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THE APÉRY NUMBERS, THE ALMKVIST-ZUDILIN NUMBERS AND NEW SERIES FOR $1/\pi$

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Abstract. This paper concerns series for $1/\pi$, such as Sato's series (1.4), and the series of H.H. Chan, S.H. Chan and Z.-G. Liu (1.8) below. The examples of Sato, Chan, Chan and Liu are related to two index 2 subgroups of $\Gamma_0(6)_+$. These examples motivate us to look at a third subgroup of $\Gamma_0(6)_+$. We give a new method of constructing such series using the theory of modular forms and conclude our work with several new examples.

1. Introduction

For $n \geq 1$, define

$$\alpha_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}^2. \tag{1.1}$$

These numbers are known as Apéry's numbers, because of their appearance in Apéry's proof of the irrationality of $\zeta(3)$ [2]. The first few terms are $1, 5, 73, 1445$ (this is sequence A005259 in [18]). Let $q = e^{2\pi i\tau}$ and

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

be the Dedekind eta function. From [14, Theorem 5], it follows that if

$$t_1(\tau) = \left(\frac{\eta(6\tau)\eta(\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12} \quad \text{and} \quad F_1(\tau) = \frac{(\eta(2\tau)\eta(3\tau))^7}{(\eta(\tau)\eta(6\tau))^5}, \tag{1.2}$$

then provided $|t_1(\tau)|$ is sufficiently small,

$$F_1(\tau) = \sum_{n=0}^{\infty} \alpha_n t_1^n(\tau). \tag{1.3}$$

T. Sato [16] used (1.3) to obtain the following series for $1/\pi$:

$$\frac{1}{\pi} = \left(72\sqrt{15} - 160\sqrt{3}\right) \sum_{n=0}^{\infty} \alpha_n \left(\frac{1}{2} - \frac{3\sqrt{5}}{20} + n\right) \left(\frac{1-\sqrt{5}}{2}\right)^{12n}. \tag{1.4}$$

Sato's work motivated H.H. Chan, S.H. Chan and Z.-G. Liu [7] to consider

$$t_2(\tau) = \left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^{6} \quad \text{and} \quad F_2(\tau) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2}. \tag{1.5}$$

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It turns out that when $|t_2(\tau)|$ is sufficiently small, we have the following analogue of (1.3):

$$F_2(\tau) = \sum_{n=0}^{\infty} \beta_n t_2^n(\tau) \tag{1.6}$$

where $\beta_n = (-1)^n \tilde{\beta}_n$, where

$$\tilde{\beta}_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2(n-j)}{n-j} = \sum_{p+q+r+s=n} \binom{n}{p,q,r,s}^2, \tag{1.7}$$

where

$$\binom{n}{a_1, a_2, \cdots, a_r} = \frac{n!}{a_1! a_2! \cdots a_r!}, \text{with } a_1 + a_2 + \cdots a_r = n.$$

Note that the second equality here follows from writing

$$\sum_{p+q+r+s=n} \binom{n}{p,q,r,s}^2 = \sum_{k+j=n} \binom{n}{k,j}^2 \sum_{p+q=k} \binom{k}{p,q}^2 \sum_{r+s=j} \binom{j}{r,s}^2$$

and using the identity $\sum_{i=1}^{s} \binom{s}{i}^2 = \binom{2s}{s}$. The first few values of $\tilde{\beta}_n$ are $1, 4, 28, 256$; this is sequence A002895 in [18].

An example (given in [7]) of a series for $1/\pi$ in terms of the $\tilde{\beta}_n$ is

$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} \tilde{\beta}_n (1+5n) \left(\frac{1}{64}\right)^n. \tag{1.8}$$

The functions $t_1(\tau)$ and $t_2(\tau)$ are Hauptmoduls of $\Gamma_0(6)_{+6}$ and $\Gamma_0(6)_{+3}$ respectively, which are two of the three index 2 subgroups lying between $\Gamma_0(6)$ and its normalizer in $\mathrm{SL}_2(\mathbf{R})$. These are given by $\Gamma_0(6)_{+k} = \langle \Gamma_0(6), w_k \rangle$, where $k = 2, 3, 6$, and $w_2 = \frac{1}{\sqrt{2}} \left(\begin{smallmatrix} 2 & -1 \\ 6 & -2 \end{smallmatrix}\right)$, $w_3 = \frac{1}{\sqrt{3}} \left(\begin{smallmatrix} 3 & -2 \\ 6 & -3 \end{smallmatrix}\right)$, $w_6 = \frac{1}{\sqrt{6}} \left(\begin{smallmatrix} 0 & -1 \\ 6 & 0 \end{smallmatrix}\right)$.

It is natural to ask whether there exist analogues of (1.3) and (1.6) for the third such group, $\Gamma_0(6)_{+2}$. We will show that the answer is affirmative. In fact, when

$$t_3(\tau) = \left(\frac{\eta(3\tau)\eta(6\tau)}{\eta(\tau)\eta(2\tau)}\right)^4, \quad \text{and} \quad F_3(\tau) = \frac{(\eta(\tau)\eta(2\tau))^3}{\eta(3\tau)\eta(6\tau)},$$

and $|t_3(\tau)|$ sufficiently small, we show that

$$F_3(\tau) = \sum_{n=0}^{\infty} \gamma_n t_3^n(\tau),$$

where γ_n are the Almkvist-Zudilin numbers [1] given by

$$\gamma_n = \sum_{j=0}^{n} (-1)^{n-j} \frac{3^{n-3j}(3j)!}{(j!)^3} \binom{n}{3j} \binom{n+j}{j}. \tag{1.9}$$

The first few terms of this sequence are $1, -3, 9, -3, -279, 2997$. In this paper, we prove that we have the following analogues of (1.4) and (1.8):

$$\frac{3\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} \gamma_n (4n+1) \left(\frac{1}{81}\right)^n, \tag{1.10}$$

$$\frac{3\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \gamma_n (4n+1) \left(\frac{-1}{27}\right)^n . \tag{1.11}$$

Experimentally, we find further series such as

$$\frac{3\sqrt{3}}{7 \cdot 2^3 \pi}\left(5+2\sqrt{6}\right) = \sum_{n=0}^{\infty} \gamma_n \left(2n+1-\frac{2\sqrt{6}}{7}\right)\left(\frac{49-20\sqrt{6}}{9}\right)^n . \tag{1.12}$$

In this article, in the course of proving (1.10) and (1.11), we give a general strategy for finding such series (Theorem 2.1). A number of other series are derived experimentally, and they are given in Tables 1, 2 and 3.

