapidly convergent sequences for π can be constructed from (2.4). These iterations are given as follow:

The Borweins Iteration [8]**.** Let $t_0=\frac{1}{3}$, $s_0=\frac{\sqrt{3}-1}{2}$,

$$s_n=\frac{1-(1-s_{n-1}^3)^{1/3}}{1+2(1-s_{n-1}^3)^{1/3}} \quad \text{and} \quad t_n=(1+2s_n)^2t_{n-1}-3^{n-1}((1+2s_n)^2-1).$$

Then t_n^{-1} converges cubically to π.

Chan's iteration [6]**.** Let $k_0=0$, $s_0=\frac{1}{2^{1/3}}$,

$$s_n=\frac{1-(1-s_{n-1}^3)^{1/3}}{1+2(1-s_{n-1}^3)^{1/3}} \quad \text{and} \quad k_n=(1+2s_n)^2k_{n-1}+8\cdot 3^{n-2}\sqrt{3}s_n\frac{1-s_n^3}{1+2s_n}.$$

Then k_n^{-1} converges cubically to π.

Since the above iterations are constructed from (2.4), it is therefore natural to construct new cubic iteration to π from (2.2). In the following two sections, we will establish the following result:

Theorem 2.1. *Let* $k_0 = 0$ *and* $s_0 = \sqrt[3]{\frac{3\sqrt{2}}{4} - 1}$*. Set*

$$s_n = \frac{(1+s_{n-1}^3)^{1/3} - (1-8s_{n-1}^3)^{1/3}}{2(1+s_{n-1}^3)^{1/3} + (1-8s_{n-1}^3)^{1/3}}.$$

If

$$k_n = \frac{(1+2s_n+4s_n^2)(1+s_n)^2}{(1-s_n+s_n^2)} k_{n-1} + \frac{2 \cdot 3^{n-1}}{\sqrt{6}} \frac{s_n(1-2s_n)(8s_n^4 - 10s_n^3 + 6s_n^2 + 11s_n + 5)}{1+s_n^3},$$

then k_n^{-1} *converges cubically to* π*.*

3. New identities satisfied by $G(q)$

We first relate $G(q)$ with the Borweins' cubic singular modulus $\alpha(q)$ (see (2.3)) and deduce results associated with $G(q)$ using Ramanujan-Borweins' theory of elliptic functions to the cubic base.

Lemma 3.1. *Let*

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$X = G^3(q)$$

and

$$z = \frac{\varphi^3(-q^3)}{\varphi(-q)}.$$

Then

$$a(q) = z(1+4X) \tag{3.1}$$

and

$$\alpha(q) = 27\frac{X}{(1+4X)^3}. \tag{3.2}$$

Proof. From [2, p. 460, Entry 3(ii)], we find that

$$a(q^2) = \frac{\varphi^4(-q) + 3\varphi^4(-q^3)}{\varphi(-q)\varphi(-q^3)} = z\left(\frac{1}{4}\frac{\varphi^4(-q)}{\varphi^4(-q^3)} + \frac{3}{4}\right).$$

(3.3)

Since [2, p. 347]

$$\frac{\varphi^4(-q)}{\varphi^4(-q^3)} = 1 - 8X, \tag{3.4}$$

we deduce that

$$a(q^2) = z\left(1 - 2X\right). \tag{3.5}$$

On the other hand, we know that [3, p. 4189]

$$a(q) = 3\frac{\varphi^3(-q^3)}{\varphi(-q)} - 2a(q^2).$$

Hence, by (3.5), we find that

$$a(q) = z\left(1 + 4X\right),$$

which yields (3.1).

To prove (3.2), we recall the identity [2, p. 345, Entry 1 (iv)]

$$1 + \frac{1}{27}\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12} = \frac{(1+4X)^3}{27X}.$$

Using (2.3), we immediately deduce (3.2).

□

Corollary 3.2. *The functions z and X satisfy the following differential equations:*

$$q\frac{dX}{dq} = z^2(X - 7X^2 - 8X^3) \tag{3.6}$$

Proof. We recall the differential equation satisfied by $a := a(q)$ and $\alpha := \alpha(q)$ [5, (4.7)]:

$$q\frac{d\alpha}{dq} = a^2\alpha(1-\alpha). \tag{3.7}$$

Differentiating (3.2) with respect to q and using (3.7) and (3.1), we immediately deduce (3.6). □

4. Proof of Theorem 2.1

We begin our proof with the following transformation formula:

$$(1 + X(e^{-2\pi/\sqrt{6t}}))(1 + X(e^{-2\pi\sqrt{t/6}})) = \frac{9}{8}. \tag{4.1}$$

