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On the Equivalence of Ramanujan's Partition Identities and a Connection with the Rogers–Ramanujan Continued Fraction

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A famous identity of Ramanujan connected with partitions modulo 5 is shown to be equivalent to another identity of Ramanujan. The latter identity is used to establish a differential equation for the Rogers–Ramanujan continued fraction found in Ramanujan's lost notebook. We also prove that two other identities of Ramanujan are equivalent, one of which is associated with Ramanujan's partition congruence modulo 7. Last, we give a new proof of the transformation formula for the Dedekind eta-function, which is used in our proofs of equivalence.

1. INTRODUCTION

Let

$$(a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1}), \qquad |q|<1,$$

and

$$f(-q) := (q;q)_\infty.$$

Note that if $q=e^{2\pi iz}$, Im $z>0$, then $q^{1/24}f(-q)=\eta(z)$, where $\eta(z)$ is the Dedekind eta-function.

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In a famous manuscript on the partition function $p(n)$ and the tau-function $\tau(n)$, recently published with the lost notebook, Ramanujan [9, p. 139] recorded the identities

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{q^n}{(1-q^n)^2} = q\frac{f^5(-q^5)}{f(-q)} \tag{1.1}$$

and

$$1-5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^n}{1-q^n} = \frac{f^5(-q)}{f(-q^5)} \tag{1.2}$$

where $(n/5)$ is the Legendre symbol. There are several proofs of (1.1) and (1.2), and references may be found in our paper [2], where a new proof of (1.1) is given. In proving these two identities using modular forms of Nebentypus, Raghavan [8] remarked that his proofs ''throw some light on the pair of identities being ''allied''.'' In this paper, we employ Hecke's theory of correspondence between Fourier series and Dirichlet series to show that (1.1) and (1.2) are equivalent. Thus, as Ramanujan implicitly implied, (1.1) and (1.2) are truly companion identities.

Identity (1.1) was employed by Ramanujan [9, pp. 139–140] to give a short proof of his famous congruence $p(5n+4) \equiv 0 \pmod 5$. See also our paper [2]. In Section 2, we show that a differential equation satisfied by the Rogers–Ramanujan continued fraction follows easily from (1.2). This differential equation was stated by Ramanujan (without proof and in a slightly different form) in [9, p. 363].

Our work is motivated by the transformation formula for the Dedekind eta-function. In Section 3, we utilize the Weierstrass $\wp$-function to provide a new, short proof of this transformation formula.

In Section 4, we prove the equivalence of (1.1) and (1.2), and in Section 5 we prove the equivalence of Ramanujan's famous partition identity connected with partitions modulo 7 and a companion identity found in [9, p. 145]. These two identities were also discussed in Raghavan's paper [8].

2. IDENTITY (1.2) AND ROGERS–RAMANUJAN'S CONTINUED FRACTION

Denote the Rogers–Ramanujan continued fraction by

$$F(q) = \frac{q^{1/5}}{1}\;+\;\frac{q}{1}\;+\;\frac{q^2}{1}\;+\;\frac{q^3}{1}\;+\cdots, \qquad |q|<1.$$

It is well known that [13; 5, pp. 290–294]

$$F(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{2.1}$$

On page 363 of his Lost Notebook [9], Ramanujan claimed that $F(q)$ satisfies the differential equation

$$5q \frac{dF(q)}{dq} = \frac{f^5(-q)}{f(-q^5)} F(q). \tag{2.2}$$

We now give a proof of (2.2). From (2.1), we deduce that

$$\log F(q) = \log q^{1/5} + \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \log(1-q^n). \tag{2.3}$$

Differentiating (2.3) with respect to q, we have

$$\frac{1}{F(q)} \frac{dF(q)}{dq} = \frac{1}{5q} + \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{-nq^n}{1-q^n}.$$

Therefore,

$$5q \frac{dF(q)}{dq} = \left(1 - 5 \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^n}{1-q^n}\right) F(q).$$

Using (1.2), we deduce (2.2).

3. DEDEKIND η TRANSFORMATION FORMULA

We have already seen how (1.1) and (1.2) have connections with partition theory and the Rogers–Ramanujan continued fraction, respectively. It is therefore interesting to see that they are in fact equivalent.

We first establish a clue which will lead to the result.

LEMMA 3.1. *Let* Im $z > 0$. *We have*

$$e^{2\pi i z/24} f(-e^{2\pi i z}) = \frac{1}{\sqrt{-iz}} e^{-2\pi i/24z} f(-e^{-2\pi i/z}). \tag{3.1}$$

Lemma 3.1 gives the famous transformation formula for the Dedekind η-function for which there are many proofs. For example, see [11]. Perhaps the following proof is new.