1.1. Comparison with earlier works. Our methods are similar to those of [7]. Our series also have the form

$$\frac{1}{\pi} = \sum a_n (An+B) t_0^n, \tag{1.13}$$

where A, B are constants, and t_0 is the value of a Hauptmodul t for some genus zero subgroup of $\mathrm{SL}_2(\mathbf{R})$ at some point z_0 in the upperhalf complex plane. Our t plays the same role as X in [7, (2.7)], and $e^{-2\pi\sqrt{N/s}}$ of [7, (2.12)] is now replaced by $-\exp(2\pi i z_0)$, i.e., setting z_0 of this paper to $z_0 = i\sqrt{N/s}+1/2$, for integers N and s, will lead towards the formulas of [7, (2.12)]. However, in this paper, z_0 does not have to have this form. In theory, z_0 just has to be in an imaginary quadratic extension of $\mathbf{Q}$, though in practice, to actually be able to compute the values of $t(z_0)$, the possible values of z_0 are still restricted. The series (1.10), proved in Theorem 3.14 corresponds to $z_0 = i/\sqrt{2}$, and (1.11), proved in Theorem 3.15 corresponds to $z_0 = (i+1)/2$ (see Lemma 3.12). In the experimental results in Tables 1, 2 and 3, we take $z_0 = i\sqrt{\ell/6}$, for various integers ℓ.

Another difference of this work compared with [7] is that in the earlier work, the constants A and B in (1.13) were given in terms of the derivative of a modular form $f(\tau)$ with respect to the variable q, where $q = \exp(2\pi i \tau)$. In this paper, the model for our formulas is (2.17), where the derivative is now with respect to the variable t. In this paper, the ingredients of the main result, Theorem 2.1, are a Hauptmodul t, a modular form of weight 2, F, and an order 2 element g of $\mathrm{SL}_2(\mathbf{R})$. In order to apply the theorem to obtain explicit formulas for $1/\pi$, we need to be able to determine the value of $t(z_0)$, where z_0 is the fixed point of g, and we need to be able to express the weight 0 modular function $R = F|_{(2,g)}/(tF)$ as an algebraic function of t, where $F|_{(2,g)}$ is given by the usual action of $\mathrm{SL}_2(\mathbf{R})$ on modular forms (2.1). This determines the value of the term p_g in (3.12), (4.2), (5.1). We also need to express $\Theta t/F$ as an algebraic function of t, where $\Theta = q\frac{d}{dq}$. Thus, for fixed t and F, once t_0 and R are determined, in order to determine a series for $1/\pi$ corresponding to a choice of g reduces to determining the value of $t(z_0)$ and $F|_{(2,g)}/F$ as an algebraic function of t, e.g., as in Lemma 3.9. An empirical process to find p_g can be obtained from [4].

In the examples in Tables 1, 2 and 3, one can observe that in most cases t_0 is a unit. This phenomenon is explained in [8, Cor. 5.2, page 87].

2. A strategy for finding series for $1/\pi$ using the theory of modular forms

2.1. Modular forms.

Let Γ be a subgroup of $\mathrm{SL}_2(\mathbf{R})$, commensurable with $\mathrm{SL}_2(\mathbf{Z})$, and χ a character of Γ of finite order, namely, a group homomorphism

$$\chi : \Gamma \to \mathbf{C}^*$$

with finite image. A modular form of weight k, character χ for $\Gamma \subset \mathrm{SL}_2(\mathbf{R})$ is a holomorphic function $F(z)$ on the upper half complex plane $\mathfrak{h}$ satisfying the following conditions:

(a) For every $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbf{R})$,

$$F|_{(k,g)}(z) = \chi(g)F(z),$$

where

$$F|_{(k,g)}(z) = \det(g)^{k/2} j(g,z)^{-k} F(gz), \tag{2.1}$$

with $j(g,z) = cz + d$ and

$$gz := \frac{az+b}{cz+d}.$$

For the rest of this article we always assume $\det(g) = 1$.

(b) $F(z)$ is holomorphic at cusps.

The space of all such modular forms is denoted by $\mathcal{M}_k(\Gamma,\chi)$, and χ is omitted when it is the trivial character. We write $\mathcal{A}_k(\Gamma,\chi)$ to denote the space of meromorphic forms satisfying

$$F|_{(k,g)}(z) = \chi(g)F(z).$$

See for example [12, §2.1] or [17, Chapter 2] for complete definitions.

2.2. Differentiation and the appearance of π. In general, the derivative of a modular form is not a modular form [22]. Differentiating (2.1) with respect to z, and assuming $\det(g) = 1$, gives

$$\tfrac{d}{dz}(F|_{(k,g)}(z)) \quad = \quad \frac{1}{j(g,z)^{k+2}}\tfrac{d}{dz}F(gz) - \frac{ck}{j(g,z)^{k+1}}F(gz)$$

i.e., writing $c = c_g$, $f_z = \frac{df}{dz}$, and dividing through by $F|_{k,g}$, we have

$$\left(F|_{(k,g)}\right)_z \quad = \quad (F_z)|_{(k+2,g)} - \frac{c_g k}{j(g,z)}F|_{(k,g)}. \tag{2.2}$$

This shows that if $F|_{(0,g)} = F$, then $F_z = (F_z)|_{(2,g)}$, i.e., the derivative of a weight $k = 0$ function has weight 2. We will be interested in the case $k = 2$.

Provided that $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$, and $\chi(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = 1$, then a modular form $F(z) \in \mathcal{M}_k(\Gamma,\chi)$ can be written as a function of $q(z) = \exp(2\pi i z)$, and we define $\widetilde{F}(q)$ to be a function such that $F(z) = \widetilde{F}(q(z))$. The condition that F be holomorphic at cusps means that $\widetilde{F}(q)$ has a Taylor series expansion

$$\widetilde{F}(q) = \sum_{n\geq 0} a_n q^n$$

for some constants a_n. This function is called the q-expansion of F. When $q\frac{d}{dq}$ is applied to a Taylor series in q with integral coefficients, the resulting series also has integral coefficients. This is our motivation for setting

$$\Theta = \frac{1}{2\pi i}\frac{d}{dz},$$

so that $\Theta F(z) = q(z)\frac{d}{dq}\widetilde{F}(q(z))$. In terms of Θ, (2.2) becomes

$$\Theta\big(F|_{(k,g)}\big) \quad = \quad (\Theta F)|_{(k+2,g)} - \frac{c_g k}{j(g,z)2\pi i} F|_{(k,g)}. \tag{2.3}$$

This equation will lead to a series for $1/\pi$.

