Journal of Number Theory 129 (2009) 1786–1797

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

The Rogers–Ramanujan continued fraction and a new Eisenstein series identity ☆

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ARTICLE INFO

Article history:
Received 31 July 2008
Revised 7 December 2008
Available online 21 March 2009
Communicated by David Goss

MSC:
33E05
11F11
11F20
11F27

Keywords:
Elliptic function
Theta function
Eisenstein series
Rogers–Ramanujan's continued fraction

ABSTRACT

With two elementary trigonometric sums and the Jacobi theta function θ_1, we provide a new proof of two Ramanujan's identities for the Rogers–Ramanujan continued fraction in his lost notebook. We further derive a new Eisenstein series identity associated with the Rogers–Ramanujan continued fraction.

☆ This project is partially funded by National University of Singapore Academic Research Fund R-146-000-103-112. The third author is also supported by NSFC, PCSIRT, Innovation Program of Shanghai Municipal Education Commission, and Shanghai Leading Academic Discipline Project, Project No. B407.

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0022-314X/$ – see front matter
doi:10.1016/j.jnt.2009.01.014

1. Introduction

Throughout this paper we will take $q = e^{2\pi i\tau}$, where $\operatorname{Im}\tau > 0$. Let $G(z)$ be defined as

$$G(z) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2} z^n}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{1.1}$$

Then the celebrated Rogers–Ramanujan identities are given as follows [2, Chapter 7], [13, p. 290]:

$$G(1) = \left(q;q^5\right)_\infty^{-1}\left(q^4;q^5\right)_\infty^{-1} \quad \text{and} \quad G(q) = \left(q^2;q^5\right)_\infty^{-1}\left(q^3;q^5\right)_\infty^{-1}, \tag{1.2}$$

where

$$(a;q)_\infty := \prod_{n=0}^{\infty}\left(1 - aq^n\right).$$

The Rogers–Ramanujan continued fraction $R(\tau)$ is defined by

$$R(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}. \tag{1.3}$$

With the help of the Rogers–Ramanujan identities, both Rogers [22] and Ramanujan [20, Vol. II, Chapter 16, Section 15], [6, p. 30] prove that

$$R(\tau) = q^{1/5}\frac{G(q)}{G(1)} = q^{1/5}\frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.4}$$

In his first two letters to G.H. Hardy [19, pp. xxvii, xxviii], [7, pp. 21–30, 53–62], S. Ramanujan made several claims about $R(\tau)$. In particular, in his first letter, he asserted that

$$R(i) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}. \tag{1.5}$$

Moreover, in his notebooks [20] and "lost notebook" [21], Ramanujan recorded without proofs many results about $R(\tau)$, and many studies on these results have appeared, see for examples, [3,4,8,9,18,24,25].

In this paper, we establish several new identities associated with the Rogers–Ramanujan continued fraction $R(\tau)$. Using two elementary trigonometric sums and the Jacobi theta function θ_1, we will derive the following theorem:

Theorem 1. *Let* $|q| < 1$, $\alpha = \frac{1-\sqrt{5}}{2}$ *and* $\beta = \frac{1+\sqrt{5}}{2}$. *Then*

$$\prod_{n=1}^{\infty}\frac{1}{(1+\alpha q^n + q^{2n})} - \prod_{n=1}^{\infty}\frac{1}{(1+\beta q^n + q^{2n})} = q^{1/12}\sqrt{\frac{5\eta(25\tau)}{\eta(5\tau)}}R(5\tau), \tag{1.6}$$

$$\beta\prod_{n=1}^{\infty}\frac{1}{(1+\alpha q^n + q^{2n})} - \alpha\prod_{n=1}^{\infty}\frac{1}{(1+\beta q^n + q^{2n})} = q^{1/12}\sqrt{\frac{5\eta(25\tau)}{\eta(5\tau)R(5\tau)}}, \tag{1.7}$$

where

$$\eta(\tau)=q^{1/24}(q;q)_\infty.$$

Identities (1.6) and (1.7) appear to be new. We observe that by replacing τ by $\tau/5$ in (1.6) and (1.7) and solving the resulting equations, we immediately deduce the following identities of S. Ramanujan [21, p. 206].