This identity can be proved by rearranging the identity [1]

$$\left(1 + \frac{1}{X(e^{-2\pi/\sqrt{6t}})}\right)\left(1 - 8X(e^{-2\pi\sqrt{t/6}})\right) = 9. \tag{4.2}$$

Differentiating (4.1) with respect to t and using (3.6), we find that

$$tZ(e^{-2\pi\sqrt{t/6}})X(e^{-2\pi\sqrt{t/6}})(1-8X(e^{-2\pi\sqrt{t/6}})) = Z(e^{-2\pi/\sqrt{6t}})X(e^{-2\pi/\sqrt{6t}})(1-8X(e^{-2\pi/\sqrt{6t}})), \tag{4.3}$$

where

$$Z(q) = z^2.$$

From (4.2), we have

$$X(e^{-2\pi/\sqrt{6t}}) = \frac{1}{9}\left(1+X(e^{-2\pi/\sqrt{6t}})\right)\left(1-8X(e^{-2\pi\sqrt{t/6}})\right) \tag{4.4}$$

and

$$X(e^{-2\pi\sqrt{t/6}}) = \frac{1}{9}\left(1+X(e^{-2\pi\sqrt{t/6}})\right)\left(1-8X(e^{-2\pi/\sqrt{6t}})\right). \tag{4.5}$$

Substituting (4.4) and (4.5) into (4.3), we find that

$$tZ(e^{-2\pi\sqrt{t/6}})(1+X(e^{-2\pi\sqrt{t/6}})) = Z(e^{-2\pi/\sqrt{6t}})(1+X(e^{-2\pi/\sqrt{6t}})). \tag{4.6}$$

The above transformation formula motivates us to set

$$A(q) = Z(q)(1+X(q)).$$

Consequently, we can express (4.6) as

$$tA(e^{-2\pi\sqrt{t/6}}) = A(e^{-2\pi/\sqrt{6t}}). \tag{4.7}$$

Define

$$\kappa(t) = \frac{1}{\pi A(e^{-2\pi\sqrt{t/6}})} - 2\sqrt{\frac{t}{6}}\frac{\widetilde{A}}{A^2}(e^{-2\pi\sqrt{t/6}}), \tag{4.8}$$

where

$$\widetilde{f} := q\frac{df}{dq}.$$

Differentiating both sides of (4.7) with respect to t, we find that

$$\sqrt{\frac{1}{6t}}\frac{\widetilde{A}}{A}(e^{-2\pi\sqrt{t/6}}) + \sqrt{\frac{t}{6}}\frac{\widetilde{A}}{A}(e^{-2\pi/\sqrt{6t}}) = \frac{1}{\pi}. \tag{4.9}$$

Rewriting (4.9) in terms of $\kappa(t)$ yields

$$\kappa(t) + t\kappa\left(\frac{1}{t}\right) = 0. \tag{4.10}$$

When $t = 1$, (4.10) implies that

$$\kappa(1) = 0. \tag{4.11}$$

Next, let

$$M_N(q) = \frac{A(q)}{A(q^N)}. \tag{4.12}$$

Setting $q = e^{2\pi\sqrt{t/6}}$ and differentiating (4.12) with respect to t, we find using (4.8) that

$$\kappa(N^2t) = 2\sqrt{\frac{t}{6}}\frac{\widetilde{M_N}}{M_N}(e^{-2\pi\sqrt{t/6}})\frac{1}{A(e^{-2\pi\sqrt{N^2t/6}})} - M_N(e^{-2\pi\sqrt{t/6}})\kappa(t). \tag{4.13}$$

Note that $\kappa(N^2t)$ tends to $\frac{1}{\pi}$ at the rate of order N as N tends to ∞.

In order to obtain a cubic iteration to $1/\pi$ from (4.13), let $N = 3$. If $y = G(q^3)$ then from [4, (2.9)], we have

$$\frac{\varphi(-q^9)}{\varphi(-q)} = \frac{1}{1-2y}. \tag{4.14}$$

Using (3.4) and (4.14), we deduce that

$$\begin{aligned}\frac{Z(q)}{Z(q^3)} &= \frac{\varphi^6(-q^3)}{\varphi^2(-q)}\frac{\varphi^2(-q^3)}{\varphi^6(-q^9)}\\ &= \frac{\varphi^8(-q^3)}{\varphi^8(-q^9)}\frac{\varphi^2(-q^9)}{\varphi^2(-q)}\\ &= \left(\frac{1-8y^3}{1-2y}\right)^2 = (1+2y+4y^2)^2.\end{aligned}$$

Hence,

$$M_3 = (1+2y+4y^2)^2\frac{(1+X)}{(1+y^3)} = \frac{(1+2y+4y^2)(1+y)^2}{1-y+y^2}, \tag{4.15}$$

by (1.1).

