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CONGRUENCES SATISFIED BY APÉRY-LIKE NUMBERS

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In this article, we investigate congruences satisfied by Apéry-like numbers.

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1. Introduction: Apéry Numbers

In his proof of the irrationality of $\zeta(3)$, R. Apéry introduced the numbers

$$\alpha_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}^2, \quad n \in \mathbf{N}.$$

These numbers are now known as the Apéry numbers. Since the appearance of Apéry's work, properties of α_n were gradually discovered. One of these is the observation that for primes $p \geq 5$,

$$\alpha_p \equiv \alpha_1 \pmod{p^3}. \tag{1.1}$$

The congruence (1.1) was conjectured by Chowla *et al.* [6] and proved by Gessel [7], who established the stronger result

$$\alpha_{pn} \equiv \alpha_n \pmod{p^3}. \tag{1.2}$$

In this article, we investigate other sequences of integers $\{f_n\}_{n=1}^{\infty}$ that satisfy relations similar to (1.2).

Let

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = \exp(2\pi i \tau)$ and $\mathrm{Im}(\tau) > 0$. It can be shown [10] that if

$$t_1(\tau) = \left(\frac{\eta(6\tau)\eta(\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12} \quad \text{and} \quad F_1(\tau) = \frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)},$$

then

$$F_1(\tau) = \sum_{n=0}^{\infty} \alpha_n t_1^n(\tau) \tag{1.3}$$

for suitably small $|t_1(\tau)|$. The identification of α_n as the coefficients of certain power series serves as a starting point for us in our search of other sequences $\{f_n\}_{n=1}^{\infty}$ satisfying congruences similar to (1.2).

2. The Domb Numbers

Consider the functions

$$t_2(\tau) = \left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^{6} \quad \text{and} \quad F_2(\tau) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2}.$$

It can be shown [2, (4.14)] that when $|t_2(\tau)|$ is sufficiently small, we have

$$F_2(\tau) = \sum_{n=0}^{\infty} (-1)^n \beta_n t_2^n(\tau) \tag{2.1}$$

where

$$\beta_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2(n-j)}{n-j}.$$

The sequence $\{\beta_n\}_{n=1}^{\infty}$ turns out to satisfy the congruence

Theorem 2.1. *For primes $p \geq 5$,*

$$\beta_{pn} \equiv \beta_n \pmod{p^3}.$$

Proof. The method of proof given here is due to Gessel [7]. For a prime $p \geq 5$, we find that

$$\begin{aligned} \beta_{pn} &= \sum_{j=0}^{pn} \binom{pn}{j}^2 \binom{2j}{j} \binom{2(pn-j)}{pn-j} \\ &= S_1 + S_2, \end{aligned} \tag{2.2}$$

where

$$S_1 = \sum_{j=0}^{n} \binom{pn}{pj}^2 \binom{2pj}{pj} \binom{2p(n-j)}{p(n-j)}$$

and

$$S_2 = \sum_{k=1}^{p-1} \sum_{m=0}^{n-1} \binom{pn}{k+pm}^2 \binom{2(k+pm)}{k+pm} \binom{2(pn-k-pm)}{pn-k-pm}.$$

Now,

$$S_1 \equiv \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2(n-j)}{n-j} \pmod{p^3}$$

since [8]

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p^3} \quad \text{for primes } p \geq 5. \tag{2.3}$$

Therefore,

$$S_1 \equiv \beta_n \pmod{p^3}. \tag{2.4}$$

For $0 < k < p$, we have [7]

$$\binom{pn}{k+pm} \equiv (-1)^k \frac{pn}{k} \binom{n-1}{m} \pmod{p^2}.$$

Hence,

$$S_2 \equiv p^2 n^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{m=0}^{n-1} \binom{n-1}{m}^2 \times \binom{2k+2pm}{k+pm} \binom{2(pn-k-pm)}{pn-k-pm} \pmod{p^3}. \tag{2.5}$$