Corollary 2. *Suppose $|q|<1$. Let α, β be given as in Theorem 1. Then*

$$\prod_{n=1}^{\infty}\frac{1}{(1+\alpha q^{n/5}+q^{2n/5})}=q^{1/60}\sqrt{\frac{\eta(5\tau)}{\eta(\tau)}}\left(\sqrt{\frac{1}{R(\tau)}}-\alpha\sqrt{R(\tau)}\right), \tag{1.8}$$

$$\prod_{n=1}^{\infty}\frac{1}{(1+\beta q^{n/5}+q^{2n/5})}=q^{1/60}\sqrt{\frac{\eta(5\tau)}{\eta(\tau)}}\left(\sqrt{\frac{1}{R(\tau)}}-\beta\sqrt{R(\tau)}\right). \tag{1.9}$$

Corollary 2 was first discussed in a paper by K.G. Ramanathan [18, p. 220, (46) and (47)]. To prove Corollary 2, he stated without proofs the two identities [18, (50)]

$$g(q)-\alpha q^{1/5}h(q)=\prod_{n=1}^{\infty}\left(1-q^{n/5}\right)\left(1+\beta q^{n/5}+q^{2n/5}\right), \tag{1.10}$$

$$g(q)-\beta q^{1/5}h(q)=\prod_{n=1}^{\infty}\left(1-q^{n/5}\right)\left(1+\alpha q^{n/5}+q^{2n/5}\right), \tag{1.11}$$

where

$$h(q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(5n^2+n)/2}\quad\text{and}\quad g(q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{(5n^2+3n)/2}.$$

The proofs of (1.10) and (1.11) were subsequently given by Berndt, Huang, Sohn, and Son [9] (see also [5, p. 21]). Our proofs of Corollary 2, which relies on (1.6) and (1.7), are different from those given in [9].

In [17], the third author established the following Eisenstein series identity involving $R(\tau)$ by using (1.6).

Theorem 3. *Let $|q|<1$. Then*

$$\sum_{\substack{n=1\\ n\equiv 2 \ (\mathrm{mod}\ 5)}}^{\infty}\frac{(q^n-q^{2n}-q^{3n}+q^{4n})}{1-q^{5n}}-\sum_{\substack{n=1\\ n\equiv 3 \ (\mathrm{mod}\ 5)}}^{\infty}\frac{(q^n-q^{2n}-q^{3n}+q^{4n})}{1-q^{5n}}$$
$$=\eta(\tau)\eta(25\tau)\sqrt{\frac{R(5\tau)\eta(25\tau)}{\eta(5\tau)}}. \tag{1.12}$$

It turns out that (1.12) has the following companion:

Theorem 4. *Let $|q| < 1$. Then*

$$\sum_{\substack{n=1 \\ n\equiv 1 \pmod 5}}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1-q^{5n}} - \sum_{\substack{n=1 \\ n\equiv 4 \pmod 5}}^{\infty} \frac{(q^n - q^{2n} - q^{3n} + q^{4n})}{1-q^{5n}}$$
$$= \eta(\tau)\eta(25\tau)\sqrt{\frac{\eta(25\tau)}{\eta(5\tau)R(5\tau)}}. \tag{1.13}$$

The rest of this paper is organized as follows. In Section 2, we will prove Theorem 1. Section 3 is devoted to the proof of Theorem 4.

2. The Jacobi theta function θ_1 and the proof of Theorem 1

The Jacobi triple product identity [10, p. 10], [13, pp. 282–283] is perhaps the most important identity in the theory of elliptic theta functions, which states that

$$(q;q)_\infty (z;q)_\infty (q/z;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^n, \quad z \neq 0. \tag{2.1}$$

The Jacobi triple product identity was established by Jacobi in his famous Fundamenta Nova [14], but, in fact, was first proved by Gauss [12]. There are now several elementary proofs of (2.1) which do not use elliptic functions, see for examples, [1,11,15].

