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Eisenstein series and theta functions to the septic base

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Abstract

We develop a theory for Eisenstein series to the septic base, which was started by S. Ramanujan in his "Lost Notebook." We show that two types of septic Eisenstein series may be parameterized in terms of the septic theta function $\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2}$ and the eta quotient $\eta^4(7\tau)/\eta^4(\tau)$. This is accomplished by constructing elliptic functions which have the septic Eisenstein series as Taylor coefficients. The elliptic functions are shown to be solutions of a differential equation, and this leads to a recurrence relation for the septic Eisenstein series.

MSC: 11F11; 11F20; 14K25; 33D52; 33E05

1. Introduction

Let $q = \exp(2\pi i\tau)$ where $\mathrm{Im}(\tau) > 0$ and define

$$z = z(q) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2},$$

$$h = h(q) = q\prod_{n=1}^{\infty}\frac{(1-q^{7n})^4}{(1-q^n)^4},$$

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$$\begin{aligned} k = k(q) &= \prod_{n=1}^{\infty} \frac{(1-q^n)^7}{(1-q^{7n})}, \\ x = x(q) &= \frac{7h}{1+7h}. \end{aligned} \tag{1.1}$$

S. Ramanujan [20, pp. 53, 355, 357] stated without proof results which are equivalent to

$$1 + 2\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{q^j}{1-q^j} = z, \tag{1.2}$$

$$8 - 7\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{j^2 q^j}{1-q^j} = k(49h+8), \tag{1.3}$$

and Z.-G. Liu [17, Eq. (1.18)] proved that

$$16 + \sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{j^4 q^j}{1-q^j} = k^{5/3}(49h+16)\bigl(49h^2+13h+1\bigr)^{2/3}. \tag{1.4}$$

Here $(\frac{j}{7})$ is the Legendre symbol. If we use the result [12, Lemma 2.2]

$$z^3 = k\bigl(49h^2+13h+1\bigr),$$

then (1.3) and (1.4) may be rewritten as

$$8 - 7\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{j^2 q^j}{1-q^j} = z^3 \frac{49h+8}{49h^2+13h+1}, \tag{1.5}$$

$$16 + \sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{j^4 q^j}{1-q^j} = z^5 \frac{49h+16}{49h^2+13h+1}. \tag{1.6}$$

Motivated by these examples, we empirically discovered the relation

$$584 - \sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{j^6 q^j}{1-q^j} = z^7 \frac{(16807h^3+25039h^2+7007h+584)}{(49h^2+13h+1)^2} \tag{1.7}$$

as well as a similar formula for

$$282592 + 7\sum_{j=1}^{\infty} \left(\frac{j}{7}\right) \frac{j^8 q^j}{1-q^j}.$$

The goal of this paper is to show that (1.2) and (1.5)–(1.7) are the first few instances of an infinite family of results. In order to make a general statement, we require some definitions. The Bernoulli numbers B_n are defined by

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}.$$

Let p be an odd prime and define the generalized Bernoulli numbers $B_{n,p}$ by [1, p. 59]

$$\frac{x}{e^{px}-1}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)e^{kx}=\sum_{n=0}^{\infty}B_{n,p}\frac{x^n}{n!}. \tag{1.8}$$

The classical Eisenstein series $E_{2n}(q)$ are defined, for positive integers n, by

$$E_{2n}(q)=-\frac{B_{2n}}{4n}+\sum_{j=1}^{\infty}\frac{j^{2n-1}q^j}{1-q^j}. \tag{1.9}$$

Define the septic Eisenstein series $E_{2n+1}^{\infty}(q;\chi_7)$ and $E_{2n+1}^{0}(q;\chi_7)$, for non-negative integers n, by [18, Eq. (6)]

$$E_{2n+1}^{\infty}(q;\chi_7)=-\frac{B_{2n+1,7}}{4n+2}+\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{j^{2n}q^j}{1-q^j}, \tag{1.10}$$

and

$$E_{2n+1}^{0}(q;\chi_7)=\frac{\delta_{n,0}}{2}+\sum_{j=1}^{\infty}\frac{j^{2n}(q^j+q^{2j}-q^{3j}+q^{4j}-q^{5j}-q^{6j})}{1-q^{7j}}, \tag{1.11}$$

where

$$\delta_{n,0}=\begin{cases}1 & \text{if } n=0,\\ 0 & \text{otherwise}\end{cases}$$

is the Kronecker delta. Let us also define

$$E_{2n}^{\infty}(q;\chi_7)=E_{2n}(q)-7^{2n}E_{2n}\big(q^7\big), \tag{1.12}$$

and

$$E_{2n}^{0}(q;\chi_7)=E_{2n}(q)-E_{2n}\big(q^7\big). \tag{1.13}$$

Throughout this work, $p_k(h)$ will denote a polynomial in h of degree $\leqslant k$.

We will prove the following two theorems:

Theorem 1.1. *Let n be a positive integer, and let $k = \lfloor (2n+1)/3 \rfloor$. There exists a polynomial $p_{2k-1}(h)$ of degree $\leqslant 2k-1$ such that*

$$E_{2n+1}^{\infty}(q;\chi_7) = z^{2n+1}\frac{p_{2k-1}(h)}{(49h^2+13h+1)^k}.$$

When $n=0$ we have

$$E_1^{\infty}(q;\chi_7) = \frac{z}{2}.$$

Theorem 1.2. *Let n be a non-negative integer, and let $k = \lfloor (2n+1)/3 \rfloor$. There exists a polynomial $p_{2k}(h)$ of degree $\leqslant 2k$ such that*

$$E_{2n+1}^{0}(q;\chi_7) = z^{2n+1}\frac{p_{2k}(h)}{(49h^2+13h+1)^k}.$$

Equations (1.5)–(1.7) are the cases $n = 1, 2$ and 3 of Theorem 1.1, respectively. The cases $n = 0, 1, 2$ and 3 of Theorem 1.2 will be given in full at the end of Section 5.

The analogous results for classical Eisenstein series were given by Ramanujan [19, Chapter 17, Entries 13–17] and [21, pp. 140–141]. See [5, pp. 126–139] for a proof of Ramanujan's results, and see [14] for a detailed analysis.