In order to prove that

$$S_2 \equiv 0 \pmod{p^3},$$

it suffices to show that

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{m=0}^{n-1} \binom{n-1}{m}^2 \binom{2k+2pm}{k+pm} \binom{2(pn-k-pm)}{pn-k-pm} \equiv 0 \pmod{p}. \tag{2.6}$$

By Lucas' congruence [9],

$$\binom{a+pb}{c+pd} \equiv \binom{a}{c} \binom{b}{d} \pmod{p}. \tag{2.7}$$

Hence, we deduce that

$$\begin{aligned}
&\sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{m=0}^{n-1} \binom{n-1}{m}^2 \binom{2k+2pm}{k+pm} \binom{2(pn-k-pm)}{pn-k-pm} \\
&\quad = \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{m=1}^{n} \binom{n-1}{m-1}^2 \binom{2k+2p(m-1)}{k+p(m-1)} \binom{2(pn-k-p(m-1))}{pn-k-p(m-1)} \\
&\quad \equiv \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{m=1}^{n} \binom{n-1}{m-1}^2 \binom{2k}{k} \binom{2(m-1)}{(m-1)} \binom{2(n-m)}{n-m} \binom{2(p-k)}{p-k} \pmod{p}.
\end{aligned}$$

But for $1 \le k \le p-1$,

$$p \,\Big|\, \binom{2k}{k} \quad \text{or} \quad p \,\Big|\, \binom{2(p-k)}{p-k}.$$

Hence,

$$\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv 0 \pmod{p},$$

and we deduce (2.6). ☐

A simple corollary of Theorem 2.1 is that

$$\beta_p \equiv \beta_1 \equiv 4 \pmod{p^3}$$

for all prime numbers $p > 3$.

3. Almkvist–Zudilin Sequence

The study of the sequence $\{\beta_n\}_{n=1}^{\infty}$ is inspired by the fact that α_n appears as the coefficients of the power series given by (1.3). As we have seen above, β_n are coefficients of the power series given by (2.1). There is a third sequence that behaves similarly to both α_n and β_n. To motivate our discovery of this third sequence, we observe that F_1 and F_2 are modular forms associated with $\Gamma_0(6)_{+6}$ and $\Gamma_0(6)_{+3}$ respectively. Naturally, one would expect to have a third sequence arising from $\Gamma_0(6)_{+2}$. Indeed, in [5] it was shown that if

$$t_3(\tau) = \left(\frac{\eta(3\tau)\eta(6\tau)}{\eta(\tau)\eta(2\tau)}\right)^4 \quad \text{and} \quad F_3(\tau) = \frac{(\eta(\tau)\eta(2\tau))^3}{\eta(3\tau)\eta(6\tau)},$$

and $|t_3(\tau)|$ is sufficiently small, then

$$F_3(\tau) = \sum_{n=0}^{\infty} (-1)^n \gamma_n t_3^n(\tau),$$

where γ_n are the Almkvist–Zudilin numbers [1], given by

$$\gamma_n = \sum_{j=0}^{\lfloor n/3 \rfloor} (-1)^j \frac{3^{n-3j}(3j)!}{(j!)^3} \binom{n}{3j}\binom{n+j}{j}. \tag{3.1}$$

The numbers γ_n appear to satisfy the congruence

Conjecture 3.1.

$$\gamma_{pn} \equiv \gamma_n \pmod{p^3}$$

for all primes $p > 3$.

We have been unable to give a proof of Conjecture 3.1 as Gessel's method does not seem to work in this case.

4. Yang–Zudilin Sequence

For positive integers k and n, let

$$y_{k,n} = \sum_{j=0}^{n} \binom{n}{j}^k.$$

Around 2003, Zudilin realized that $y_{4,n}$ is associated with a certain modular form and modular function as in the case for the Apéry numbers, Domb numbers and the Almkvist–Zudilin numbers. This form and function were eventually obtained by Yang [11] (see [4] for the explicit forms of the form and function).