2.3. Genus zero subgroups of $SL_2(\mathbf{R})$ commensurable with $SL_2(\mathbf{Z})$.

We denote by $\mathfrak{h}$ the upper-half complex plane, and $\mathfrak{h}^* = \mathfrak{h} \cup \mathbf{Q} \cup \{\infty\}$. If the Riemann surface $X(\Gamma) := \Gamma \setminus \mathfrak{h}^*$ has *genus zero* then there is a function t, known as a Hauptmodul, satisfying $\mathcal{A}_0(\Gamma) = \mathbf{C}(t)$. In other words, any element of $\mathcal{A}_0(\Gamma)$ can be written as a rational function in t. In this case we say that Γ has genus 0. For the remainder of this article we will always assume Γ has genus zero and that t is a Hauptmodul associated with Γ.

2.4. Power series for $F(t)$.

If $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right) \in \Gamma$ and $\chi(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)) = 1$, so that $F \in \mathcal{M}_k(\Gamma,\chi)$ and $t \in \mathcal{A}_0(\Gamma,\chi)$ both have q-expansions. If we also have $t(i\infty) = 0$, then by a series inversion, for $\mathrm{Im}(z)$ sufficiently large, we have $F(z) = \widehat{F}(t(z))$, where for some sequence γ_n,

$$\widehat{F}(t) = \sum_{n=0}^{\infty} \gamma_n t^n. \tag{2.4}$$

The γ_n satisfy a recurrence relation, which follows from the existence of differential equations solved in terms of modular forms [19], [21].

2.5. Conjugation of subgroups of $\mathrm{SL}_2(\mathbf{R})$.

Let $g, h \in \mathrm{SL}_2(\mathbf{R})$. Since the action defined by (2.1) is a group action, we have

$$F|_{(k,g)}|_{(k,g^{-1}hg)} = F|_{(k,h)}|_{(k,g)}.$$

If $F|_{(k,h)} = \chi(h)F$, then

$$F|_{(k,g)}|_{(k,h^g)} = \chi(h)F|_{(k,g)},$$

where $h^g = g^{-1}hg$. This shows that if $F \in \mathcal{M}_k(\Gamma,\chi)$, then

$$F|_{(k,g)} \in \mathcal{M}_k(g^{-1}\Gamma g, \chi^{g^{-1}}),$$

where

$$\chi^{g^{-1}}(h) = \chi(ghg^{-1}).$$

Hence

$$\frac{F|_{(k,g)}}{F} \in \mathcal{A}_0(\Gamma \cap \Gamma^g, \chi^{g^{-1}}\chi^{-1}), \tag{2.5}$$

where $\Gamma^g = g^{-1}\Gamma g$. We define F^g by

$$F^g := \frac{F|_{(k,g)}}{F}. \tag{2.6}$$

When Γ has genus 0, Hauptmodul t, if $\Gamma^g = \Gamma$ and $\chi^{g^{-1}} = \chi$, then F^g is a rational function of t, since in this case (2.5) says that $F^g \in \mathcal{A}_0(\Gamma)$. Provided that Γ and Γ^g are commensurable, $F^g(z)$ will be an algebraic function of t, since $F^g \in \mathcal{A}_0(H)$ where

$H \subset \Gamma \cap \Gamma^g$ is the kernel of $\chi^{g^{-1}}\chi^{-1}$, which is a subgroup of Γ of finite index. We will suppose that we are in a situation where there is some algebraic function P^g with

$$P^g(t(z)) = F^g(z). \tag{2.7}$$

Applying $\Theta \circ \ln$ to (2.6) gives

$$\frac{\Theta F^g}{F^g} = \frac{\Theta(F|_{(k,g)})}{F|_{(k,g)}} - \frac{\Theta F}{F}. \tag{2.8}$$

From (2.3) we have

$$\frac{\Theta\big(F|_{(k,g)}\big)}{F|_{(k,g)}} \quad = \quad \frac{(\Theta F)|_{(k+2,g)}}{F|_{(k,g)}} - \frac{c_g k}{2\pi i j(g,z)}. \tag{2.9}$$

Together with (2.8), this gives the following identity

$$\frac{c_g k}{2\pi i j(g,z)} = \left(\frac{\Theta F}{F}\right)\bigg|_{(2,g)} - \left(\frac{\Theta F}{F} + \frac{\Theta F^g}{F^g}\right). \tag{2.10}$$

Note that we have used the fact that $\frac{A|_{(k_1,g)}}{B|_{(k_2,g)}} = (\frac{A}{B})|_{(k_1-k_2,g)}$.

Take $\widehat{F}(t)$ as in (2.4). In a neighbourhood of $z = i\infty$ where $F(z) = \widehat{F}(t(z))$ we have

$$\frac{\Theta F(z)}{F(z)} = \frac{(\Theta t(z))}{F(z)}\left(\frac{d}{dt}\widehat{F}\right)(t(z)). \tag{2.11}$$

From (2.7) we have

$$\frac{\Theta F^g(z)}{F^g(z)} = \Theta t(z)\frac{\frac{d}{dt}P^g(t(z))}{P^g(t(z))} \tag{2.12}$$

substituting (2.12) and (2.11) into (2.10) gives

(2.13)
$$\frac{c_g k}{2\pi i j(g,z)} = \left(\frac{\Theta t(z)}{F(z)}\frac{d}{dt}\widehat{F}(t(z))\right)\bigg|_{(2,g)} - \frac{\Theta t(z)}{F(z)}\left(\frac{d}{dt}\widehat{F}(t(z)) + \frac{\frac{d}{dt}P^g(t(z))}{P^g(t(z))}F(z)\right).$$

2.6. The case $k = 2$.

From § 2.2, if $t \in \mathcal{A}_0(\Gamma)$ then $\Theta t \in \mathcal{A}_2(\Gamma)$, so if $F \in \mathcal{M}_2(\Gamma,\chi)$, then $\frac{\Theta t(z)}{t(z)F(z)} \in \mathcal{A}_0(\Gamma,\chi)$. Suppose that there is an algebraic function R with

$$\frac{\Theta t(z)}{t(z)F(z)} = R(t(z)). \tag{2.14}$$

Substituting (2.14) and (2.4) into (2.13), we obtain

$$\left(R(t(z))\sum_{n=0}^{\infty} n\gamma_n t(z)^n\right)\bigg|_{(2,g)} - R(t(z))\sum_{n=0}^{\infty}\left(n + \frac{t(z)\frac{d}{dt}P^g(t(z))}{P^g(t(z))}\right)\gamma_n t(z)^n$$
$$= \frac{c_g}{\pi i j(g,z)}. \tag{2.15}$$