The theory for cubic analogues was started by Ramanujan [19, p. 257], and has been developed further in [6,9,10] and [13].

The quartic and sextic theories were started by Ramanujan. See [6] for an analysis of Ramanujan's results. The quartic theory has been extended in [7].

The quintic theory has been developed recently by H.H. Chan and Z.-G. Liu [11].

Some results on septic theta functions have been given in [12] and [17]. Equation (1.2) has been proved in [2, pp. 404–405]. Equations (1.3) and (1.4) have been proved in [17]. Equation (1.7) and Theorems 1.1 and 1.2 are new.

This work is organized as follows. Four elliptic functions are defined in Section 2, and their periodicity properties and poles are studied. A differential equation relating two of the elliptic functions is given in Section 3. In Section 4, the results from Sections 2 and 3 are used to prove Theorem 1.1. The imaginary transformation $\tau \to -1/(7\tau)$ is analyzed in Section 5, and the result is used to prove Theorem 1.2. We conclude with some remarks in Section 6.

2. Elliptic functions

Let us define the generating functions L, ℓ, M and m by

$$L(u) = L(u|\tau) = \sum_{n=0}^{\infty} E_{2n+1}^{\infty}(q;\chi_7)\frac{(-1)^n(2u)^{2n}}{(2n)!}, \tag{2.1}$$

$$\ell(u) = \ell(u|\tau) = \sum_{n=0}^{\infty} E_{2n+1}^{0}(q;\chi_7)\frac{(-1)^n(2u)^{2n}}{(2n)!}, \tag{2.2}$$

$$M(u) = M(u|\tau) = \sum_{n=0}^{\infty} E_{2n+2}^{\infty}(q;\chi_7)\frac{(-1)^n(2u)^{2n+1}}{(2n+1)!}, \tag{2.3}$$

$$m(u)=m(u|\tau)=\sum_{n=0}^{\infty}E^0_{2n+2}(q;\chi_7)\frac{(-1)^n(2u)^{2n+1}}{(2n+1)!}. \tag{2.4}$$

In this section we will show that L and ℓ are elliptic functions of order 6, and calculate their periods, poles and residues. Although M and m are not elliptic functions, their derivatives are. We determine the periods and poles.

If we substitute (1.10) into (2.1), we obtain

$$L(u)=-\sum_{n=0}^{\infty}\frac{B_{2n+1,7}}{(4n+2)}\frac{(-1)^n(2u)^{2n}}{(2n)!}+\sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{q^j}{(1-q^j)}\frac{(-1)^n(2ju)^{2n}}{(2n)!}.$$

Now interchange the order of summation and use (1.8) to obtain the Fourier expansion

$$L(u)=\frac{\sin 5u+\sin 3u-\sin u}{2\sin 7u}+\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{q^j}{1-q^j}\cos 2ju. \tag{2.5}$$

In the Fourier series, write

$$\frac{q^j}{1-q^j}=\sum_{k=1}^{\infty}q^{jk}\quad\text{and}\quad\cos 2ju=\frac{e^{2iju}+e^{-2iju}}{2},$$

and reverse the order of summation. The result is the analytic continuation formula

$$\begin{aligned}L(u)&=\frac{1}{2}\sum_{k=0}^{\infty}\frac{q^ke^{2iu}+q^{2k}e^{4iu}-q^{3k}e^{6iu}+q^{4k}e^{8iu}-q^{5k}e^{10iu}-q^{6k}e^{12iu}}{1-q^{7k}e^{14iu}}\\&\quad+\frac{1}{2}\sum_{k=1}^{\infty}\frac{q^ke^{-2iu}+q^{2k}e^{-4iu}-q^{3k}e^{-6iu}+q^{4k}e^{-8iu}-q^{5k}e^{-10iu}-q^{6k}e^{-12iu}}{1-q^{7k}e^{-14iu}}.\end{aligned} \tag{2.6}$$

This implies the periodicity properties

$$L(u+\pi)=L(u)\quad\text{and}\quad L(u+\pi\tau)=L(u).$$

Furthermore, it shows that L has simple poles at $u=\pi j/7+\pi\tau k$, where j and k are integers and j is not a multiple of 7, and is analytic at all other points in the complex plane. The residue of L at the point $u=j\pi/7$ may be calculated using the Fourier series (2.5), and the residues at the other poles follow from the periodicity properties. The result is

$$\operatorname{Res}\left(L(u);u=\frac{j\pi}{7}+k\pi\tau\right)=-\frac{\sqrt{7}}{28}\left(\frac{j}{7}\right), \tag{2.7}$$

where j and k are integers and $j\not\equiv 0 \pmod 7$.

A similar analysis may be performed for the function ℓ. The Fourier series is

$$\ell(u)=\frac{1}{2}+\sum_{j=1}^{\infty}\frac{q^j+q^{2j}-q^{3j}+q^{4j}-q^{5j}-q^{6j}}{1-q^{7j}}\cos 2ju;$$

the analytic continuation is

$$\ell(u) = \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{k}{7}\right)\frac{q^k e^{2iu}}{1-q^k e^{2iu}} + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{k}{7}\right)\frac{q^k e^{-2iu}}{1-q^k e^{-2iu}};$$

and the periodicity properties are

$$\ell(u+\pi) = \ell(u) \quad \text{and} \quad \ell(u+7\pi\tau) = \ell(u).$$

The function ℓ has simple poles at $u = \pi j + \pi\tau k$ where j and k are integers and k is not a multiple of 7, and ℓ is analytic for all other values of u. The residues are

$$\operatorname{Res}\bigl(\ell(u); u = j\pi + k\pi\tau\bigr) = -\frac{i}{4}\left(\frac{k}{7}\right). \tag{2.8}$$

Here are the corresponding results for M and m. The function M has simple poles at $j\pi/7 + k\pi\tau$, where j and k are integers and j is not a multiple of 7, and no other singularities. It satisfies the properties

$$M(u+\pi) = M(u) \quad \text{and} \quad M(u+\pi\tau) = M(u) + 3i.$$

Thus M is not an elliptic function, but its derivative is. Lastly, the function m has simple poles at $j\pi + k\pi\tau$, where j and k are integers and k is not a multiple of 7, and no other singularities. It satisfies the properties

$$m(u+\pi) = m(u) \quad \text{and} \quad m(u+7\pi\tau) = m(u) - 3i,$$

so m is not an elliptic function, but its derivative is.