If we multiply both sides of (2.1) with $z^{-1/2}$, then we can find that

$$\left(z^{-1/2} - z^{1/2}\right)(q;q)_\infty (qz;q)_\infty (q/z;q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} z^{n-1/2}.$$

Replacing z by e^{2iz} and e^{-2iz} respectively in the equation above, we conclude that

$$2i(\sin z)(q;q)_\infty \left(qe^{2iz};q\right)_\infty \left(qe^{-2iz};q\right)_\infty = -\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} e^{(2n-1)iz},$$
$$2i(\sin z)(q;q)_\infty \left(qe^{2iz};q\right)_\infty \left(qe^{-2iz};q\right)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} e^{-(2n-1)iz}.$$

Making the index change $n \to n+1$ in the right-hand sides of the two equations above, adding the two resulting equations together, we arrive at

$$2(\sin z)(q;q)_\infty \left(qe^{2iz};q\right)_\infty \left(qe^{-2iz};q\right)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z$$
$$= 2\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \sin(2n+1)z. \tag{2.2}$$

The well-known Jacobi theta function θ_1 is defined as follows

$$\theta_1(z|\tau) = 2\sum_{n=0}^{\infty}(-1)^n q^{(2n+1)^2/8}\sin(2n+1)z = 2q^{1/8}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\sin(2n+1)z. \tag{2.3}$$

Combining (2.2) and (2.3), we immediately find the infinite product representation of θ_1:

$$\theta_1(z|\tau) = 2q^{1/8}(\sin z)(q;q)_\infty\left(qe^{2iz};q\right)_\infty\left(qe^{-2iz};q\right)_\infty. \tag{2.4}$$

Let $z = \frac{\pi}{5}$ and $z = \frac{2\pi}{5}$ respectively in (2.4). Then we have

$$\theta_1\left(\frac{\pi}{5}\middle|\tau\right) = 2q^{1/8}\left(\sin\frac{\pi}{5}\right)(q;q)_\infty\left(qe^{\frac{2\pi i}{5}};q\right)_\infty\left(qe^{\frac{-2\pi i}{5}};q\right)_\infty, \tag{2.5}$$

and

$$\theta_1\left(\frac{2\pi}{5}\middle|\tau\right) = 2q^{1/8}\left(\sin\frac{2\pi}{5}\right)(q;q)_\infty\left(qe^{\frac{4\pi i}{5}};q\right)_\infty\left(qe^{\frac{-4\pi i}{5}};q\right)_\infty. \tag{2.6}$$

Multiplying (2.5) and (2.6) together, and using the elementary identities

$$\sin\frac{\pi}{5}\sin\frac{2\pi}{5} = \frac{\sqrt{5}}{4}$$

and

$$\left(1-xe^{\frac{2\pi i}{5}}\right)\left(1-xe^{-\frac{2\pi i}{5}}\right)\left(1-xe^{\frac{4\pi i}{5}}\right)\left(1-xe^{-\frac{4\pi i}{5}}\right) = 1-x^5$$

in the resulting equation, and simplifying, we deduce the following useful identity:

$$\theta_1\left(\frac{\pi}{5}\middle|\tau\right)\theta_1\left(\frac{2\pi}{5}\middle|\tau\right) = \sqrt{5}\eta(\tau)\eta(5\tau). \tag{2.7}$$

We now give a proof of Theorem 1.

Proof. It is easily seen that we may rewrite (2.5) and (2.6) as follows

$$\theta_1\left(\frac{\pi}{5}\middle|\tau\right) = 2q^{1/12}\left(\sin\frac{\pi}{5}\right)\eta(\tau)\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right), \tag{2.8}$$

and

$$\theta_1\left(\frac{2\pi}{5}\middle|\tau\right) = 2q^{1/12}\left(\sin\frac{2\pi}{5}\right)\eta(\tau)\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right), \tag{2.9}$$

where

$$\alpha = -2\cos\frac{2\pi}{5} = \frac{1-\sqrt{5}}{2} \quad \text{and} \quad \beta = -2\cos\frac{4\pi}{5} = 2\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{2}.$$

It follows that

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)=q^{-1/12}\frac{1}{\eta(\tau)}\frac{\theta_1(\frac{\pi}{5}|\tau)}{2\sin\frac{\pi}{5}} \tag{2.10}$$

and

$$\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right)=q^{-1/12}\frac{1}{\eta(\tau)}\frac{\theta_1(\frac{2\pi}{5}|\tau)}{2\sin\frac{2\pi}{5}}. \tag{2.11}$$

Multiplying the two equations above together and then using (2.7), we deduce that

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)\left(1+\beta q^n+q^{2n}\right)=q^{-1/6}\frac{\eta(5\tau)}{\eta(\tau)}. \tag{2.12}$$

