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THE ROGERS-RAMANUJAN CONTINUED FRACTION AND A QUINTIC ITERATION FOR $1/\pi$

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Abstract. Properties of the Rogers-Ramanujan continued fraction are used to obtain a formula for calculating $1/\pi$ with quintic convergence.

1. Introduction

Let q be a complex number satisfying $|q| < 1$. The Rogers-Ramanujan continued fraction is

$$R(q) = \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \, \cdots .$$

The purpose of this article is to use properties of the Rogers-Ramanujan continued fraction to derive the following iteration for $1/\pi$.

Theorem 1.1. *Let $g = (1+\sqrt{5})/2$. Define sequences by*

$$\begin{aligned}
s_0 &= \left(\sqrt{g^{10}+1} - g^5\right)^{1/5}, \\
k_0 &= 0, \\
r_{n+1} &= \left(\frac{1-g^5 s_n^5}{g^5 + s_n^5}\right)^{1/5}, \\
s_{n+1} &= \frac{1-g r_{n+1}}{g + r_{n+1}}, \\
k_{n+1} &= \frac{(s_{n+1}+g)^4 (g^2 s_{n+1}^2 + g^2 s_{n+1} + 1)}{g^2 (s_{n+1}^2 - g^2 s_{n+1} + g^2)} k_n \\
&\quad + \frac{2 \times 5^{n-1/2} g^2 s_{n+1} (1 - g s_{n+1})(g^2 s_{n+1}^2 - s_{n+1} + 1)}{(s_{n+1}+g)(s_{n+1}^2 - g^2 s_{n+1} + g^2)} f(s_{n+1}),
\end{aligned}$$

where

$$f(s) = 4s^4 - (2+5g)s^3 + (5-3g)s^2 + (6+7g)s + (5+3g).$$

Then k_n converges to $1/\pi$, and the rate of convergence is order 5.

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A cubic iteration for $1/\pi$, based on Ramanujan's cubic continued fraction

$$G(q) = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots,$$

has been given by H. H. Chan and K. P. Loo [11]. Our Theorem 1.1 is the analogue of [11, Theorem 2.1], for which the Rogers-Ramanujan continued fraction takes the role that Ramanujan's cubic continued fraction played in [11]. The method, in both the present work and in [11], is based on ideas developed in [10].

Theorem 1.1 is different from the quintic iterations of J. M. and P. B. Borwein in [6, p. 175], [7] and [8, p. 202], which were obtained using quintic modular equations. Other iterations for $1/\pi$ based on Dedekind's η-function and modular functions were given by J. M. Borwein and F. G. Garvan [9], and iterations based on elliptic functions were given by Chan [10].

2. Some preliminary results

In this section, we collect some important results concerning the Rogers-Ramanujan continued fraction and some allied functions. Two good sources of information about the Rogers-Ramanujan continued fraction are the last chapter of the introductory book by B. C. Berndt [4] and the expository article by W. Duke [13].

The first significant fact about the Rogers-Ramanujan continued fraction is its expression in terms of an infinite product:

$$R(q) = q^{1/5} \prod_{j=1}^{\infty} \frac{(1-q^{5j-4})(1-q^{5j-1})}{(1-q^{5j-3})(1-q^{5j-2})}. \tag{2.1}$$

An outline of a proof of this result, together with references, can be found in Berndt's book [4].

Let

$$X(q) = R^5(q), \tag{2.2}$$

$$y(q) = R(q^5), \tag{2.3}$$

and

$$Z(q) = \prod_{j=1}^{\infty} \frac{(1-q^j)^5}{(1-q^{5j})}. \tag{2.4}$$

When it is not necessary to emphasize the parameter q, we will simply write R, X, y and Z for $R(q)$, $X(q)$, $y(q)$ and $Z(q)$, respectively.

We will use the golden ratio, which we denote by

$$g = \frac{1+\sqrt{5}}{2}.$$

We will require the formulas

$$\frac{1}{R} - 1 - R = \frac{1}{q^{1/5}} \prod_{j=1}^{\infty} \frac{(1-q^{j/5})}{(1-q^{5j})} \tag{2.5}$$

and

$$\frac{1}{X} - 11 - X = \frac{1}{q} \prod_{j=1}^{\infty} \frac{(1-q^j)^6}{(1-q^{5j})^6}. \tag{2.6}$$

Simple proofs of these results, using only the Jacobi triple product identity, have been given by M. Hirschhorn [15]. More information about the identities (2.5) and (2.6), and references to other proofs, can be found in the book by G. E. Andrews and B. C. Berndt [1, pp. 11–12].