The functions M and m are related to the logarithmic derivative of the theta function. The precise identifications are given in the next section in Lemma 3.4.

3. A differential equation

The main result of this section is the following differential equation involving L and M:

Theorem 3.1.

$$\begin{aligned} L^{(4)} &= 20L\bigl(M''' - 48E_4(q)\bigr) + 40L''\bigl(M' + 12E_2(q)\bigr) - 1120L''L^2 \\ &\quad - 240L\bigl(M' + 12E_2(q)\bigr)^2 + 4480L^3\bigl(M' + 12E_2(q)\bigr) - 12544L^5. \end{aligned}$$

Observe that each term $L^{(4)}$, $20L(M''' - 48E_4(q))$, etc., in Theorem 3.1 is an even elliptic function with periods π and $\pi\tau$, and (counting according to multiplicity) has 30 poles in a period parallelogram.

In order to prove Theorem 3.1, we first define the theta function θ_1, and establish several lemmas.

Definition 3.2. Let

$$\theta_1(z|\tau) := -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(2n+1)^2/8} e^{(2n+1)iz},$$

and write

$$\frac{\theta_1'}{\theta_1}(z|\tau) := \frac{\theta_1'(z|\tau)}{\theta_1(z|\tau)},$$

where the prime denotes differentiation with respect to z.

Lemma 3.3. *The theta function satisfies the following properties*:

$$\begin{aligned}
\theta_1(z+\pi|\tau) &= -\theta_1(z|\tau), \\
\theta_1(z+\pi\tau|\tau) &= -q^{-1/2}e^{-2iz}\theta_1(z|\tau), \\
\theta_1(z|\tau) &= 2q^{1/8}\sin z \prod_{n=1}^{\infty}\left(1-q^n e^{2iz}\right)\left(1-q^n e^{-2iz}\right)\left(1-q^n\right), \\
\frac{\theta_1'}{\theta_1}(z|\tau) &= \cot z + 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz, \quad \textit{and} \\
\frac{\theta_1'}{\theta_1}(z|\tau) &= \frac{1}{z} + 4\sum_{n=0}^{\infty} E_{2n+2}(q)\frac{(-1)^n(2z)^{2n+1}}{(2n+1)!}.
\end{aligned}$$

Proof. These are standard results. The first four results can be found in [22, pp. 465, 470, 489].[1] The fifth result follows from the fourth by expanding in powers of z and using (1.9). □

Lemma 3.4. *The functions M and m satisfy the properties*

$$\begin{aligned}
M(u) &= \frac{1}{4}\left(\frac{\theta_1'}{\theta_1}(u|\tau) - 7\frac{\theta_1'}{\theta_1}(7u|7\tau)\right), \\
m(u) &= \frac{1}{4}\left(\frac{\theta_1'}{\theta_1}(u|\tau) - \frac{\theta_1'}{\theta_1}(u|7\tau)\right).
\end{aligned}$$

Proof. These follow by substituting (1.12) and (1.13) into (2.3) and (2.4), and comparing with the last result in Lemma 3.3. □

[1] In [22] the convention $q = \exp(i\pi\tau)$ is used instead of our $q = \exp(2\pi i\tau)$, and so the formulas in [22] have q^2 in place of our q.

Lemma 3.5. *Let $F(z)$ be any meromorphic function, and define $\phi(z) = \frac{F'(z)}{F(z)}$. Then*

$$\begin{aligned}
F' &= \phi F, \\
F'' &= \left(\phi' + \phi^2\right)F, \\
F''' &= \left(\phi'' + 3\phi'\phi + \phi^3\right)F, \\
F^{(4)} &= \left(\phi''' + 4\phi''\phi + 3(\phi')^2 + 6\phi'\phi^2 + \phi^4\right)F, \\
F^{(5)} &= \left(\phi^{(4)} + 5\phi'''\phi + 10\phi''\phi' + 10\phi''\phi^2 + 15(\phi')^2\phi + 10\phi'\phi^3 + \phi^5\right)F.
\end{aligned}$$

Proof. Each identity follows from the preceding one by the product rule for differentiation. □

Before stating the next lemma, we recall that the set of quadratic residues modulo 7 is $\{1, 2, 4\}$.

Lemma 3.6. *Let*

$$f(z, u|\tau) = \frac{\prod_{j\in\{1,2,4\}} \theta_1(z + u + \frac{j\pi}{7}|\tau)\theta_1(z - u + \frac{j\pi}{7}|\tau)}{\theta_1^6(z|\tau)}.$$

Then f is an elliptic function of z with periods π and $\pi\tau$. Furthermore, f has poles of order 6 at $z = m\pi + n\pi\tau$ for all integers m and n, and no other singularities.

Proof. All of these follow from properties of the function θ_1 given in Lemma 3.3. □

Lemma 3.7. *Let $f(z, u|\tau)$ be as defined in Lemma 3.6. Let $F(z) = z^6 f(z, u|\tau)$ and $\phi(z) = F'(z)/F(z)$. Then*

$$\begin{aligned}
&F(0) \not\equiv 0, \\
&\phi(z) = 6\left(\frac{1}{z} - \frac{\theta_1'}{\theta_1}(z|\tau)\right) \\
&\qquad + \sum_{j\in\{1,2,4\}} \left\{\frac{\theta_1'}{\theta_1}\left(z + u + \frac{j\pi}{7}\middle|\tau\right) + \frac{\theta_1'}{\theta_1}\left(z - u + \frac{j\pi}{7}\middle|\tau\right)\right\},
\end{aligned}$$

and

$$\left(\phi^{(4)} + 5\phi'''\phi + 10\phi''\phi' + 10\phi''\phi^2 + 15(\phi')^2\phi + 10\phi'\phi^3 + \phi^5\right)\Big|_{z=0} = 0.$$

Proof. First,

$$F(0) = \frac{1}{(\theta_1'(0))^6} \prod_{j\in\{1,2,4\}} \theta_1\left(u + \frac{j\pi}{7}\middle|\tau\right)\theta_1\left(-u + \frac{j\pi}{7}\middle|\tau\right) \not\equiv 0.$$