Now let z_0 be a fixed point of g and set $t_0 = t(z_0)$. Then (2.15) becomes

$$\frac{-c_g}{\pi i j(g,z_0)} = R(t_0)\sum_{n=0}^{\infty}\left(n\left[1 - \frac{1}{j(g,z_0)^2}\right] + \frac{t_0\frac{d}{dt}P^g(t_0)}{P^g(t_0)}\right)\gamma_n t_0^n. \tag{2.16}$$

In the case that $g^2 = -I$, we have $j(g, z_0) = i$ and so we obtain

$$\frac{c_g}{\pi R(t_0)} = \sum_{n=0}^{\infty} \left(2n + \frac{t_0 \frac{d}{dt} P^g(t_0)}{P^g(t_0)} \right) \gamma_n t_0^n. \tag{2.17}$$

To summarize, we have now proved the following:

Theorem 2.1. *Let $t(z)$ be a Hauptmodul for some genus 0 subgroup $\Gamma \subset \mathrm{SL}_2(\mathbf{R})$ commensurable with $\mathrm{SL}_2(\mathbf{Z})$, and let $F \in \mathcal{M}_2(\Gamma')$, where Γ' is a finite index subgroup of Γ. Let $g \in \mathrm{SL}_2(\mathbf{R})$ with $g^2 = -I$. Let z_0 be a point in the upperhalf complex plane fixed by g. Let $t_0 = t(z_0)$. Suppose that $t(i\infty) = 0$. Let $\widehat{F}(t) = \sum_{n\geq 0} \gamma_n t^n$ be a function such that for some $M > 0$, provided $\mathrm{Im}(z) > M$, we have $\widehat{F}(t(z)) = F(z)$. Suppose that there are algebraic functions $R(t)$ and $P^g(t)$ so that $R(t(z)) = \frac{\Theta t(z)}{t(z)F(z)}$ and $P^g(t(z)) = \frac{F(z)|_{(2,g)}}{F(z)}$. Then provided $\mathrm{Im}(z_0) > M$, (2.17) holds.*

3. Series for $1/\pi$ arising from modular forms for $\Gamma_0(6)_{+2}$

We now take $\Gamma = \Gamma_0(6)_{+2}$, the group generated by $\Gamma_0(6)$ and w_2, where

$$\Gamma_0(6) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) | c \equiv 0 \mod 6 \right\}, \qquad w_2 = \tfrac{1}{\sqrt{2}} \left(\begin{smallmatrix} 2 & -1 \\ 6 & -2 \end{smallmatrix}\right).$$

We will apply Theorem 2.1 with $g = \begin{pmatrix} 0 & -\sqrt{\ell/6} \\ \sqrt{6/\ell} & 0 \end{pmatrix}$, $F = F_3$, $t = t_3$, where

$$\begin{aligned} t_3(\tau) &= \left(\frac{\eta(6\tau)\eta(3\tau)}{\eta(\tau)\eta(2\tau)} \right)^4 = q + 4q^2 + 18q^3 + 52q^4 + 159q^5 + \cdots \\ F_3(\tau) &= \frac{(\eta(\tau)\eta(2\tau))^3}{\eta(6\tau)\eta(3\tau)} = 1 - 3q - 3q^2 + 15q^3 - 3q^4 - 18q^5 + \cdots \end{aligned} \tag{3.1}$$

Lemma 3.1. *$F_3 \in \mathcal{M}_2(\Gamma_0(6)_{+2}, \chi)$, where $\chi(g) = -1$ if $g \in \Gamma_0(6)_{+2} \setminus \Gamma_0(6)$, and $\chi(g) = 1$ otherwise, and t_3 is a Hauptmodul for $\Gamma_0(6)_{+2}$.*

Proof. Since $\eta(\tau)$ is a modular form of weight 1/2 for $\mathrm{SL}_2(\mathbf{Z})$, $\eta(N\tau)$ is a modular form of weight 1/2 for $\Gamma_0(N)$, and so F_3 is a modular form of weight 2 for $\Gamma_0(6)$. Its transformation behavior under the action of w_2 can be determined by using Dedekind's functional equation [3, Theorem 3.4]

$$\eta\left(\frac{az+b}{cz+d}\right) = \eta(z)\sqrt{-i(cz+d)} \cdot \epsilon^{s(a,b,c,d)}, \tag{3.2}$$

where $\epsilon = \exp(2\pi i/24)$,

$$s(a,b,c,d) = \frac{a+d}{c} + 12 \sum_{r=1}^{c-1} \frac{r}{c} \left(\frac{-dr}{c} - \left[\frac{-dr}{c} \right] - \frac{1}{2} \right)$$

and a, b, c, d are integers with $ad - bc = 1$, and $c > 0$. The transformation properties of t_3 can similarly be checked; t_3 is also given in Conway and Norton's tables [9] as a Hauptmodul for $\Gamma_0(6)_{+2}$. □

Lemma 3.2. *If δ_n is a sequence such that in a neighborhood of $\tau = i\infty$,*

$$F_3(\tau) = \sum_{n=0}^{\infty} \delta_n t_3^n(\tau),$$

then δ_n satisfy a recurrence relation

$$n^3\delta_n + (2n-1)(7n^2 - 7n + 3)\delta_{n-1} + 81(n-1)^3\delta_{n-2} = 0. \tag{3.3}$$

Proof. First note that $F_3 t_3 = s^2 r$, where s and r are modular functions for $\Gamma_0(6)$ defined by

$$s(\tau) = \frac{\eta(3\tau)^6\eta(2\tau)}{\eta(\tau)^2\eta(6\tau)^3}, \qquad r(\tau) = \frac{\eta(\tau)^3\eta(6\tau)^9}{\eta(2\tau)^3\eta(3\tau)^9}. \tag{3.4}$$

Let $S(\mathfrak{r})$ be a series defined by

$$S(\mathfrak{r}) = \sum_{n=0}^{\infty} \zeta_n \mathfrak{r}^n \qquad \text{where} \qquad \zeta_n = \sum_{j=0}^{n} \binom{n}{j}^3.$$

Then by interpreting s as a differential form on a family of elliptic curves, (as in e.g., [14]) near $\tau = i\infty$ we have $s(\tau) = S(r(\tau))$ (see [20, Table 2]).