The function Z has a simple Lambert series expansion:

$$Z = 1 - 5\sum_{j=1}^{\infty} \left(\frac{j}{5}\right) \frac{jq^j}{1-q^j}, \tag{2.7}$$

where $\left(\frac{j}{5}\right)$ is the Legendre symbol. This formula was given by Ramanujan [16, Chapter 19, Entry 9 (v)]. For proofs, see Berndt's book [2, pp. 257–261] or the papers by J. M. Dobbie [12] and Hirschhorn [14]. References to other proofs are given in [2] and [14].

The functions R and R^5 satisfy the modular properties [13, eqs. (3.2) and (7.3)]

$$R\left(e^{-2\pi/\alpha}\right) = \frac{1 - gR(e^{-2\pi\alpha})}{g + R(e^{-2\pi\alpha})}, \tag{2.8}$$

$$R^5\left(e^{-2\pi/5\alpha}\right) = \frac{1 - g^5R^5(e^{-2\pi\alpha})}{g^5 + R^5(e^{-2\pi\alpha})}, \tag{2.9}$$

where α is any complex number satisfying $\mathrm{Re}(\alpha) > 0$. If we let $\alpha = \sqrt{t/5}$ and rearrange, then (2.9) may be rewritten as

$$\left(g^5 + X\left(e^{-2\pi\sqrt{t/5}}\right)\right)\left(g^5 + X\left(e^{-2\pi/\sqrt{5t}}\right)\right) = 1 + g^{10}. \tag{2.10}$$

This result appears in Ramanujan's lost notebook [1, p. 91], [17, p. 364]. If we replace α with 5α in (2.8) and combine the result with (2.9), we obtain a relation between $u = R(q)$ and $v = R(q^5)$ given by

$$\left(\frac{1-gv}{g+v}\right)^5 = \frac{1-g^5u^5}{g^5+u^5}. \tag{2.11}$$

If we solve for u^5, we obtain

$$u^5 = v\frac{1 - 2v + 4v^2 - 3v^3 + v^4}{1 + 3v + 4v^2 + 2v^3 + v^4}. \tag{2.12}$$

On the other hand, if we solve (2.11) for v, we obtain

$$v = \frac{1 - g\left(\frac{1-g^5u^5}{g^5+u^5}\right)^{1/5}}{g + \left(\frac{1-g^5u^5}{g^5+u^5}\right)^{1/5}}. \tag{2.13}$$

Equation (2.12) was given by Ramanujan in his first letter to Hardy [5, p. 29]. Equation (2.13) will be used in our iteration for $1/\pi$.

3. A formula for $1/\pi$

3.1. The functions $A(q)$ and $\kappa(t)$. Let

$$q = \exp\left(\frac{-2\pi\sqrt{t}}{\sqrt{5}}\right), \quad p = \exp\left(\frac{-2\pi}{\sqrt{5t}}\right), \quad t > 0.$$

If we logarithmically differentiate (2.2) and use (2.7), we obtain

$$q\frac{dX}{dq} = X\left(1 - 5\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j}\right) = ZX. \tag{3.1}$$

Differentiating (2.10) and using (3.1), we get

$$t\,\frac{Z(q)X(q)}{g^5 + X(q)} = \frac{Z(p)X(p)}{g^5 + X(p)}. \tag{3.2}$$

We may rewrite (2.10) as

$$X(q) = \frac{(g^5 + X(q))(1 - g^5X(p))}{g^{10}+1}, \tag{3.3}$$

and replacing t with $1/t$, we obtain

$$X(p) = \frac{(g^5 + X(p))(1 - g^5X(q))}{g^{10}+1}. \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we deduce that

$$t\,\frac{Z(q)}{1 - g^5X(q)} = \frac{Z(p)}{1 - g^5X(p)}. \tag{3.5}$$

If we define

$$A(q) = \frac{Z(q)}{1 - g^5X(q)}, \tag{3.6}$$

then (3.5) reduces to

$$tA(q) = A(p). \tag{3.7}$$

Differentiating (3.7) with respect to t, we find that

$$A(q) - \frac{\pi\sqrt{t}}{\sqrt{5}}\widetilde{A}(q) = \frac{\pi}{\sqrt{5t^3}}\widetilde{A}(p),$$

where

$$\widetilde{f}(z) = z\frac{df}{dz}.$$

Multiplying both sides by $2/\pi A(q)$, we deduce that

$$\left(\frac{1}{\pi} - \frac{2\sqrt{t}}{\sqrt{5}}\frac{\widetilde{A}(q)}{A(q)}\right) + \left(\frac{1}{\pi} - \frac{2}{\sqrt{5t}}\frac{\widetilde{A}(p)}{A(p)}\right) = 0. \tag{3.8}$$