Second, the formula for $\phi(z)$ follows from the definitions of $f(z)$, $F(z)$ and $\phi(z)$. Third, consider $f(z,u|\tau)$ as a function of z. In a fundamental period parallelogram, f has a sixth order pole at $z=0$ and no other singularities. Thus the residue of f at this pole is zero, and therefore

$$\frac{1}{5!}\frac{d^5}{dz^5}z^6 f(z,u|\tau)\bigg|_{z=0}=0.$$

Hence

$$\frac{d^5}{dz^5}F(z)\bigg|_{z=0}=0.$$

Now invoke Lemma 3.5 and use the fact that $F(0)\not\equiv 0$ to complete the proof. □

We will require the following sums:

Lemma 3.8. *Let p be an odd prime, and define*

$$c_p=\begin{cases}1, & \text{if } p\equiv 1 \pmod 4,\\ i, & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Let m and n be positive integers, and let u be a complex variable. Then

$$\sum_{j=0}^{p-1} e^{2ij^2n\pi/p}=c_p\sqrt{p}\left(\frac{n}{p}\right); \tag{3.1}$$

$$\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)\cot\left(u+\frac{j\pi}{p}\right)=\frac{2ic_p\sqrt{p}}{e^{2ipu}-1}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)e^{2iku}; \tag{3.2}$$

$$\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)\sin 2n\left(u+\frac{j\pi}{p}\right)=\begin{cases}\sqrt{p}(\frac{n}{p})\sin 2nu, & \text{if } p\equiv 1 \pmod 4,\\ \sqrt{p}(\frac{n}{p})\cos 2nu, & \text{if } p\equiv 3 \pmod 4;\end{cases} \tag{3.3}$$

$$\sum_{j=0}^{m-1}\csc^2\left(u+\frac{j\pi}{m}\right)=m^2\csc^2 mu; \tag{3.4}$$

$$\sum_{j=0}^{m-1}\cos 2n\left(u+\frac{j\pi}{m}\right)=\begin{cases}0, & \text{if } n\not\equiv 0 \pmod m,\\ m\cos 2nu, & \text{if } n\equiv 0 \pmod m.\end{cases} \tag{3.5}$$

Proof. The first of these is a standard result, for example, see [3, p. 195]. Let us prove (3.2). To begin with, assume $\operatorname{Im} u>0$. Using

$$\cot x=-i\left(1+\frac{2e^{2ix}}{1-e^{2ix}}\right),$$

we have

$$\begin{aligned}\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)\cot\left(u+\frac{j\pi}{p}\right) &= -2i\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)\frac{e^{2i(u+j\pi/p)}}{1-e^{2i(u+j\pi/p)}}\\ &= -2i\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)\sum_{k=1}^{\infty}e^{2ik(u+j\pi/p)}\\ &= -2i\sum_{k=1}^{\infty}e^{2iku}\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)e^{2ijk\pi/p}\\ &= -2ic_p\sqrt{p}\sum_{k=1}^{\infty}\left(\frac{k}{p}\right)e^{2iku}\\ &= \frac{2ic_p\sqrt{p}}{(e^{2ipu}-1)}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)e^{2iku}.\end{aligned}$$

The identity in (3.1) has been used to obtain the penultimate step. This proves (3.2) for $\operatorname{Im} u > 0$. The result holds true for other complex values of u by analytic continuation, as both sides are meromorphic functions of u. This completes the proof of (3.2).

Let us prove (3.3). Equation (3.1) implies

$$\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)e^{2in(u+j\pi/p)} = e^{2inu}\sum_{j=0}^{p-1}e^{2ij^2n\pi/p} = e^{2inu}c_p\sqrt{p}\left(\frac{n}{p}\right).$$

Equation (3.3) now follows for real values of u by equating imaginary parts of both sides. The result holds for complex values of u by analytic continuation. This completes the proof of (3.3).

To prove (3.4), observe that

$$\prod_{j=0}^{m-1}\sin\left(u+\frac{j\pi}{m}\right) = \prod_{j=0}^{m-1}\frac{e^{i(u+j\pi/m)}-e^{-i(u+j\pi/m)}}{2i} = \frac{e^{imu}-e^{-imu}}{2^m i} = \frac{\sin mu}{2^{m-1}}.$$

Now take logarithms and differentiate twice with respect to u to obtain (3.4). Finally we prove (3.5). By geometric series, we have

$$\sum_{j=0}^{m-1}e^{2in(u+j\pi/m)} = \begin{cases}0, & \text{if } n\not\equiv 0 \pmod{m},\\ me^{2inu}, & \text{if } n\equiv 0 \pmod{m}.\end{cases}$$

The result now follows for real values of u by equating real parts, and for complex values of u by analytic continuation.

This completes the proof of Lemma 3.8. □

Lemma 3.9. *Let $\phi(z)$ be the function that was defined in Lemma* 3.7. *Then*

$$\begin{aligned}\phi(0)&=4\sqrt{7}L(u),\\ \phi'(0)&=-4M'(u)-48E_2(q),\\ \phi''(0)&=4\sqrt{7}L''(u),\\ \phi'''(0)&=-4M'''(u)-192E_4(q),\\ \phi^{(4)}(0)&=4\sqrt{7}L^{(4)}(u).\end{aligned}$$

Proof. Since $\theta_1'(z|\tau)/\theta_1(z|\tau)$ is an odd function of z with period π we have

$$\begin{aligned}\phi(0)&=\sum_{j\in\{1,2,4\}}\left\{\frac{\theta_1'}{\theta_1}\left(u+\frac{j\pi}{7}\middle|\tau\right)+\frac{\theta_1'}{\theta_1}\left(-u+\frac{j\pi}{7}\middle|\tau\right)\right\}\\ &=\sum_{j=1}^{6}\left(\frac{j}{7}\right)\frac{\theta_1'}{\theta_1}\left(u+\frac{j\pi}{7}\middle|\tau\right)\\ &=\sum_{j=1}^{6}\left(\frac{j}{7}\right)\left\{\cot\left(u+\frac{j\pi}{7}\right)+4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2n\left(u+\frac{j\pi}{7}\right)\right\}.\end{aligned}$$