The sequence ζ_n is classical, (see e.g., Sloane's sequence A000172 [18] for several references) and $S(\mathfrak{r})$ satisfies the differential equation

$$(8\mathfrak{r}-1)(\mathfrak{r}+1)\Theta_{\mathfrak{r}}^2 S + \mathfrak{r}(16\mathfrak{r}+7)\Theta_{\mathfrak{r}} S + 2\mathfrak{r}(4\mathfrak{r}+1)S = 0, \tag{3.5}$$

where $\Theta_{\mathfrak{r}} = \mathfrak{r}\frac{d}{d\mathfrak{r}}$. We will now substitute r for $\mathfrak{r}$ and s for S. By replacing s by $r^{-1/2}u$, we find that $u = r^{1/2}s$ satisfies $\mathcal{E}u = 0$ where

$$\mathcal{E} = (8r-1)(r+1)\Theta_r^2 + (8r^2+1)\Theta_r + \tfrac{1}{4}(8r^2 + r - 1). \tag{3.6}$$

Note that $u^2 = t_3 F_3$. Since r is a Hauptmodul for $\Gamma_0(6)$ (taken from [9]), our Hauptmodul $t = t_3$ for $\Gamma_0(6)_{+2}$ is given by a degree 2 rational function in r, namely

$$t = \frac{r}{(r+1)(1-8r)}. \tag{3.7}$$

Make a substitution to write (3.6) in terms of t, which transforms $\mathcal{E}$ to

$$\Theta_t^2 + \left[2 - \frac{(r+1)(8r-1)(8r^2-1)}{(8r^2+1)^2}\right]\Theta_t - \frac{(r+1)(8r-1)(8r^2+r-1)}{4(8r^2+1)^2},$$

which can be written as

$$\Theta_t^2 + \left[2 - \frac{7t+1}{81t^2+14t+1}\right]\Theta_t - \frac{6t+1}{4(81t^2+14t+1)}. \tag{3.8}$$

A substitution shows that $K = t^{-1/2}u = \sqrt{F_3}$ satisfies $\mathcal{G}K = 0$ where

$$\mathcal{G} := \Theta_t^2 + t(14\Theta_t^2 + 7\Theta_t + \tfrac{3}{2}) + 81t^2(\Theta_t + \tfrac{1}{2})^2. \tag{3.9}$$

As explained in [1, Proposition 9], the symmetric square of $\mathcal{G}$ is

$$\mathcal{D} = \Theta_t^3 - t(2\Theta_t + 1)(7\Theta + 7\Theta + 3) + 81t^2(\Theta + 1)^3, \tag{3.10}$$

and so $K^2 = \sum_{n=0}^{\infty} \delta_n t^n$ satisfies $\mathcal{D}(K^2) = 0$. By the Frobenius method, this is equivalent to the δ_n satisfying the recurrence relation (3.3) [10, Chapter XVI, §16.11]. □

Lemma 3.3. *With δ_n as in Lemma 3.2, for $n \geq 0$ we have $\delta_n = \gamma_n$, where the Almkvist-Zudilin numbers γ_n are given by (1.9).*

Proof. By Lemma 3.2, the δ_n satisfy the recurrence (3.3). According to [1, (δ) p. 498], the $\sum_{n\geq 0} \gamma_n t^n$ satisfy the differential equation (3.10), which is equivalent to the γ_n satisfying (3.3). In more detail, the summand of γ_n has the simple form

$$c_{n,j} = (-3)^n \frac{(n+j)!}{j!^4(n-3j)!} \left(\frac{-1}{27}\right)^j,$$

and so Zeilberger's telescoping sum algorithm for finding recurrences, described in [15, Chapter 6], applies, and the book's accompanying Maple package *Ekhad*, computes the easily verifiable relation

$$\begin{aligned} 81\,(n+1)^3\, c_{n,j} + (2n+3)\left(7n^2+21n+17\right) c_{n+1,j} + (n+2)^3\, c_{n+2,j} \\ = G(n,j+1) - G(n,j), \end{aligned}$$

$$\text{where} \qquad G(n,j) = -324 \frac{(4n+5)\, j^4}{(n+2-3j)\,(n+1-3j)} c_{n,j},$$

which via Zeilberger's method (i.e., summing over all j) leads to (after substituting $n-2$ for n) the recurrence (3.3).

A substitution of the inversion of the q-expansion of t_3 into the q-expansion for F_3 shows that the first few δ_n are $1, -3, 9, -3, -279$, and so are equal to the γ_n, and so all terms agree. □

Lemma 3.4. *With the Almkvist-Zudilin numbers γ_n as in (1.9), $\sum_{n=0}^{\infty} \gamma_n t^n$ has radius of convergence $\frac{1}{9}$.*

Proof. This follows from [10, Chapter XVI, §16.2] by writing (3.10) as

$$(81t^2+14t+1)t^3 \frac{d^3}{dt^3} + 3(162t^2+21t+1)t^2 \frac{d^2}{dt^2} + (21t+1)(27t+1)t \frac{d}{dt} + 3(27t+1)t$$

and noting that the roots of $81t^2 + 14t + 1 = 0$ have absolute value $\frac{1}{9}$. □

Lemma 3.5. *As (multivalued) functions on $X(\Gamma_0(6))$, F_3 and t_3 have orders of vanishing at cusps of $\Gamma_0(6)$, denoted by O_c, as in the following table:*

cusp c	∞	0	1/2	1/3
width	1	6	3	1
$O_c(t_3)$	1	-1	-1	1
$O_c(F_3)$	0	1	1	0

Proof. Quoting [11, (5)], the Fourier series of $\eta_g(z)$ at the cusp $\frac{a}{c}$ is

$$\eta_g(z)|_{\frac{1}{2}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)} = C \exp\left(\frac{2\pi i z}{24} \sum_{j=1}^{s} \frac{\gcd(t_j, c)^2}{t_j} r_j\right) G_{a/c}(z),$$

where C is a constant and $G_{a/c}(z)$ is holomorphic and nonvanishing at $i\infty$. This formula allows us to compute the leading term of the $q = \exp(2\pi i z)$ expansion. If this is q^d for cusp c, then we must multiply d by the cusp width to obtain the order of vanishing.

□

Lemma 3.6. *Let F and t be defined as in (3.1). Using the notation of (2.14),*

$$R(t) = \sqrt{81t^2 + 14t + 1} \tag{3.11}$$

Proof. Since $F(z) \in \mathcal{M}_2(\Gamma_0(6))$, $t(z) \in \mathcal{A}_0(\Gamma_0(6))$ and $\Theta t(z) \in \mathcal{A}_2(\Gamma_0(6))$, we have $R(t(z)) \in \mathcal{A}_0(\Gamma_0(6))$, and so $R(t(z))$ may be expressed as a rational function in terms of the Hauptmodul r (given by (3.4)) for $\Gamma_0(6)$. Since $\Theta t(z)$ has exactly the same orders of vanishing at cusps as t, the orders of vanishing of $R(t(z))$ at cusps are determined by those of F, so from Lemma 3.5, $R(t(z))$ has poles of order one at 0 and $\frac{1}{2}$, and no other poles (since t is holomorphic on $\mathfrak{h}$, any poles of $R(t(z))$ are at cusps). Thus R is a degree 2 rational function of r. By comparing q expansions, we find that $R = \frac{1+8r^2}{(1+r)(1-8r)}$. Using expression (3.7) leads to (3.11). □