In this section, we will deduce that for primes $p \geq 7$,

$$y_{4,p} \equiv y_{4,1} \equiv 2 \pmod{p^5}$$

by showing the following more general result:

Theorem 4.1. *Suppose k is even, and $p > 3$ is a prime number for which $p-1 \nmid k$. Then*

$$y_{k,p} \equiv 2 \pmod{p^{k+1}}.$$

Proof. Observe that

$$p \,\Big|\, \binom{p}{j} \quad \text{for } 1 \leq j \leq p-1.$$

Hence it suffices to show that

$$p \,\Bigg|\, \sum_{j=1}^{p-1} \left(\frac{(p-1)!}{j!(p-j)!}\right)^k. \tag{4.1}$$

Now

$$\frac{(p-1)!}{j!(p-j)!} = \frac{1}{j} \prod_{i=1}^{p-j} \frac{p-i}{i} \equiv \frac{1}{j}(-1)^{p-j} \pmod{p}.$$

Thus, since k is even,

$$\sum_{j=1}^{p-1} \left(\frac{(p-1)!}{j!(p-j)!}\right)^k \equiv \sum_{j=1}^{p-1} \frac{1}{j^k} \equiv \sum_{j=1}^{p-1} j^k \pmod{p}. \tag{4.2}$$

But

$$\sum_{j=1}^{p-1} j^k \equiv \begin{cases} 0 \pmod{p}, & \text{if } p-1 \nmid k, \\ -1 \pmod{p}, & \text{if } p-1 \mid k. \end{cases} \tag{4.3}$$

By hypothesis $p-1 \nmid k$, therefore (4.1) follows from (4.2) and (4.3). This completes the proof. □

Theorem 4.1 does not have a generalization modulo p^{k+1} similar to Theorem 2.1. However, we have the following result:

Theorem 4.2. *Let $p > 3$ be prime and let $k > 1$ be an integer. Then*

$$y_{k,pn} \equiv y_{k,n} \pmod{p^3}.$$

Proof. When $k = 2$ we have

$$y_{2,n} = \sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n},$$

so

$$y_{2,pn} = \binom{2pn}{pn} \equiv \binom{2n}{n} \equiv y_{2,n} \pmod{p^3}$$

by (2.3). This establishes the result for $k = 2$. For the remainder of the proof, suppose $k \geq 3$ and write

$$y_{pn} = \sum_{j=0}^{pn} \binom{pn}{j}^k = T_1 + T_2,$$

where

$$T_1 = \sum_{j=0}^{n} \binom{pn}{jp}^k,$$

and

$$T_2 = \sum_{j=1}^{p-1} \sum_{m=0}^{n-1} \binom{pn}{j+pm}^k.$$

Using (2.3), we deduce

$$T_1 \equiv y_n \pmod{p^3}.$$

Next, we rewrite T_2 as

$$T_2 = \sum_{j=1}^{p-1} \sum_{m=0}^{n-1} \binom{pn}{j+pm}^k = \sum_{j=1}^{p-1} \sum_{m=0}^{n-1} \binom{p+p(n-1)}{j+pm}^k. \tag{4.4}$$

By (2.7), we find that

$$\binom{p+p(n-1)}{j+pm} \equiv \binom{n-1}{m}\binom{p}{j} \equiv 0 \pmod{p}.$$

This implies that for $k \geq 3$ and $1 \leq j \leq p-1$,

$$\binom{p+p(n-1)}{j+pm}^k \equiv 0 \pmod{p^3}. \tag{4.5}$$

Substituting (4.5) into (4.4), we conclude that

$$T_2 \equiv 0 \pmod{p^3}$$

and this completes the proof of Theorem 4.2. □

5. Other Sequences

We hope that we have illustrated that sequences arising from the study of modular forms serve as a good source of numbers satisfying interesting congruences modulo certain power of primes. We end this article with a series of conjectures associated with various modular forms. The letter p always denotes a prime number.