Using (3.6) with q replaced by q^3, we have

$$\widetilde{y} = A(q^3)y(1-8y^3).$$

This allows us to differentiate both sides of (4.15) and conclude that

$$\frac{1}{M_3(q)A(q^3)}\widetilde{M_3}(q) = \frac{(1-2y)y(8y^4-10y^3+6y^2+11y+5)}{(y+1)(1-y+y^2)}. \tag{4.16}$$

We are now ready to construct our sequence k_n. Let $s_n = G(e^{-2\pi\sqrt{3^{2n}/6}})$ and $k_n = \kappa(3^n)$. Writing (4.13) in terms of s_n and k_n, we find that

$$\begin{aligned}k_n = &\frac{(1+2s_n+4s_n^2)(1+s_n)^2}{(1-s_n+s_n^2)}k_{n-1}\\ &+ \frac{2\cdot 3^{n-1}}{\sqrt{6}}\frac{s_n(1-2s_n)(8s_n^4-10s_n^3+6s_n^2+11s_n+5)}{1+s_n^3},\end{aligned} \tag{4.17}$$

From (4.11), we know that the initial value of k_n is

$$k_0 = 0.$$

By letting $t = 1$ in (4.1), we find that the initial value of s_0 is

$$s_0 = G(e^{-2\pi/\sqrt{6}}) = \left(\frac{3\sqrt{2}}{4} - 1\right)^{1/3}. \tag{4.18}$$

We can then evaluate s_n from s_{n-1} using (2.2). Substituting s_n into (4.17), we construct the sequence $\{k_n\}$ which converges cubically to $1/\pi$ and this completes the proof of Theorem 2.1.

5. A series for $\frac{1}{\pi}$

Set $t = 1$ in (4.9). We find that

$$\frac{\widetilde{A}}{A}(e^{-2\pi/\sqrt{6}}) = \frac{\sqrt{6}}{2\pi}. \tag{5.1}$$

Using the relation (3.1) and (3.2) in the differential equation

$$\alpha(1-\alpha)\frac{d^2a}{d\alpha^2} + (1-2\alpha)\frac{da}{d\alpha} - \frac{2}{9}a = 0,$$

we deduce that

$$X(8X-1)(1+X)\frac{d^2z}{dX^2} + (24X^2+14X-1)\frac{dz}{dX} + 2(1+4X)z = 0. \tag{5.2}$$

If

$$z = \sum_{k=0}^{\infty} c_k X^k,$$

then from (5.2), we know that a_k satisfies the recurrence

$$k^2 c_k - (7k^2 - 7k + 2)c_{k-1} - 8(k-1)^2 c_{k-2} = 0.$$

The solution of the above recurrence with $c_0 = 1, c_1 = 2$ is given by [10, Table 2] [1]

$$c_k = \sum_{j=0}^{k} \binom{k}{j}^3.$$

Hence,

$$z = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \binom{k}{j}^3 X^k.$$

Therefore,

$$Z = z^2 = \sum_{k=0}^{\infty} C_k X^k,$$

where C_k is given by (1.3), or

$$A = \sum_{k=0}^{\infty} C_k X^k (1+X). \tag{5.3}$$

[1] According to H.A. Verrill, the solution to the recurrence is due to D. Zagier.

From (5.3), we deduce that

$$\begin{aligned}\frac{\widetilde{A}}{A} &= \frac{1}{A}\frac{dA}{dX}\widetilde{X} \\ &= (1-8X)\sum_{k=0}^{\infty} C_k X^k (k(1+X)+X),\end{aligned} \tag{5.4}$$

by (3.6).

Set $q = e^{-2\pi/\sqrt{6}}$ in (5.4). From (4.18), we know that

$$X(e^{-2\pi/\sqrt{6}}) = x_1 = -1 + \frac{3\sqrt{2}}{4}.$$

Hence, we have

$$(1-8x_1)\sum_{k=0}^{\infty} C_k x_1^k (k(1+x_1)+x_1) = \frac{\sqrt{6}}{2\pi}.$$

Simplifying the above yields (1.2).

6. Conclusions

1. We have seen here that (4.1) plays an important role for our determination of $A(q)$. In general, if we have a modular function (i.e. a Hauptmodul) associated to congruence subgroup Γ of $SL_2(\mathbf{Z})$ with genus zero, we need to determine a "nice" modular form of weight 2 on Γ in order to derive new series for $1/\pi$. It is therefore possible to derive new series for $1/\pi$ associated with the Rogers-Ramanujan continued fraction.
2. We can also obtain another cubic iteration to $1/\pi$ if we use the alternative formula [1]

$$\left(1+\frac{1}{G^3(-e^{-\pi t})}\right)\left(1+\frac{1}{G^3(-e^{-\pi/t})}\right) = 9.$$

   We leave this as an exercise for the readers.

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