Corollary 3.7. *Let ℓ be a positive integer. Let the Almkvist-Zudilin numbers γ_n be as in (1.9), and $t_0 = t(i\sqrt{\ell/6})$, where t is as in (3.1), and let $p_g = \frac{t_0 \frac{d}{dt} P^g(t_0)}{P^g(t_0)}$ with P^g as in (2.7). Then provided $|t_0| < 1/9$,*

$$\frac{1}{\pi} \frac{\sqrt{6}}{\sqrt{\ell(81t_0^2 + 14t_0 + 1)}} = \sum_{n=0}^{\infty} \big(2n + p_g\big)\gamma_n t_0^n \tag{3.12}$$

Proof. This is an application of Theorem 2.1 with F and t as in (3.1) and $g = \begin{pmatrix} 0 & -\sqrt{\ell/6} \\ \sqrt{6/\ell} & 0 \end{pmatrix}$. Values of γ_n come from Lemma 3.3, $R(t_0)$ is given by Lemma 3.6, and the radius of convergence is in Lemma 3.4. □

Example 3.8. Since t_3 is an eta product, the value of t_0 in (3.12) can be found to many decimal places using a computer program such as PARI [13]. Now (3.12) has the form $A(t_0) = B(t_0) + p_g C(t_0)$, so an approximation to t_0 allows us to determine the expected value of p_g. Experimentally, we can search for values of ℓ where t_0 and p_g both appear to be roots of quadratic equations, obtaining the series in Table 1. We will only prove the case $\ell = 3$.

$\ell = 2$	$\dfrac{3(3+\sqrt{3})}{10\pi} = \sum_{n=0}^{\infty} \left(2n+1-\dfrac{\sqrt{3}}{5}\right)\gamma_n \left(\dfrac{2-\sqrt{3}}{9}\right)^n$
$\ell = 3$	$\dfrac{9}{4\sqrt{3}\pi} = \sum_{n=0}^{\infty} \left(2n+\dfrac{1}{2}\right)\gamma_n \left(\dfrac{1}{81}\right)^n$
$\ell = 5$	$\dfrac{3\sqrt{6}(3+2\sqrt{2})}{40\pi} = \sum_{n=0}^{\infty} \left(2n+1-\dfrac{9\sqrt{2}}{20}\right)\gamma_n \left(\dfrac{17-12\sqrt{2}}{9}\right)^n$
$\ell = 7$	$\dfrac{3\sqrt{3}(5+2\sqrt{6})}{56\pi} = \sum_{n=0}^{\infty} \left(2n+1-\dfrac{2\sqrt{6}}{7}\right)\gamma_n \left(\dfrac{49-20\sqrt{6}}{9}\right)^n$
$\ell = 13$	$\dfrac{3\sqrt{3}(17+12\sqrt{2})}{260\pi} = \sum_{n=0}^{\infty} \left(2n+1-\dfrac{36\sqrt{2}}{65}\right)\gamma_n \left(\dfrac{577-408\sqrt{2}}{9}\right)^n$
$\ell = 17$	$\dfrac{3\sqrt{51}(33+8\sqrt{17})}{2380\pi} = \sum_{n=0}^{\infty} \left(2n+1-\dfrac{117}{35\sqrt{17}}\right)\gamma_n \left(\dfrac{2177-528\sqrt{17}}{9}\right)^n$

TABLE 1. Series for $\frac{1}{\pi}$ in terms of the Almkvist-Zudilin numbers

Lemma 3.9. *With t and F as in (3.1), $g = \begin{pmatrix} 0 & -\sqrt{\frac{1}{2}} \\ \sqrt{2} & 0 \end{pmatrix}$, and $P = \frac{F_3|_{(2,g)}(\tau)}{F_3(\tau)}$,*

$$\begin{aligned} P^3 &= -27t(1-P+P^2), \\ p_g(t) := \frac{t\frac{d}{dt}P(t)}{P(t)} &= \frac{P^2-P+1}{P^2-2P+3}. \end{aligned}$$

Proof. We have $g(z) = -\frac{1}{2z}$, and by Dedekind's functional equation, (with a numerical computation to be sure of the sign of the square root),

$$P(t(\tau)) = \frac{F_3|_{(2,g)}(\tau)}{F_3(\tau)} = -3\frac{\eta(6\tau)\eta(3\tau)}{\eta(\frac{\tau}{3})\eta(\frac{2\tau}{3})}.$$

So, $P(3\tau) = \frac{-3}{x(\tau)}$, where $x(\tau) = \frac{\eta(\tau)\eta(2\tau)}{\eta(18\tau)\eta(9\tau)}$. Note that $x(\tau)^4 t(3\tau)t(\tau) = 1$. Since x is a Hauptmodul for $\Gamma_0(18)$ [9], t for $\Gamma_0(6)$, and $[\Gamma_0(6) : \Gamma_0(18)] = 3$, it is possible to write t as a degree 3 rational function in x, which we can determine, by comparision of q-expansions, to be $tx^3 = (x^2+3x+9)$. From this we have $1 = x(\tau)^4 t(3\tau)t(\tau) = x(\tau)t(3\tau)(x(\tau)^2+3x(\tau)+9)$. By substituting $x(\tau) = -\frac{3}{P(3\tau)}$, and replacing τ by $\tau/3$, this gives the first statement, from which the second follows. □

Lemma 3.10. *With $t = t_3$ we have $t\left(\frac{1}{6\tau}\right) \cdot t(\tau) \cdot 81 = 1$.*

Proof. This follows from Dedekind's functional equation for η. □

To determine $t\left(\frac{i}{\sqrt{2}}\right)$, we use the relationship between $\Gamma_0(6)_{+2}$ and $\Gamma_0(2)_+$.

Lemma 3.11. *$\Gamma_0(2)_+$ has a Hauptmodul t_{2A} satisfying*

$$t_{2A} = \frac{(1+27t)^4}{t}, \tag{3.13}$$

where t is as in (3.1).