Conjecture 5.1. *If*

$$z_2 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+n^2} \quad \textit{and} \quad x_2 = \frac{\eta^{12}(2\tau)}{z_2^6}$$

and

$$z_2 = \sum_{n=0}^{\infty} f_{2,n} x_2^n,$$

then

$$f_{2,pn} \equiv f_{2,n} \pmod{p^2} \quad \textit{when } p \equiv 1 \pmod 4.$$

Conjecture 5.2. *If*

$$z_3 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \quad \textit{and} \quad x_3 = \frac{\eta^6(\tau)\eta^6(3\tau)}{z_3^6}$$

and

$$z_3 = \sum_{n=0}^{\infty} f_{3,n} x_3^n,$$

then

$$f_{3,pn} \equiv f_{3,n} \pmod{p^2} \quad \textit{when } \left(\frac{p}{3}\right) = 1.$$

Conjecture 5.3. *If*

$$z_5 = \frac{\eta^5(\tau)}{\eta(5\tau)} \quad \textit{and} \quad x_5 = \frac{\eta^6(5\tau)}{\eta^6(\tau)}$$

and

$$z_5 = \sum_{n=0}^{\infty} f_{5,n} x_5^n,$$

then

$$f_{5,pn} \equiv f_{5,n} \pmod{p^3} \quad \textit{for all primes } p, \textit{ including } p = 2.$$

Conjecture 5.4. *If*

$$z_7 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+2n^2} \quad \textit{and} \quad x_7 = \frac{\eta^3(\tau)\eta^3(7\tau)}{z_7^3}$$

and

$$z_7 = \sum_{n=0}^{\infty} f_{7,n} x_7^n,$$

then

$$f_{7,pn} \equiv f_{7,n} \pmod{p^2} \quad \textit{when} \left(\frac{p}{7}\right) = 1.$$

Conjecture 5.5. *If*

$$z_{11} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+3n^2} \quad \textit{and} \quad x_{11} = \frac{\eta^2(\tau)\eta^2(11\tau)}{z_{11}^2}$$

and

$$z_{11} = \sum_{n=0}^{\infty} f_{11,n} x_{11}^n,$$

then

$$f_{11,pn} \equiv f_{11,n} \pmod{p^2} \quad \textit{when} \left(\frac{p}{11}\right) = 1.$$

Conjecture 5.6. *If*

$$z_{23} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+6n^2} \quad \textit{and} \quad x_{23} = \frac{\eta(\tau)\eta(23\tau)}{z_{23}}$$

and

$$z_{23} = \sum_{n=0}^{\infty} f_{23,n} x_{23}^n,$$

then

$$f_{23,pn} \equiv f_{23,n} \pmod{p} \quad \textit{when} \left(\frac{p}{23}\right) = 1.$$

Conjecture 5.7. *If*

$$Z_{23} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{2m^2+mn+3n^2} \quad \textit{and} \quad X_{23} = \frac{\eta(\tau)\eta(23\tau)}{Z_{23}}$$

and

$$Z_{23} = \sum_{n=0}^{\infty} F_{23,n} X_{23}^n,$$

then

$$F_{23,pn} \equiv F_{23,n} \pmod{p} \quad \textit{when} \left(\frac{p}{23}\right) = 1.$$

Remarks. One can verify that

$$f_{2,n} = 64^n \frac{\left(\frac{1}{4}\right)_n^2}{(n!)^2} \quad \text{and} \quad f_{3,n} = 108^n \frac{\left(\frac{1}{6}\right)_n \left(\frac{1}{3}\right)_n}{(n!)^2},$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$. There are no known closed forms for $f_{r,n}$ for $r = 5, 7, 11$ and 23 but they satisfy certain recurrence relations. The functions z_r and x_r, for $r = 3$, 7, 11 and 23, were studied in [3].

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