We observe that by multiplying (1.8) and (1.9) and using (2.12), we obtain the following identity of Ramanujan (see [6, pp. 265–267], [16]):

$$\frac{1}{R(\tau)}-1-R(\tau)=\frac{\eta(\frac{\tau}{5})}{\eta(5\tau)}. \tag{2.13}$$

If we subtract (2.11) from (2.10), then we find that

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)-\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right)=q^{-1/12}\frac{1}{\eta(\tau)}\left(\frac{\theta_1(\frac{\pi}{5}|\tau)}{2\sin\frac{\pi}{5}}-\frac{\theta_1(\frac{2\pi}{5}|\tau)}{2\sin\frac{2\pi}{5}}\right). \tag{2.14}$$

Using (2.3), we simplify the right-hand side of the equation above as

$$q^{-1/12}\frac{1}{\eta(\tau)}\sum_{n=0}^{\infty}(-1)^n A(n) q^{(2n+1)^2/8}, \tag{2.15}$$

where

$$A(n)=\frac{\sin(2n+1)\frac{\pi}{5}}{\sin\frac{\pi}{5}}-\frac{\sin(2n+1)\frac{2\pi}{5}}{\sin\frac{2\pi}{5}}. \tag{2.16}$$

Direct computations show that

$$A(5n+1)=\sqrt{5},\qquad A(5n+3)=-\sqrt{5},\qquad A(5n)=A(5n+2)=A(5n+4)=0.$$

Hence,

$$\sum_{n=0}^{\infty}(-1)^n A(n) q^{(2n+1)^2/8}=-\sqrt{5}\sum_{n=-\infty}^{\infty}(-1)^n q^{(10n-3)^2/8},$$

and the Jacobi triple product identity tells us that

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{(10n-3)^2/8} = q^{9/8}\left(q^5;q^{25}\right)_\infty\left(q^{20};q^{25}\right)_\infty\left(q^{25};q^{25}\right)_\infty.$$

Thus, we have

$$\sum_{n=0}^{\infty}(-1)^n A(n) q^{(2n+1)^2/8} = -\sqrt{5}q^{9/8}\left(q^5;q^{25}\right)_\infty\left(q^{20};q^{25}\right)_\infty\left(q^{25};q^{25}\right)_\infty. \tag{2.17}$$

Combining (2.14)–(2.17), we conclude that

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right) - \prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right) = -\sqrt{5}\frac{\eta(25\tau)}{\eta(\tau)}\left(q^5;q^{25}\right)_\infty\left(q^{20};q^{25}\right)_\infty. \tag{2.18}$$

Dividing both sides of the equation above by $\prod_{n=1}^{\infty}(1+\alpha q^n+q^{2n})(1+\beta q^n+q^{2n})$, we deduce that

$$\begin{aligned}&\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)^{-1} - \prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right)^{-1}\\ &\quad=\sqrt{5}\frac{\eta(25\tau)}{\eta(\tau)}\frac{(q^5;q^{25})_\infty(q^{20};q^{25})_\infty}{\prod_{n=1}^{\infty}(1+\alpha q^n+q^{2n})(1+\beta q^n+q^{2n})}.\end{aligned}$$

Substituting (2.12) into the right-hand side of the equation above, we find that

$$\begin{aligned}&\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)^{-1} - \prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right)^{-1}\\ &\quad=\sqrt{5}q^{1/6}\frac{\eta(25\tau)}{\eta(5\tau)}\left(q^5;q^{25}\right)_\infty\left(q^{20};q^{25}\right)_\infty\\ &\quad=q^{1/12}\sqrt{\frac{5\eta(25\tau)}{\eta(5\tau)}R(5\tau)},\end{aligned}$$

which is (1.6).