Equations (3.2) and (3.3) of Lemma 3.8 with $p=7$ imply

$$\begin{aligned}\sum_{j=1}^{6}\left(\frac{j}{7}\right)\cot\left(u+\frac{j\pi}{7}\right)&=\frac{-2\sqrt{7}}{e^{14iu}-1}\sum_{k=1}^{6}\left(\frac{k}{7}\right)e^{2iku}\\ &=2\sqrt{7}\frac{(\sin 5u+\sin 3u-\sin u)}{\sin 7u}\end{aligned}$$

and

$$\sum_{j=1}^{6}\left(\frac{j}{7}\right)\sin 2n\left(u+\frac{j\pi}{7}\right)=\sqrt{7}\left(\frac{n}{7}\right)\cos 2nu.$$

Therefore

$$\begin{aligned}\phi(0)&=2\sqrt{7}\frac{(\sin 5u+\sin 3u-\sin u)}{\sin 7u}+4\sqrt{7}\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n}{1-q^n}\cos 2nu\\ &=4\sqrt{7}L(u).\end{aligned}$$

In a similar way we find

$$\phi'(0)=-48E_2(q)-\sum_{j=1}^{6}\csc^2\left(u+\frac{j\pi}{7}\right)+8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\sum_{j=1}^{6}\cos 2n\left(u+\frac{j\pi}{7}\right).$$

By Eqs. (3.4) and (3.5) of Lemma 3.8 with $m = 7$ we obtain

$$\phi'(0) = -48E_2(q) - 4M'(u).$$

The results for $\phi''(0)$, $\phi'''(0)$ and $\phi^{(4)}(0)$ are proved similarly. □

Theorem 3.1 may now be proved by substituting the results of Lemma 3.9 into the last part of Lemma 3.7.

4. Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1, which expresses $E_{2n+1}^{\infty}(q;\chi_7)$ as a product of z^{2n+1} and a rational function of h. We shall accomplish this by first proving an analogous result for $E_{2n}^{\infty}(q;\chi_7)$, and then using the differential equation in Theorem 3.1 to obtain the result for $E_{2n+1}^{\infty}(q;\chi_7)$. We will require some preparatory lemmas.

Lemma 4.1. *Let z, h, k and E_2 be defined by* (1.1) *and* (1.9). *Then*

$$z = 1 + 2\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{q^j}{1-q^j}, \tag{4.1}$$

$$z^2 = 4\big(E_2(q) - 7E_2\big(q^7\big)\big), \tag{4.2}$$

$$z^3 = k\big(49h^2 + 13h + 1\big). \tag{4.3}$$

Proof. Equation (4.1) was stated without proof by Ramanujan [20, pp. 355, 357] and has been proved by G.E. Andrews and B.C. Berndt [2, p. 404]. It is equivalent to a special case of a result of Dirichlet; see [16, Theorem 204].

Equation (4.2) was known to Ramanujan [20, pp. 355, 357]. Proofs of (4.2) have been given by Andrews and Berndt [2, p. 405], S. Cooper and P.C. Toh [15], Liu [17, Eq. (1.17)] and Williams [23]. The proofs in [15] and [17] use the Lambert series representation for z given by Eq. (4.1) instead of the double series in Eq. (1.1). H.H. Chan and Y.L. Ong [12] pointed out that (4.2) is equivalent to

$$q\frac{dx}{dq} = z^2x(1-x).$$

Equation (4.3) was known to Ramanujan. It is a combination of [19, Chapter 21, Entry 5(i)] and (4.1). For proofs, see Berndt [5, pp. 467–471], Chan and Ong [12, Lemma 2.2] and Liu [17, Eq. (1.16)]. □

It will be convenient to use Ramanujan's Eisenstein series Q and R defined by

$$Q(q) = 240E_4(q) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n},$$

$$R(q) = -504E_6(q) = 1 - 504\sum_{n=1}^{\infty}\frac{n^5q^n}{1-q^n}. \tag{4.4}$$

Lemma 4.2.

$$\begin{aligned} Q(q) &= k^{4/3}\left(1+245h+2401h^2\right)\left(1+13h+49h^2\right)^{1/3}, \\ Q\left(q^7\right) &= k^{4/3}\left(1+5h+h^2\right)\left(1+13h+49h^2\right)^{1/3}, \\ R(q) &= k^2\left(1-490h-21609h^2-235298h^3-823543h^4\right), \\ R\left(q^7\right) &= k^2\left(1+14h+63h^2+70h^3-7h^4\right). \end{aligned}$$

Proof. These results were given by Ramanujan [20, p. 53]. Proofs have been given by Berndt et al. [8], Cooper and Toh [15], Liu [17] and S. Raghavan and S.S. Rangachari [18]. □

Lemma 4.3.

$$Q(q) = z^4\frac{1+245h+2401h^2}{1+13h+49h^2}, \tag{4.5}$$

$$Q\left(q^7\right) = z^4\frac{1+5h+h^2}{1+13h+49h^2}, \tag{4.6}$$

$$R(q) = z^6\frac{1-490h-21609h^2-235298h^3-823543h^4}{(1+13h+49h^2)^2}, \tag{4.7}$$

$$R\left(q^7\right) = z^6\frac{1+14h+63h^2+70h^3-7h^4}{(1+13h+49h^2)^2}. \tag{4.8}$$

Proof. Use (4.3) to eliminate k from the results in Lemma 4.2. □

Lemma 4.4. *Let $E_{2n}(q)$ be as in* (1.9)*. Then for $n \geqslant 2$ we have*

$$E_{2n}(q) = \sum_{\substack{2\alpha+3\beta=n \\ \alpha,\beta\geqslant 0}} c_{\alpha,\beta}\left(Q(q)\right)^{\alpha}\left(R(q)\right)^{\beta},$$

where $c_{\alpha,\beta}$ are positive rational numbers.

