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TRIPLE PRODUCT IDENTITY, QUINTUPLE PRODUCT IDENTITY AND RAMANUJAN'S DIFFERENTIAL EQUATIONS FOR THE CLASSICAL EISENSTEIN SERIES

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Abstract. In this article, we use the triple product identity and the quintuple product identity to derive Ramanujan's famous differential equations for the Eisenstein series.

1. Introduction

In his famous paper [10], S. Ramanujan gave elementary proofs to two trigonometric identities

$$\begin{aligned}(1.1)\qquad &\left(\frac{1}{4}\cot\frac{u}{2}+\sum_{k=1}^{\infty}\frac{q^k}{1-q^k}\sin ku\right)^2\\ &=\left(\frac{1}{4}\cot\frac{u}{2}\right)^2+\sum_{k=1}^{\infty}\frac{q^k}{(1-q^k)^2}\cos ku+\frac{1}{2}\left(\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}(1-\cos ku)\right)\end{aligned}$$

and

$$\begin{aligned}(1.2)\qquad &\left(\frac{1}{8}\cot^2\frac{u}{2}+\frac{1}{12}+\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}(1-\cos ku)\right)^2\\ &=\left(\frac{1}{8}\cot^2\frac{u}{2}+\frac{1}{12}\right)^2+\frac{1}{12}\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}(5+\cos ku),\end{aligned}$$

where $q=e^{2\pi i\tau}, \operatorname{Im}\tau>0$. He then deduced from (1.1) and (1.2) many identities satisfied by the classical Eisenstein series

$$E_{2k}=1-\frac{4k}{B_{2k}}\sum_{n\geq 1}\frac{n^{2k-1}q^n}{1-q^n}, \tag{1.3}$$

where B_{2k} is the Bernoulli number defined by

$$\frac{x}{e^x-1}=\sum_{k\geq 0}B_k\frac{x^k}{k!}. \tag{1.4}$$

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In particular, he showed that if $P = E_2(q), Q = E_4(q)$ and $R = E_6(q)$, then

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \tag{1.5}$$

$$q\frac{dQ}{dq} = \frac{PQ - R}{3} \tag{1.6}$$

and

$$q\frac{dR}{dq} = \frac{PR - Q^2}{2}. \tag{1.7}$$

Using (1.5), (1.6) and (1.7), Ramanujan derived the famous identity

$$\eta^{24}(\tau) := q\prod_{k=1}^{\infty}(1-q^k)^{24} = \frac{1}{1728}\left(Q^3 - R^2\right). \tag{1.8}$$

Identity (1.8) together with the fact that Q and R are respectively the normalization of the Eisenstein series

$$G_4(\tau) := \sum_{\substack{(m,n)\in\mathbf{Z}\times\mathbf{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^4}$$

and

$$G_6(\tau) := \sum_{\substack{(m,n)\in\mathbf{Z}\times\mathbf{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^6}$$

implies the transformation formula

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau). \tag{1.9}$$

The purpose of this article is to derive new proofs of (1.5)–(1.7) using only the Jacobi triple product identity and the quintuple product identity.

For other proofs of (1.5)–(1.7) with combinatorial flavor, see [12], [13] and [7].

For proofs of (1.5)–(1.7) using the theory of modular forms, see for example [8, p. 161, Theorem 5.3]. These proofs, however, require the knowledge of (1.9), as well as the dimension of the space of modular forms of weight 4, 6, and 8 on $\mathrm{SL}_2(\mathbf{Z})$.

2. The Jacobi triple product identity

The Jacobi theta function is defined by

$$\vartheta_1(u|\tau) = 2\sum_{k=0}^{\infty}(-1)^k q^{(2k+1)^2/8}\sin(2k+1)u. \tag{2.1}$$

By Jacobi's triple product identity [1, p. 497, Theorem 10.4.1], we know that

$$\vartheta_1(u|\tau) = 2q^{1/8}\sin u\prod_{k=1}^{\infty}\left(1-q^k\right)\left(1-q^ke^{2iu}\right)\left(1-q^ke^{-2iu}\right).$$

By logarithmically differentiating the above with respect to u, we find that

$$\frac{\vartheta_1'(u|\tau)}{\vartheta_1(u|\tau)} = \cot u + 4\sum_{k\geq 1}\frac{q^k}{1-q^k}\sin 2ku. \tag{2.2}$$

One can rewrite the above as (see [9, Lemma 2])

$$\frac{\vartheta_1'(u|\tau)}{\vartheta_1(u|\tau)} = \frac{1}{u} + \sum_{k\geq 1}(-1)^k \frac{2^{2k}B_{2k}}{(2k)!}E_{2k}u^{2k-1},$$

where E_{2k} and B_{2k} are given by (1.3) and (1.4), respectively.