Proof. From [9] we have $t_{2B} = t_{2A} + 4096/t_{2A}$, where $t_{2B} = (\eta(\tau)/\eta(2\tau))^{24}$. From this we find enough terms of the q-expansion to determine the relation between t and t_{2A}, which has degree 4 in t, since $[\Gamma_0(2)_+ : \Gamma_0(6)_{+2}] = 4$. □

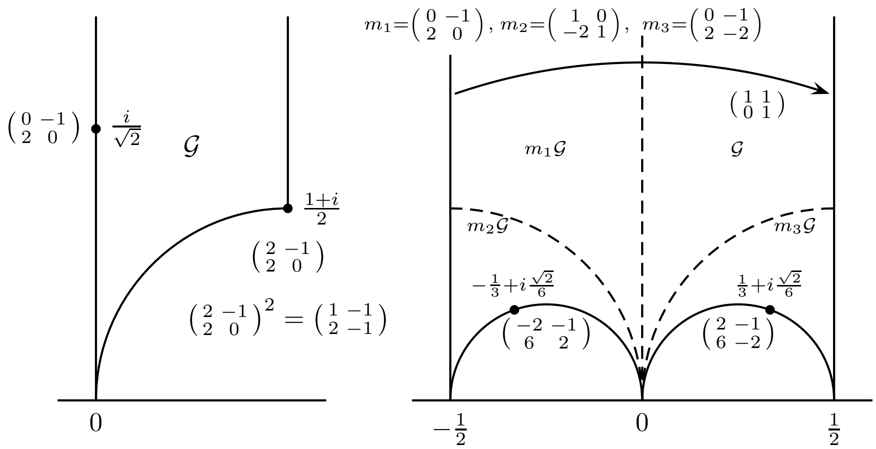

FIGURE 1. Fundamental domain $\mathcal{G}$ for $\Gamma_0(2)_+$ and domain for $\Gamma_0(6)_{+2}$ in terms of $\mathcal{G}$. Matrices I, m_1, m_2, m_3 are coset representatives for $\Gamma_0(6)_{+2}$ in $\Gamma_0(2)_+$. Matrices written near elliptic points fix these points, and identify adjacent edges. The edge identifying matrices in each case generate the groups. Matrices are only given up to a real constant.

Lemma 3.12. *With t as in (3.1), we have*

$$t\left(\frac{1+i}{2}\right) = -\frac{1}{27}\ , \quad t\left(\frac{i}{\sqrt{2}}\right) = \frac{1}{81} \quad \text{and} \quad t\left(\frac{i}{\sqrt{18}}\right) = 1$$

Proof. Motivated by (3.13), define

$$f_\alpha(y) := (1+27y)^4 - y\alpha.$$

Note that $\frac{1}{2}(1+i)$ and $\frac{i}{\sqrt{2}}$ are elliptic points of $\Gamma_0(2)_+$, but not of the subgroup $\Gamma_0(6)_{+2}$. This is illustrated in Figure 1 showing fundamental domains for these groups, from which we see that the image of $(1+i)/2$ in $X(\Gamma_0(2)_+)$ has only one preimage in $X(\Gamma_0(6)_{+2})$, whereas the image of $i/\sqrt{2}$ has three preimages, two being elliptic points, the other mapping to $i/\sqrt{2}$ with multiplicity 2. This means that $f_\alpha(y) = 0$ has a quadruple root when $\alpha = t_{2A}((1+i)/2)$, and a double root when $\alpha = t_{2A}(i/\sqrt{2})$. The discriminant of $f_\alpha(y)$ is $-3^{27}\alpha^3(\alpha - 2^8)$, and so for $f_\alpha(y) = 0$ to have a multiple root, we require $\alpha = 0$ or $\alpha = 2^8$. If $\alpha = 0$, we have a quadruple root, $y = -1/27$, so $t_{2A}((1+i)/2) = -1/27$.

With $\alpha = 2^8$, we have

$$f_{2^8}(y) = (81y-1)^2(81y^2 + 14y + 1),$$

from which we obtain $t(i/\sqrt{2}) = 1/81$, as well as the fact that the values of t at the elliptic points of $\Gamma_0(6)_{+2}$ are roots of $81t^2+14t+1$. The value for $t(i/\sqrt{18}) = t(i\sqrt{2}/6)$ now follows from Lemma 3.10. □

Lemma 3.13. *With P as in Lemma 3.9, $P(i/\sqrt{2}) = -1$.*

Proof. From Lemma 3.9, we have $P^3 = -27t(1 - P + P^2)$, and $P(3\tau)^4 = (-3)^4 t(3\tau)t(\tau)$. For $\tau = i/\sqrt{18}$, the latter implies $P(i/\sqrt{2})^4 = 1$, by Lemma 3.10. The first relation, together with $t(i/\sqrt{2}) = 1/81$ (Lemma 3.12) implies $(P+1)(3P^2 - 2P + 1) = 0$, so the only possibility is $P\left(\frac{i}{\sqrt{2}}\right) = -1$. □

Theorem 3.14. *With the Almkvist-Zudilin numbers γ_n as in (1.9), we have*

$$\frac{9}{2\sqrt{3}\pi} = \sum_{n=0}^{\infty} \gamma_n (4n+1) \left(\frac{1}{81}\right)^n .$$

Proof. We apply Corollary 3.7 with $\ell = 3$. From Lemma 3.12, $t_0 = 1/81$, and from Lemma 3.13 and Lemma 3.9, $p_g = 1/2$. Plugging these values into (3.12) and multiplying by 2, gives the result. □

In Lemma 3.12, we saw that $t((1+i)/2) = -1/27$. Since $m = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$ fixes $(1+i)/2$, we apply Theorem 2.1 with $g = m$. The function $R(t)$ is still as in Lemma 3.6, and $R\left(\frac{-1}{27}\right) = \frac{4}{3\sqrt{3}}$, and in Lemma 3.16 below we show that $p_g = p_m = 1/2$. Plugging these values into (2.17) we obtain the following

Theorem 3.15.

$$\frac{3\sqrt{3}}{\pi} = \sum_{n=0}^{\infty} \Big(4n+1\Big) \frac{\gamma_n}{(-27)^n}. \tag{3.14}$$

Lemma 3.16. *With t and F as in (3.1), $m = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right)$, and $P = \frac{F_3|_{(2,m)}(\tau)}{F_3(\tau)}$,*

$$\begin{aligned} P^3\alpha^3 &= 81t(\alpha^2P^2 + 3\alpha P + 9) \\ p_m(t) := \frac{t\frac{d}{dt}P(t)}{P(t)} &= \frac{\alpha^2P^2 + 3\alpha P + 9}{\alpha^3P^2 + 6\alpha P + 27}, \end{aligned}$$

where $\alpha = 3\sqrt{-i}$. Moreover, $p_m(t((1+i)/2)) = \frac{1}{2}$.