Next, we prove (1.7). Multiplying (2.10) by $\cos\frac{2\pi}{5}$ and (2.11) by $\cos\frac{\pi}{5}$ and then adding the two resulting equations together, we find that

$$\begin{aligned}&2\cos\frac{2\pi}{5}\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)+2\cos\frac{\pi}{5}\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right)\\ &\quad=q^{-1/12}\frac{1}{\eta(\tau)}\left(\frac{\cos\frac{2\pi}{5}}{\sin\frac{\pi}{5}}\theta_1\left(\frac{\pi}{5}\middle|\tau\right)+\frac{\cos\frac{\pi}{5}}{\sin\frac{2\pi}{5}}\theta_1\left(\frac{2\pi}{5}\middle|\tau\right)\right).\end{aligned} \tag{2.19}$$

Using (2.3), we rewrite (2.19) as

$$q^{-1/12}\frac{1}{\eta(\tau)}\sum_{n=0}^{\infty}(-1)^n B(n) q^{(2n+1)^2/8}, \tag{2.20}$$

where

$$B(n)=2\frac{\cos\frac{2\pi}{5}}{\sin\frac{\pi}{5}}\sin(2n+1)\frac{\pi}{5}+2\frac{\cos\frac{\pi}{5}}{\sin\frac{2\pi}{5}}\sin(2n+1)\frac{2\pi}{5}. \tag{2.21}$$

Direct computations show that

$$B(5n)=\sqrt{5},\qquad B(5n+4)=-\sqrt{5},\qquad B(5n+1)=B(5n+2)=B(5n+3)=0.$$

Thus, by (2.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^nB(n)q^{(2n+1)^2/8}&=\sqrt{5}\sum_{n=-\infty}^{\infty}(-1)^nq^{(10n+1)^2/8}\\&=\sqrt{5}q^{1/8}\left(q^{10};q^{25}\right)_\infty\left(q^{15};q^{25}\right)_\infty\left(q^{25};q^{25}\right)_\infty.\end{aligned}$$

It follows that

$$\begin{aligned}&2\cos\frac{2\pi}{5}\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)+2\cos\frac{\pi}{5}\prod_{n=1}^{\infty}\left(1+\beta q^n+q^{2n}\right)\\&\quad=\sqrt{5}q^{-1}\frac{\eta(25\tau)}{\eta(\tau)}\left(q^{10};q^{25}\right)_\infty\left(q^{15};q^{25}\right)_\infty.\end{aligned} \tag{2.22}$$

Proceeding through the same steps as in the proof of (1.6), we divide both sides of the equation above by

$$\prod_{n=1}^{\infty}\left(1+\alpha q^n+q^{2n}\right)\left(1+\beta q^n+q^{2n}\right)=q^{-1/6}\frac{\eta(5\tau)}{\eta(\tau)},$$

and arrive at (1.7). Thus we complete the proof of Theorem 1. □

3. A new Eisenstein series identity associated with $R(\tau)$

In this section, we will prove Theorem 4.

Proof of Theorem 4. We first recall the identity [17, p. 296, Eq. (5.8)]:

$$\sum_{n=1}^{\infty}\frac{(q^n-q^{2n}-q^{3n}+q^{4n})}{1-q^{5n}}\sin 2nz=\frac{\eta^2(\tau)\theta_1(z|5\tau)\theta_1(2z|5\tau)}{2\eta(5\tau)\theta_1(z|\tau)}. \tag{3.1}$$

Taking $z=\frac{\pi}{5}$ and then multiplying both sides by $2\beta=1+\sqrt{5}=2\cos\frac{\pi}{5}$ we find that

$$\sum_{n=1}^{\infty}\frac{(q^n-q^{2n}-q^{3n}+q^{4n})}{1-q^{5n}}\sin\frac{2\pi}{5}\sin\frac{2n\pi}{5}=\frac{\sqrt{5}}{4}q^{-1/12}\eta(\tau)\eta(25\tau)\prod_{n=1}^{\infty}\frac{\beta}{1+\alpha q^n+q^{2n}}. \tag{3.2}$$

In the same way by taking $z=\frac{2\pi}{5}$ in (3.1) and then multiplying both sides by

$$2\alpha=1-\sqrt{5}=-2\cos\frac{2\pi}{5},$$

we obtain

$$-\sum_{n=1}^{\infty}\frac{(q^n-q^{2n}-q^{3n}+q^{4n})}{1-q^{5n}}\sin\frac{4\pi}{5}\sin\frac{4n\pi}{5}=\frac{\sqrt{5}}{4}q^{-1/12}\eta(\tau)\eta(25\tau)\prod_{n=1}^{\infty}\frac{\alpha}{1+\beta q^n+q^{2n}}. \tag{3.3}$$