Now, let

$$S_{2n+1} = 2\sum_{k=0}^{\infty}(-1)^k(2k+1)^{2n+1}q^{(2k+1)^2/8}.$$

Then by expanding (2.1) in powers of u, we obtain

$$\vartheta_1(u|\tau) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}S_{2n+1}u^{2n+1}. \tag{2.3}$$

Using (2.3) and (2.2), we find that

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}S_{2n+1}u^{2n} = \left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}S_{2n+1}u^{2n+1}\right) \times \left(\frac{1}{u} + \sum_{k\geq 1}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!}E_{2k}u^{2k-1}\right).$$

This implies that

$$S_{2n+1} = \frac{1}{2n}\sum_{j=1}^{n}2^{2j}\binom{2n+1}{2j}B_{2j}E_{2j}S_{2(n-j)+1}.$$

The first four identities derived from this recurrence are:

$$S_3 = PS_1, \tag{2.4}$$

$$S_5 = \frac{5P^2-2Q}{3}S_1, \tag{2.5}$$

$$S_7 = \frac{35P^3-42PQ+16R}{9}S_1 \tag{2.6}$$

and

$$S_9 = \left(\frac{35}{3}P^4 - 28P^2Q + \frac{64}{3}PR + \frac{28}{5}Q^2 - \frac{48}{5}E_8\right)S_1, \tag{2.7}$$

where we have replaced E_2, E_4 and E_6 by P, Q, and R, respectively.

Note that

$$8q\frac{dS_n}{dq} = S_{n+2}.$$

Hence, by applying the operator $q\dfrac{d}{dq}$ to (2.4), we find that

$$S_5 = 8q\frac{dP}{dq}S_1 + PS_3,$$

which implies that

$$S_5 = \left(8q\frac{dP}{dq} + P^2\right)S_1. \tag{2.8}$$

Comparing (2.8) with (2.5), we deduce (1.5).

Similarly, by using (1.5), (2.5), (2.6) and (2.7), we deduce (1.6) and the identity

$$-112Q^2 - 320PR + 640q\frac{dR}{dq} + 432E_8 = 0. \tag{2.9}$$

To prove (1.7), we need another relation similar to (2.9), and this will be given in the next section.

3. The quintuple product identity

The quintuple product identity states that

$$q^{1/24}\prod_{k=1}^{\infty}(1-q^k)\frac{\vartheta_1(2u|\tau)}{\vartheta_1(u|\tau)} = 2\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}\cos(6n+1)u. \tag{3.1}$$

One of the simplest proofs of an equivalent form of (3.1) is due to L. Carlitz and M.V. Subbarao [5]. Their proof involves only the Jacobi triple product identity. For more details of the history and the different forms of (3.1), see [2, p. 83] or [6].

By expanding the right-hand side of (3.1) in powers of u, we deduce that

$$q^{1/24}\prod_{k=1}^{\infty}(1-q^k)\frac{\vartheta_1(2u|\tau)}{\vartheta_1(u|\tau)} = \sum_{k=0}^{\infty}(-1)^k\frac{T_{2k}}{(2k)!}u^{2k}, \tag{3.2}$$

where

$$T_{2k} = 2\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)^{2k}q^{(6n+1)^2/24}.$$

By logarithmically differentiating both sides of (3.2), we find that

$$\left(2\frac{\vartheta_1'}{\vartheta_1}(2u|\tau) - \frac{\vartheta_1'}{\vartheta_1}(u|\tau)\right)\left(\sum_{k=0}^{\infty}(-1)^k\frac{T_{2k}}{(2k)!}u^{2k}\right) = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{T_{2k+2}}{(2k+1)!}u^{2k+1}. \tag{3.3}$$

Using (2.2) and comparing the coefficients of powers of u in (3.3), we find that for $n \geq 0$,

$$T_{2n+2} = \frac{1}{2n+2}\sum_{j=0}^{n}\binom{2n+2}{2j+2}2^{2j+2}\left(2^{2j+2}-1\right)B_{2j+2}E_{2j+2}T_{2(n-j)}. \tag{3.4}$$

We construct from (3.4) the following identities for T_{2k}, $1 \leq k \leq 4$:

$$\begin{aligned} T_2 &= PT_0, \\ T_4 &= (3P^2-2Q)T_0, \\ T_6 &= \left(15P^3 - 30PQ + 16R\right)T_0 \end{aligned} \tag{3.5}$$

and

$$T_8 = \left(105P^4 - 420P^2Q + 448PR + 140Q^2 - 272E_8\right)T_0. \tag{3.6}$$

In the construction of the above identities, we have followed Z.G. Liu [9].[1]

Using (1.5), (1.6), (3.5), (3.6) and the relation

$$24q\frac{dT_n}{dq} = T_{n+2},$$

[1] There is a misprint in the formula for T_8. The coefficient of RP should be replaced by 336.

we find that

$$-192PR - 80Q^2 + 384q\frac{dR}{dq} + 272E_8 = 0. \tag{3.7}$$

Solving the simultaneous equations (2.9) and (3.7), we deduce (1.7) and the identity

$$E_8 = Q^2.$$

4. Conclusion

The functions S_{2k+1} and T_{2k} were studied by Ramanujan on page 369 of his Lost Notebook [11]. Ramanujan showed that S_{2k+1}/S_1 and T_{2k}/T_0 can be expressed in terms of P, Q and R. Further discussions of Ramanujan's proofs can be found in [14, pp. 31–32]. A discussion of S_{2k+1} can also be found in [3], while that of T_{2k} can be found in [4] and [9]. All the proofs discussed above use (1.5)-(1.7). We have shown here that in fact the first few identities satisfied by S_{2k+1}/S_1 $(k = 1, 2, 3, 4)$ and T_{2k}/T_0 $(k = 1, 2, 3, 4)$ are enough for us to deduce Ramanujan's results for these functions.

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