Proof. First note that $2(m\tau) = g_1(2\tau)$, $2(m\tau) = g_2(u)$, $6(m\tau) = g_3(2u)$, where $g_1 = \left(\begin{smallmatrix} 1 & -2 \\ 1 & -1 \end{smallmatrix}\right)$, $g_2 = \left(\begin{smallmatrix} 3 & -2 \\ 2 & -1 \end{smallmatrix}\right)$ and $g_3 = \left(\begin{smallmatrix} 3 & -4 \\ 1 & -1 \end{smallmatrix}\right)$ are all in $\mathrm{SL}_2(\mathbf{Z})$, and $u = \frac{1}{3}(z+1)$. From Dedekind's functional equation, (3.2), it follows that

$$F_3|_{(2,m)}(z) = 3\sqrt{-i}\frac{(\eta(z)\eta(2z))^3}{\eta(u)\eta(2u)}.$$

Since $z = 3u - 1$, we have $P^m(z) = \frac{\eta(9u)\eta(18u)}{\eta(u)\eta(2u)} = 3\frac{\sqrt{-i}}{x(u)}$, with x as in Lemma 3.9. Proceeding as in Lemma 3.9, using $t(3u+1) = t(3u) = (x(u)^4 t(u))^{-1}$, we obtain the expression relating P and t, from which the second expression follows.

By Lemma 3.12, $t((1+i)/2) = -\frac{1}{27}$, and $y = \alpha P(-1/27)$ satisfies $y^3 + 3(y^2 + 3y + 9) = (y+3)(y^2+9) = 0$. One can check numerically that the correct root is $\alpha P(-1/27) = -3$, from which we obtain $p_m(-1/27) = 1/2$. □

4. Series for $1/\pi$ arising from modular forms for $\Gamma_0(6)_{+3}$

Exactly the same method as used for $\Gamma_0(6)_{+2}$ and F_3, t_3 in Section 3 can be applied to $\Gamma_0(6)_{+3}$ and F_2, t_2 (given by (1.5)), using the same g. The value of r, given by Lemma 3.6 for the t_3, F_3 case, is now replaced by

$$R(t) = \sqrt{(4t+1)(16t+1)}, \tag{4.1}$$

$$\frac{\sqrt{6}}{\pi\sqrt{\ell(4t_0+1)(16t_0+1)}} = \sum_{n=0}^{\infty} \Big(2n + p_g\Big)\beta_n t_0^n, \tag{4.2}$$

As explained in Example 3.8, one can now search for series for $1/\pi$. Experimentally, we obtain the series in Table 2. The exact details of proof should be similar to the above methods, with the $\ell = 2$ case being the simplest.

$\ell = 2$ $$\frac{4}{3\pi} = \sum_{n=0}^{\infty}\left(2n+\frac{2}{3}\right)\beta_n\left(\frac{1}{32}\right)^n$$

$\ell = 3$ $$\frac{2(\sqrt{3}+\sqrt{2})}{5\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{\sqrt{6}}{5}\right)\beta_n\left(\frac{5-2\sqrt{6}}{8}\right)^n$$

$\ell = 4$ $$\frac{2}{\sqrt{3}\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{\sqrt{3}}{3}\right)\beta_n\left(\frac{-5+3\sqrt{3}}{32}\right)^n$$

$\ell = 5$ $$\frac{2\sqrt{15}(3+\sqrt{10})}{45\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{\sqrt{10}}{5}\right)\beta_n\left(\frac{19-6\sqrt{10}}{8}\right)^n$$

$\ell = 7$ $$\frac{28\sqrt{3}+18\sqrt{7}}{105\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{16\sqrt{21}}{105}\right)\beta_n\left(\frac{55-12\sqrt{21}}{8}\right)^n$$

$\ell = 8$ $$\frac{2\sqrt{6}+8}{15\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{(4\sqrt{6}+1)}{15}\right)\beta_n\left(\frac{-22+9\sqrt{6}}{64}\right)^n$$

$\ell = 10$ $$\frac{8(4\sqrt{3}+9)}{165\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{(56\sqrt{3}+27)}{165}\right)\beta_n\left(\frac{26-15\sqrt{3}}{64}\right)^n$$

$\ell = 13$ $$\frac{2\sqrt{39}(5\sqrt{13}+18)}{663\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{48\sqrt{13}}{221}\right)\beta_n\left(\frac{649-180\sqrt{13}}{8}\right)^n$$

$\ell = 17$ $$\frac{2\sqrt{51}(35+6\sqrt{34})}{1683\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{26\sqrt{34}}{187}\right)\beta_n\left(\frac{(35-6\sqrt{34})^2}{8}\right)^n$$

5. Series for $1/\pi$ arising from modular forms for $\Gamma_0(6)_{+6}$

For F_1 and t_1 as in (1.2), we apply Theorem 2.1 with

$$g = \begin{pmatrix} 0 & -\sqrt{\ell/6} \\ \sqrt{6/\ell} & 0 \end{pmatrix}$$

Now R is given by $R(t) = \sqrt{t^2 - 34t + 1}$, and we have

$$\frac{\sqrt{6}}{\pi\sqrt{\ell}\sqrt{t_0^2 - 34t_0 + 1}} = \sum_{n=0}^{\infty} \Big(2n + p_g\Big)\alpha_n t_0^n, \tag{5.1}$$

where $t_0 = t_1(i/\sqrt{6\ell})$, and α_n are the Apéry numbers, defined by (1.1). Experimentally, we obtain series as in Table 3. For $\ell = 5$ we recover Sato's example (1.4) (cf. [16]).

$$\ell = 2 \qquad \frac{9+5\sqrt{3}}{6\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{1}{\sqrt{3}}\right)\alpha_n(2-\sqrt{3})^{3n}$$

$$\ell = 3 \qquad \frac{\sqrt{2}(5+2\sqrt{6})}{8\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{\sqrt{3}}{2\sqrt{2}}\right)\alpha_n(\sqrt{3}-\sqrt{2})^{4n}$$

$$\ell = 5 \qquad \frac{9+4\sqrt{5}}{4\sqrt{15}\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{3\sqrt{5}}{10}\right)\alpha_n(2-\sqrt{5})^{4n}$$

$$\ell = 7 \qquad \frac{15+4\sqrt{14}}{12\sqrt{7}\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{4\sqrt{14}}{21}\right)\alpha_n(15-4\sqrt{14})^{2n}$$

$$\ell = 13 \qquad \frac{\sqrt{3}(51+10\sqrt{26})}{72\sqrt{13}\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{2\sqrt{26}}{13}\right)\alpha_n(\sqrt{26}-5)^{4n}$$

$$\ell = 17 \qquad \frac{\sqrt{6}(\sqrt{2}+1)^6}{816\pi} = \sum_{n=0}^{\infty}\left(2n+1-\frac{39\sqrt{2}}{68}\right)\alpha_n(\sqrt{2}-1)^{12n}$$

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