Subtracting (3.3) from (3.2), we deduce that

$$\sum_{n=1}^{\infty}\frac{(q^n-q^{2n}-q^{3n}+q^{4n})}{1-q^{5n}}S(n)$$
$$=\frac{\sqrt{5}}{4}q^{-1/12}\eta(\tau)\eta(25\tau)\left(\prod_{n=1}^{\infty}\frac{\beta}{1+\alpha q^n+q^{2n}}-\prod_{n=1}^{\infty}\frac{\alpha}{1+\beta q^n+q^{2n}}\right), \tag{3.4}$$

where

$$S(n)=\sin\frac{2\pi}{5}\sin\frac{2n\pi}{5}+\sin\frac{4\pi}{5}\sin\frac{4n\pi}{5}. \tag{3.5}$$

Substituting (1.7) into the right-hand side of (3.4), we deduce that

$$\sum_{n=1}^{\infty}\frac{(q^n-q^{2n}-q^{3n}+q^{4n})}{1-q^{5n}}S(n)=\frac{5}{4}\eta(\tau)\eta(25\tau)\sqrt{\frac{\eta(25\tau)}{\eta(5\tau)R(5\tau)}}. \tag{3.6}$$

By direct computation, we have

$$S(5n)=S(5n+2)=S(5n+3)=0,\qquad S(5n+1)=\sin^2\frac{\pi}{5}+\sin^2\frac{2\pi}{5}=\frac{5}{4},$$
$$S(5n+4)=-\sin^2\frac{\pi}{5}-\sin^2\frac{2\pi}{5}=-\frac{5}{4}. \tag{3.7}$$

Combining (3.6) and (3.7), we complete the proof of (1.13). □

4. Some concluding remarks

4.1. Let G denotes the multiplicative group

$$(\mathbf{Z}/25\mathbf{Z})^*:=\{[k]_{25}\mid \gcd(k,5)=1,\ 1\leqslant k\leqslant 25\}.$$

Note that G is cyclic and generated by $[2]_{25}$. Let $\widetilde{\chi}$ be defined by

$$\widetilde{\chi}([2]_{25})=i$$

and extend it to a Dirichlet character on $\mathbf{Z}$ by

$$\chi(n)=\begin{cases}\widetilde{\chi}([n]_{25}) & \text{if } \gcd(n,5)=1,\\ 0, & \text{otherwise.}\end{cases}$$

Then (1.12) and (1.13) can be rewritten as

$$\sum_{n=1}^{\infty}\chi(n)\frac{q^n-q^{2n}-q^{3n}+q^{4n}}{1-q^{5n}}=\frac{\eta(\tau)\eta^{3/2}(25\tau)}{\eta^{1/2}(5\tau)}\left(\sqrt{\frac{1}{R(5\tau)}}+i\sqrt{R(5\tau)}\right).$$

Interpreting identities such as (1.12) and (1.13) with the help of group characters will certainly assist us in understanding their existence.

4.2. If we set $z=\frac{\pi}{3}$ in (3.1) and use $\theta_1(\frac{\pi}{3}\mid\tau)=\sqrt{3}\eta(3\tau)$ in the resulting equation, we find that

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{(q^n-q^{2n}-q^{3n}+q^{4n})}{1-q^{5n}}=\frac{\eta^2(\tau)\eta^2(15\tau)}{\eta(3\tau)\eta(5\tau)}. \tag{4.1}$$

We may rewrite the left-hand side of (4.1) as

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{3}\right)\left(\frac{n}{5}\right)q^{mn}. \tag{4.2}$$

Interchanging summation, we deduce that

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n-q^{2n}}{1-q^{3n}}=\frac{\eta^2(\tau)\eta^2(15\tau)}{\eta(3\tau)\eta(5\tau)}. \tag{4.3}$$

The expression (4.2) is an example of q-series associated with genus characters of the imaginary quadratic field $\mathbf{Q}(\sqrt{-15})$. Series associated with genus characters are recently used by P.C. Toh [23] to derive explicit representations of certain theta series associated with positive definite quadratic forms in terms of Lambert series. For example, using the ideas in [23], one can show that

$$\frac{1}{2}\left(\sum_{m,n\in\mathbf{Z}}q^{m^2+mn+4n^2}-\sum_{m,n\in\mathbf{Z}}q^{3m^2+3mn+2n^2}\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{m}{3}\right)\left(\frac{n}{5}\right)q^{mn}. \tag{4.4}$$