Proof. This is a well-known result, for example, see [4, Theorem 1.13 and Ex. 1.11] or [21, p. 141]. □

Theorem 4.5. *Let z, h, and $E_{2n}^{\infty}(q;\chi_7)$ be as defined in* (1.1) *and* (1.10)*. Let $n \geqslant 2$ be an integer, and let $k = \lfloor 2n/3 \rfloor$. Then there exists a polynomial $p_{2k-1}(h)$ of degree $2k-1$ such that*

$$E_{2n}^{\infty}(q;\chi_7) = z^{2n}\frac{p_{2k-1}(h)}{(49h^2+13h+1)^k}.$$

Furthermore, we have

$$E_2^{\infty}(q;\chi_7) = z^2\frac{2+13h}{49h^2+13h+1} + 3zh\frac{dz}{dh}.$$

Proof. Let us first prove the result for $n \geqslant 2$. We have

$$\begin{aligned} E_{2n}^{\infty}(q;\chi_7) &= E_{2n}(q) - 7^{2n}E_{2n}\big(q^7\big) \\ &= \sum_{\substack{2\alpha+3\beta=n \\ \alpha,\,\beta\geqslant 0}} c_{\alpha,\beta}\Big(\big(Q(q)\big)^{\alpha}\big(R(q)\big)^{\beta} - 7^{2n}\big(Q\big(q^7\big)\big)^{\alpha}\big(R\big(q^7\big)\big)^{\beta}\Big) \\ &= \sum_{\substack{2\alpha+3\beta=n \\ \alpha,\,\beta\geqslant 0}} c_{\alpha,\beta} z^{4\alpha+6\beta}\frac{f_1^{\alpha}f_2^{\beta} - 7^{2n}f_3^{\alpha}f_4^{\beta}}{(49h^2+13h+1)^{\alpha+2\beta}}, \end{aligned}$$

where

$$\begin{aligned} f_1 &= 2401h^2 + 245h + 1, \\ f_2 &= -\big(823543h^4 + 235298h^3 + 21609h^2 + 490h - 1\big), \\ f_3 &= h^2 + 5h + 1, \\ f_4 &= -\big(7h^4 - 70h^3 - 63h^2 - 14h - 1\big). \end{aligned}$$

The highest power of h occurring in $f_1^{\alpha}f_2^{\beta} - 7^{2n}f_3^{\alpha}f_4^{\beta}$ is $\leqslant 2\alpha + 4\beta - 1$, because the coefficient of the highest possible power $h^{2\alpha+4\beta}$ is

$$(2401)^{\alpha}(-823543)^{\beta} - 7^{2n}(-7)^{\beta} = (-7)^{\beta}\big(7^{4\alpha+6\beta} - 7^{2n}\big) = 0.$$

Therefore

$$\begin{aligned} E_{2n}^{\infty}(q;\chi_7) &= z^{2n}\sum_{\substack{2\alpha+3\beta=n \\ \alpha,\,\beta\geqslant 0}} c_{\alpha,\beta}\frac{p_{2\alpha+4\beta-1}(h)}{(49h^2+13h+1)^{\alpha+2\beta}} \\ &= z^{2n}\frac{p_{2\lfloor 2n/3\rfloor-1}(h)}{(49h^2+13h+1)^{\lfloor 2n/3\rfloor}}, \end{aligned}$$

because $n/2 \leqslant \alpha + 2\beta \leqslant 2n/3$. This proves the result for $E_{2n}^{\infty}(q;\chi_7)$ for $n \geqslant 2$.

Next, by (1.13) and (4.2), we have

$$E_2^{\infty}(q;\chi_7) = \frac{7}{4}z^2 - 6E_2(q). \tag{4.9}$$

By [12, Eqs. (2.23) and (2.27)], this becomes

$$\begin{aligned} E_2^{\infty}(q;\chi_7) &= \frac{7}{4}z^2 + \frac{z^2}{4}\frac{(7-53x-3x^2)}{(7-x+x^2)} + \frac{3}{z}q\frac{dz}{dq} \\ &= z^2\frac{(14-15x+x^2)}{(7-x+x^2)} + 3zx(1-x)\frac{dz}{dx}. \end{aligned}$$

Now make the change of variable $x = 7h/(1+7h)$ to obtain

$$E_2^{\infty}(q;\chi_7) = z^2 \frac{2+13h}{49h^2+13h+1} + 3zh\frac{dz}{dh}. \qquad \square$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We use induction on n. The result is known to be true for $E_1^{\infty}(q;\chi_7)$ and $E_3^{\infty}(q;\chi_7)$ by (1.2) and (1.5). Assume the result is true for $E_1^{\infty}(q;\chi_7)$, $E_3^{\infty}(q;\chi_7)$, $\ldots$, $E_{2n+3}^{\infty}(q;\chi_7)$ and consider $E_{2n+5}^{\infty}(q;\chi_7)$. If we use the expansions (2.1) and (2.3) in Theorem 3.1 and compare coefficients of u^{2n}, we find that $E_{2n+5}^{\infty}(q;\chi_7)$ is a linear combination of

$$\begin{gathered}
E_{2a}^{\infty}E_{2b+1}^{\infty}, \quad a+b=n+2,\ a\geqslant 2;\\
E_{2a+1}^{\infty}E_{2b+1}^{\infty}E_{2c+1}^{\infty}, \quad a+b+c=n+1;\\
E_{2a}^{\infty}E_{2b}^{\infty}E_{2c+1}^{\infty}, \quad a+b+c=n+2,\ a,b\geqslant 2;\\
E_{2a}^{\infty}E_{2b+1}^{\infty}E_{2c+1}^{\infty}E_{2d+1}^{\infty}, \quad a+b+c+d=n+1,\ a\geqslant 2;\\
E_{2a+1}^{\infty}E_{2b+1}^{\infty}E_{2c+1}^{\infty}E_{2d+1}^{\infty}E_{2e+1}^{\infty}, \quad a+b+c+d+e=n;\\
E_4(q)E_{2n+1}^{\infty};\\
\bigl(E_2^{\infty}+6E_2(q)\bigr)E_{2n+3}^{\infty};\\
\bigl(E_2^{\infty}+6E_2(q)\bigr)E_{2a}^{\infty}E_{2b+1}^{\infty}, \quad a+b=n+1,\ a\geqslant 2;\\
\bigl(E_2^{\infty}+6E_2(q)\bigr)^2E_{2n+1}^{\infty};\\
\bigl(E_2^{\infty}+6E_2(q)\bigr)E_{2a+1}^{\infty}E_{2b+1}^{\infty}E_{2c+1}^{\infty}, \quad a+b+c=n,
\end{gathered}$$