If we define

$$\kappa(t) = \frac{1}{\pi A(q)} - \frac{2\sqrt{t}}{\sqrt{5}}\frac{\widetilde{A}(q)}{A^2(q)}, \tag{3.9}$$

then (3.8) becomes, after dividing by $A(q)$, simply

$$\kappa(t) + t\kappa\left(\frac{1}{t}\right) = 0. \tag{3.10}$$

3.2. **The multiplier.** Let

$$M_N(q) = \frac{A(q)}{A(q^N)}. \tag{3.11}$$

We will be particularly interested in $M_5(q)$. Observe that by (3.6),

$$\begin{aligned} M_5(q) &= \frac{A(q)}{A(q^5)} \\ &= \frac{Z(q)}{(1-g^5X(q))}\frac{(1-g^5X(q^5))}{Z(q^5)} \\ &= \frac{Z(q)}{Z(q^5)}\frac{(1-g^5y^5)}{(1-g^5X)}. \end{aligned}$$

By (2.4), we have

$$\begin{aligned} M_5(q) &= \left(\prod_{j=1}^{\infty}\frac{(1-q^j)^5}{(1-q^{5j})}\right)\left(\prod_{j=1}^{\infty}\frac{(1-q^{25j})}{(1-q^{5j})^5}\right)\frac{(1-g^5y^5)}{(1-g^5X)} \\ &= \left(\frac{1}{q}\prod_{j=1}^{\infty}\frac{(1-q^j)^6}{(1-q^{5j})^6}\right)\left(q\prod_{j=1}^{\infty}\frac{(1-q^{25j})}{(1-q^j)}\right)\frac{(1-g^5y^5)}{(1-g^5X)}. \end{aligned}$$

By (2.5) and (2.6), we obtain

$$M_5(q) = \frac{(X^{-1}-11-X)}{(y^{-1}-1-y)}\frac{(1-g^5y^5)}{(1-g^5X)}.$$

Now using (2.12) and the relations $u^5 = X$ and $v = y$, we may express X in terms of y. The final result is

$$M_5(q) = \frac{(y+g)^4(g^2y^2+g^2y+1)}{g^2(y^2-g^2y+g^2)}. \tag{3.12}$$

Differentiating (3.12) gives

$$\frac{dM_5}{dy} = \frac{(y+g)^3}{(y^2-g^2y+g^2)^2}f(y), \tag{3.13}$$

where

$$f(y) = 4y^4 - (2+5g)y^3 + (5-3g)y^2 + (6+7g)y + (5+3g). \tag{3.14}$$

By the chain rule, together with (2.2), (2.3) and (3.1), we obtain

$$q\frac{dM_5}{dq} = Z(q^5)y\frac{dM_5}{dy}.$$

Therefore, using (3.6), (3.12) and (3.13), we obtain

$$\begin{aligned} \frac{\widetilde{M}_5(q)}{M_5(q)A(q^5)} &= \frac{Z(q^5)y}{M_5(q)A(q^5)}\frac{dM_5}{dy} \\ &= \frac{y(1-g^5y^5)}{M_5(q)}\frac{dM_5}{dy} \\ &= \frac{g^2y(1-gy)(g^2y^2-y+1)}{(y+g)(y^2-g^2y+g^2)}f(y). \end{aligned} \tag{3.15}$$

3.3. **A functional equation for κ.** In this section, we obtain a formula that expresses $\kappa(tN^2)$ in terms of $\kappa(t)$. The iteration for $1/\pi$ is based on this formula.