Our identity (4.1) shows that indeed the left-hand side of (4.4) has a product representation, namely,

$$\frac{1}{2}\left(\sum_{m,n\in\mathbf{Z}}q^{m^2+mn+4n^2}-\sum_{m,n\in\mathbf{Z}}q^{3m^2+3mn+2n^2}\right)=\frac{\eta^2(\tau)\eta^2(15\tau)}{\eta(3\tau)\eta(5\tau)}.$$

4.3. In this subsection, we present a different proof of Theorem 4 using (2.13) and the identity

$$\frac{(az,q/(az),q,q)_\infty}{(a,q/a,z,q/z;q)_\infty}=\sum_{n=-\infty}^{\infty}\frac{z^n}{1-aq^n}\quad\big(|q|<|z|<1\big), \tag{4.5}$$

where $(a_1,a_2,\ldots,a_n;q)_\infty:=(a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty$. Theorem 3 can also be proved similarly.

The identity (4.5) is often referred to as a corollary of Ramanujan's ${}_1\psi_1$ summation formula [6, Entry 17 p. 32] (and can be obtained by setting $b=aq$ in [6, Entry 17 p. 32]).

Second proof of Theorem 4. Noting that

$$\sum_{n=0}^{\infty}\frac{x^n}{1-yq^n}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}x^ny^mq^{mn}=\sum_{n=0}^{\infty}\frac{y^n}{1-xq^n}\quad\bigl(|x|,|y|<1\bigr),$$

the left-hand side of (1.13) simplifies to become,

$$\begin{aligned}
&\sum_{\substack{n=1\\ n\equiv 1 \pmod 5}}^{\infty}\frac{(q^n-q^{2n}-q^{3n})}{1-q^{5n}}-\sum_{\substack{n=1\\ n\equiv 4 \pmod 5}}^{\infty}\frac{(-q^{2n}-q^{3n}+q^{4n})}{1-q^{5n}}\\
&=\sum_{\substack{n=1\\ n\equiv 1 \pmod 5}}^{\infty}\frac{(q^n-q^{2n}-q^{3n})}{1-q^{5n}}-\sum_{\substack{n=-\infty\\ n\equiv 1 \pmod 5}}^{-1}\frac{(-q^{-2n}-q^{-3n}+q^{-4n})}{1-q^{-5n}}\\
&=\sum_{n=-\infty}^{\infty}\frac{q^{5n+1}}{1-q^{25n+5}}-\sum_{n=-\infty}^{\infty}\frac{q^{10n+2}}{1-q^{25n+5}}-\sum_{n=-\infty}^{\infty}\frac{q^{15n+3}}{1-q^{25n+5}}\\
&=q\frac{(q^{10},q^{15},q^{25},q^{25};q^{25})_\infty}{(q^5,q^{20};q^{25})_\infty^2}-q^2\frac{(q^{10},q^{15},q^{25},q^{25};q^{25})_\infty}{(q^5,q^{10},q^{15},q^{20};q^{25})_\infty}-q^3\frac{(q^5,q^{20},q^{25},q^{25};q^{25})_\infty}{(q^5,q^{10},q^{15},q^{20};q^{25})_\infty}\\
&=q\frac{(q^{10},q^{15},q^{25},q^{25};q^{25})_\infty}{(q^5,q^{20};q^{25})_\infty^2}-q^2\frac{(q^{25};q^{25})_\infty^2}{(q^5,q^{20};q^{25})_\infty}-q^3\frac{(q^{25};q^{25})_\infty^2}{(q^{10},q^{15};q^{25})_\infty}\\
&=q^2\frac{(q^{25};q^{25})_\infty^2}{(q^5,q^{20};q^{25})_\infty}\left\{\frac{1}{R(5\tau)}-1-R(5\tau)\right\}\\
&=\eta^2(25\tau)\sqrt{\frac{\eta(25\tau)}{\eta(5\tau)R(5\tau)}}\frac{\eta(\tau)}{\eta(25\tau)}\\
&=\eta(\tau)\eta(25\tau)\sqrt{\frac{\eta(25\tau)}{\eta(5\tau)R(5\tau)}},
\end{aligned}$$

where we applied identity (4.5) three times in the third equality and (2.13) in the penultimate equality. □

Acknowledgment

The authors would like to thank the referee for his invaluable suggestions.

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