where we have written E_k^{∞} for $E_k^{\infty}(q;\chi_7)$, and a, b, c, d and e are non-negative integers which satisfy the given conditions. Let us consider the terms in the first line. By the induction hypothesis and Theorem 4.5 we have, for $b\geqslant 1$,

$$\begin{aligned}
E_{2a}^{\infty}E_{2b+1}^{\infty} &= z^{2a}\frac{p_{2\lfloor 2a/3\rfloor-1}(t)}{(49t^2+13t+1)^{\lfloor 2a/3\rfloor}} \times z^{2b+1}\frac{p_{2\lfloor (2b+1)/3\rfloor-1}(t)}{(49t^2+13t+1)^{\lfloor (2b+1)/3\rfloor}}\\
&= z^{2n+5}\frac{p_{2\lfloor (2n+5)/3\rfloor-2}(t)}{(49t^2+13t+1)^{\lfloor (2n+5)/3\rfloor}},
\end{aligned}$$

because $\lfloor j/3\rfloor + \lfloor k/3\rfloor \leqslant \lfloor (j+k)/3\rfloor$ for all non-negative integers j, k. If $b=0$, then

$$\begin{aligned}
E_{2n+4}^{\infty}E_1^{\infty} &= z^{2n+4}\frac{p_{2\lfloor (2n+4)/3\rfloor-1}(t)}{(49t^2+13t+1)^{\lfloor (2n+4)/3\rfloor}} \times \frac{z}{2}\\
&= z^{2n+5}\frac{p_{2\lfloor (2n+5)/3\rfloor-1}(t)}{(49t^2+13t+1)^{\lfloor (2n+5)/3\rfloor}}.
\end{aligned}$$

The other terms may be analyzed similarly. For the term involving $E_4(q)$, use (4.5). For the terms involving $E_2^\infty + 6E_2(q)$, use (4.9) to write

$$E_2^\infty + 6E_2(q) = \frac{7}{4}z^2.$$

In summary, all of the terms can be shown to have the form

$$z^{2n+5}\frac{p_{2\lfloor(2n+5)/3\rfloor-1}(t)}{(49t^2+13t+1)^{\lfloor(2n+5)/3\rfloor}}$$

and therefore $E_{2n+5}^\infty(q;\chi_7)$ also has this form. This completes the proof of Theorem 1.1. □

5. The imaginary transformation $\tau \to -1/(7\tau)$, and proof of Theorem 1.2

Let $\mathrm{Im}(\tau) > 0$, and recall that Dedekind's eta-function is defined by

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{j=1}^{\infty}\left(1 - e^{2\pi ij\tau}\right).$$

Let

$$q = \exp(2\pi i\tau) \quad \text{and} \quad p = \exp(-2\pi i/7\tau).$$

In this section we will find a relation between $E_{2n+1}^\infty(q;\chi_7)$ and $E_{2n+1}^0(p;\chi_7)$, and use it to prove Theorem 2. We begin with some lemmas.

Lemma 5.1.

$$\eta(\tau) = \sqrt{\frac{i}{\tau}}\,\eta\left(-\frac{1}{\tau}\right),$$

$$h(q)h(p) = \frac{1}{49},$$

and

$$z(q) = \left(\frac{i}{\tau\sqrt{7}}\right)z(p).$$

Proof. The first result is standard, for example, see [4, p. 48]. For the second result, using (1.1), we have

$$h(q) = \frac{\eta^4(7\tau)}{\eta^4(\tau)} = \left(\frac{i}{7\tau}\right)^2\left(\frac{\tau}{i}\right)^2\frac{\eta^4(-1/7\tau)}{\eta^4(-1/\tau)} = \frac{1}{49h(p)}.$$

For the third result, first observe that

$$
\begin{aligned}
h(q)k(q) = \eta^3(\tau)\eta^3(7\tau) &= \left(\frac{i}{\tau\sqrt{7}}\right)^3 \eta^3\left(-\frac{1}{\tau}\right)\eta^3\left(\frac{-1}{7\tau}\right) \\
&= \left(\frac{i}{\tau\sqrt{7}}\right)^3 h(p)k(p).
\end{aligned}
$$

Then by (4.3), we have

$$
\begin{aligned}
z^3(q) &= h(q)k(q)\left(49h(q) + 13 + \frac{1}{h(q)}\right) \\
&= \left(\frac{i}{\tau\sqrt{7}}\right)^3 k(p)h(p)\left(\frac{1}{h(p)} + 13 + 49h(p)\right) \\
&= \left(\frac{i}{\tau\sqrt{7}}\right)^3 z^3(p).
\end{aligned}
$$

The result now follows by taking cube roots. The branch may be determined by using the fact that $q = p = \exp(-2\pi/\sqrt{7})$ when $\tau = i/\sqrt{7}$. □

Lemma 5.2.

$$z(q) = 2E_1^\infty(q;\chi_7) = 2E_1^0(q;\chi_7).$$

Proof. The first result is just (4.1). For the second result, we have

$$
\begin{aligned}
E_1^\infty(q;\chi_7) &= \frac{1}{2} + \sum_{j=1}^\infty \left(\frac{j}{7}\right)\frac{q^j}{1-q^j} \\
&= \frac{1}{2} + \sum_{j=1}^\infty \sum_{k=1}^\infty \left(\frac{j}{7}\right) q^{jk}.
\end{aligned}
$$

Interchange the order of summation to obtain

$$E_1^\infty(q;\chi_7) = \frac{1}{2} + \sum_{k=1}^\infty \frac{q^k + q^{2k} - q^{3k} + q^{4k} - q^{5k} - q^{6k}}{1-q^{7k}} = E_1^0(q;\chi_7). \qquad \square$$

Lemma 5.3.

$$L(u|\tau) = \frac{i}{\tau\sqrt{7}}\,\ell\left(\frac{u}{\tau}\middle|\frac{-1}{7\tau}\right).$$