Logarithmically differentiating (3.11), we get

$$\frac{\widetilde{M}_N(q)}{M_N(q)} = \frac{\widetilde{A}(q)}{A(q)} - N\frac{\widetilde{A}(q^N)}{A(q^N)}.$$

Divide by $A(q^N)$ and use (3.11) again to get

$$\begin{aligned}\frac{\widetilde{M}_N(q)}{M_N(q)A(q^N)} &= \frac{\widetilde{A}(q)}{A(q)A(q^N)} - N\frac{\widetilde{A}(q^N)}{A^2(q^N)} \\ &= M_N(q)\frac{\widetilde{A}(q)}{A^2(q)} - N\frac{\widetilde{A}(q^N)}{A^2(q^N)}.\end{aligned}$$

Now multiply by $2\sqrt{t/5}$ and use (3.9) to get

$$\begin{aligned}2\sqrt{\frac{t}{5}}\frac{\widetilde{M}_N(q)}{M_N(q)A(q^N)} &= M_N(q)\,2\sqrt{\frac{t}{5}}\frac{\widetilde{A}(q)}{A^2(q)} - 2\sqrt{\frac{tN^2}{5}}\frac{\widetilde{A}(q^N)}{A^2(q^N)} \\ &= M_N(q)\left(\frac{1}{\pi A(q)} - \kappa(t)\right) - \left(\frac{1}{\pi A(q^N)} - \kappa(tN^2)\right) \\ &= \kappa(tN^2) - M_N(q)\kappa(t).\end{aligned}$$

Therefore,

$$\kappa(tN^2) = M_N(q)\kappa(t) + 2\sqrt{\frac{t}{5}}\frac{\widetilde{M}_N(q)}{M_N(q)A(q^N)}. \tag{3.16}$$

3.4. **An iteration for $1/\pi$.** If we let $\alpha = 1/\sqrt{5}$ in (2.9) and solve the resulting quadratic equation in R^5, we obtain

$$R\left(e^{-2\pi/\sqrt{5}}\right) = \left(\sqrt{g^{10}+1} - g^5\right)^{1/5}.$$

Now let $t = 1$ in (3.10) to get

$$\kappa(1) = 0.$$

Define two sequences by

$$\begin{aligned}k_n &= \kappa(5^{2n}), \\ s_n &= R\left(e^{-2\pi\sqrt{5^{2n-1}}}\right),\end{aligned}$$

where n is a non-negative integer. By the calculations just done, we have

$$k_0 = 0, \quad s_0 = \left(\sqrt{g^{10}+1} - g^5\right)^{1/5}.$$

Furthermore, expanding (3.9) in a series gives

$$\kappa(t) = \frac{1}{\pi} - (1+5\sqrt{5})\left(\frac{1}{2\pi} + \sqrt{\frac{t}{5}}\right)q + O(\sqrt{t}q^2), \text{ as } t \to \infty.$$

Therefore

$$k_n - \frac{1}{\pi} \sim -(1+5\sqrt{5})\left(\frac{1}{2\pi} + \frac{5^n}{\sqrt{5}}\right)\exp\left(-\frac{2\pi}{\sqrt{5}}5^n\right), \text{ as } n \to \infty. \tag{3.17}$$

It follows that k_n converges to $1/\pi$ and the rate of convergence is order 5. The identity (2.13) with $q = e^{-2\pi\sqrt{5^{2n-1}}}$ gives

$$s_{n+1} = \frac{1 - g\left(\frac{1-g^5 s_n^5}{g^5 + s_n^5}\right)^{1/5}}{g + \left(\frac{1-g^5 s_n^5}{g^5+s_n^5}\right)^{1/5}}. \tag{3.18}$$

Let $N = 5$ and $t = 5^{2n}$ in (3.16). We find that

$$k_{n+1} = M_5\left(e^{-2\pi\sqrt{5^{2n-1}}}\right) k_n + 2 \times 5^{n-1/2} \frac{\widetilde{M}_5\left(e^{-2\pi\sqrt{5^{2n-1}}}\right)}{M_5\left(e^{-2\pi\sqrt{5^{2n-1}}}\right) A\left(e^{-2\pi\sqrt{5^{2n+1}}}\right)}.$$

Using (3.12) and (3.15), we have

$$\begin{aligned} k_{n+1} = {} & \frac{(s_{n+1}+g)^4(g^2 s_{n+1}^2 + g^2 s_{n+1} + 1)}{g^2(s_{n+1}^2 - g^2 s_{n+1} + g^2)} k_n \\ & + \frac{2 \times 5^{n-1/2} g^2 s_{n+1}(1 - g s_{n+1})(g^2 s_{n+1}^2 - s_{n+1} + 1)}{(s_{n+1}+g)(s_{n+1}^2 - g^2 s_{n+1} + g^2)} f(s_{n+1}). \end{aligned} \tag{3.19}$$

Identities (3.17), (3.18) and (3.19) imply Theorem 1.1.

Remark 3.1. The values of $1/k_1$, $1/k_2$, $1/k_3$, $1/k_4$ and $1/k_5$ give π correct to 3, 27, 148, 758 and 3808 decimal places, respectively.

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