Proof. By (2.7) and (2.8), we find that $L(u|\tau)$ and $\frac{i}{\tau\sqrt{7}}\ell(\frac{u}{\tau}|\frac{-1}{7\tau})$ both have simple poles at $u = \pi j/7 + \pi\tau k$, where j and k are integers and j is not a multiple of 7, and no other singularities. The residues of each function coincide, and the functions have the same periods. The difference is therefore an entire elliptic function, and by Liouville's theorem must be a constant. The value of the constant may be found by setting $u = 0$:

$$\begin{aligned}L(0|\tau) - \frac{i}{\tau\sqrt{7}}\ell\left(0\left|\frac{-1}{7\tau}\right.\right) &= E_1^\infty(q;\chi_7) - \frac{i}{\tau\sqrt{7}}E_1^0(p;\chi_7)\\ &= \frac{z(q)}{2} - \frac{i}{\tau\sqrt{7}}\frac{z(p)}{2}\\ &= 0,\end{aligned}$$

by Lemmas 5.1 and 5.2. This completes the proof. □

Corollary 5.4. *Let $q = \exp(2\pi i\tau)$ and $p = \exp(-2\pi i/7\tau)$. For any non-negative integer n we have*

$$E_{2n+1}^\infty(q;\chi_7) = \frac{i}{\tau^{2n+1}\sqrt{7}}E_{2n+1}^0(p;\chi_7).$$

Proof. Substitute the series expansions (2.1) and (2.2) into Lemma 5.3, and compare coefficients of u^{2n}. □

Proof of Theorem 1.2. Let us write $h = h(q)$, $z = z(q)$, $h_1 = h(p)$ and $z_1 = z(p)$. Replace τ with $-1/7\tau$ in Corollary 5.4, then apply Theorem 1.1 to get

$$\begin{aligned}E_{2n+1}^0(q;\chi_7) &= \frac{i\sqrt{7}}{(7\tau)^{2n+1}}E_{2n+1}^\infty(p;\chi_7)\\ &= \frac{i\sqrt{7}}{(7\tau)^{2n+1}}z_1^{2n+1}\frac{p_{2\lfloor(2n+1)/3\rfloor-1}(h_1)}{(49h_1^2+13h_1+1)^{\lfloor(2n+1)/3\rfloor}}.\end{aligned}$$

Now apply Lemma 5.1 to obtain

$$\begin{aligned}E_{2n+1}^0(q;\chi_7) &= \frac{(-1)^n}{7^n}z^{2n+1}\frac{p_{2\lfloor(2n+1)/3\rfloor-1}(1/49h)}{(1/49h^2+13/49h+1)^{\lfloor(2n+1)/3\rfloor}}\\ &= \frac{(-1)^n}{7^n}z^{2n+1}\frac{(49h^2)^{\lfloor(2n+1)/3\rfloor}p_{2\lfloor(2n+1)/3\rfloor-1}(1/49h)}{(49h^2+13h+1)^{\lfloor(2n+1)/3\rfloor}}.\end{aligned} \tag{5.1}$$

If we observe that $(49h^2)^j p_{2j-1}(1/49h)$ is a polynomial in h of degree $\leqslant 2j$ with no constant term, we complete the proof of Theorem 1.2. □

Equation (5.1) may be used to obtain values for $E_{2n+1}^0(q;\chi_7)$ from the corresponding values for $E_{2n+1}^\infty(q;\chi_7)$. For example, from (1.2), (1.5)–(1.7), we obtain

$$\begin{aligned}E_1^0(q;\chi_7) &= \frac{z}{2},\\ E_3^0(q;\chi_7) &= z^3\frac{8h^2+h}{49h^2+13h+1},\\ E_5^0(q;\chi_7) &= z^5\frac{16h^2+h}{49h^2+13h+1},\end{aligned}$$

$$E_7^0(q;\chi_7)=z^7\frac{4088h^4+1001h^3+73h^2+h}{(49h^2+13h+1)^2}.$$

By (4.3), the result for $E_3^0(q;\chi_7)$ is equivalent to

$$E_3^0(q;\chi_7)=k\big(8h^2+h\big).$$

This was given by Ramanujan [20, p. 53].

6. Concluding remarks

Remark 6.1.

For $n\geqslant 1$, define

$$F_{2n+1}^{\infty}(\tau|\chi_7)=\sum_j\sum_k\frac{(\frac{j}{7})}{(j+k\tau)^{2n+1}},$$

$$F_{2n+1}^{0}(\tau|\chi_7)=\sum_j\sum_k\frac{(\frac{k}{7})}{(j+k\tau)^{2n+1}},$$

where the summation is over all integers j and k except $(j,k)=(0,0)$. Clearly

$$F_{2n+1}^{0}\left(\frac{-1}{7\tau}\middle|\chi_7\right)=-(7\tau)^{2n+1}F_{2n+1}^{\infty}(7\tau|\chi_7).$$

By the techniques in [13, Section 5] it can be shown that

$$F_{2n+1}^{\infty}(\tau|\chi_7)=\frac{2(-1)^n(2\pi)^{2n+1}}{(2n)!7^{2n+\frac{1}{2}}}E_{2n+1}^{\infty}\big(q^{\frac{1}{7}};\chi_7\big),$$

$$F_{2n+1}^{0}(\tau|\chi_7)=\frac{2(-2\pi i)^{2n+1}}{(2n)!}E_{2n+1}^{0}(q;\chi_7).$$

This leads to another proof of Corollary 5.4 in the case $n\geqslant 1$.

Remark 6.2. The polynomials in the numerators of the right-hand sides of Theorems 1.1 and 1.2 can be computed by equating coefficients in the q-expansions of both sides. For example, the case $n=4$ of Theorem 1.1 implies

$$\left(282592+7\sum_{j=1}^{\infty}\left(\frac{j}{7}\right)\frac{j^8q^j}{1-q^j}\right)\big(49h^2+13h+1\big)^3$$
$$=z^9\big(c_0+c_1h+c_2h^2+c_3h^3+c_4h^4+c_5h^5\big),$$

for some constants c_0, c_1, c_2, c_3, c_4 and c_5. If we equate coefficients of q^k on both sides for $0 \leqslant k \leqslant 5$, then we obtain a triangular system of linear equations, from which we successively obtain

$$\begin{aligned} c_0 &= 282592, \\ c_1 &= 5934439, \\ c_2 &= 47477374, \\ c_3 &= 168120421, \\ c_4 &= 225650782, \\ c_5 &= 40353607. \end{aligned}$$

The calculations are easily performed using computer algebra.

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