Proof. We recall that [3, p. 69] if $q = e^{\pi i \omega_2/\omega_1}$, then

$$16q\left(\frac{\pi}{\omega_1}\right)^6 f^{12}(-q^2) = (e_1 - e_2)(e_1 - e_3)(e_3 - e_2), \tag{3.2}$$

where $e_1 = \wp(\omega_1, \omega_2; \omega_1/2)$, $e_2 = \wp(\omega_1, \omega_2; \omega_2/2)$, and $e_3 = \wp(\omega_1, \omega_2; (\omega_1 + \omega_2)/2)$. Here, $\wp(\omega_1, \omega_2; z)$ is the Weierstrass elliptic function with periods ω_1 and ω_2. Applying (3.2) with $\omega_1 = 1$ and $\omega_2 = \tau$, we have

$$16e^{\pi i\tau}\pi^6 f^{12}(-e^{2\pi i\tau}) = (e_1 - e_3)(e_1 - e_2)(e_3 - e_2), \tag{3.3}$$

where $e_1 = \wp(1, \tau; 1/2)$, $e_2 = \wp(1, \tau; \tau/2)$ and $e_3 = \wp(1, \tau; (1 + \tau)/2)$.

Next, set $\omega_1 = \tau$ and $\omega_2 = -1$, so that $q = e^{-\pi i/\tau}$. Note that

$$\wp(\tau, -1; z) = \wp(1, \tau; z)$$

since the lattice generated by -1 and τ is the same as the lattice generated by 1 and τ. Therefore,

$$e_1' = \wp\left(\tau, -1; \frac{\tau}{2}\right) = \wp\left(1, \tau; \frac{\tau}{2}\right) = e_2.$$

Similarly, $e_2' = e_1$ and $e_3' = e_3$. Applying (3.2) with $q = e^{-\pi i/\tau}$, we obtain

$$\begin{aligned} 16e^{-\pi i/\tau}\left(\frac{\pi}{\tau}\right)^6 f^{12}(-e^{-2\pi i/\tau}) &= (e_1' - e_2')(e_1' - e_3')(e_3' - e_2') \\ &= (e_2 - e_1)(e_2 - e_3)(e_3 - e_1). \end{aligned} \tag{3.4}$$

Dividing (3.3) by (3.4) and simplifying, we have

$$e^{\pi i\tau} f^{12}(-e^{2\pi i\tau}) = -\tau^{-6} e^{-\pi i/\tau} f^{12}(-e^{-2\pi i/\tau}).$$

Taking the 12th root on both sides, we have

$$e^{\pi i\tau/12} f(-e^{2\pi i\tau}) = \epsilon\tau^{-1/2} e^{-\pi i/12\tau} f(-e^{-2\pi i/\tau}),$$

where ϵ is a 24th root of unity. Setting $\tau = i$, we deduce that $\epsilon = \sqrt{i}$ to complete the proof.

4. FUNCTIONAL EQUATIONS, HECKE'S THEORY, AND THE EQUIVALENCE OF (1.1) AND (1.2)

Now, let $q = e^{2\pi i\tau}$. By Lemma 3.1, the right hand side of (1.1) is

$$\frac{\left(e^{10\pi i\tau/24} f(-e^{10\pi i\tau})\right)^5}{e^{2\pi i\tau/24} f(-e^{2\pi i\tau})} = \frac{\sqrt{-i\tau}}{\left(\sqrt{-5i\tau}\right)^5} \frac{\left(e^{-2\pi i/120\tau} f(-e^{-2\pi i/5\tau})\right)^5}{e^{-2\pi i/24\tau} f(-e^{-2\pi i/\tau})}$$

$$= \frac{-1}{5^2\sqrt{5}\,\tau^2} \frac{f^5(-e^{-2\pi i/5\tau})}{f(-e^{-2\pi i/\tau})}.$$

Setting $q_1 = e^{-2\pi i/5\tau}$, we rewrite the equality above in the form

$$q\frac{f^5(-q^5)}{f(-q)} = \frac{-1}{5^5\sqrt{5}\,\tau^2} \frac{f^5(-q_1)}{f\left(-q_1^5\right)}.$$

This shows that in some sense the right-hand side of (1.2) can be obtained from the right-hand side of (1.1) and vice versa. This provides the motivation to show the equivalence of (1.1) and (1.2). To achieve this aim, it suffices to show that

$$\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2} = \frac{-1}{5^2\sqrt{5}\,\tau^2}\left(1 - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq_1^n}{1-q_1^n}\right). \tag{4.1}$$

If we write

$$g(\tau) = \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{e^{2\pi i n\tau}}{(1-e^{2\pi i n\tau})^2} \quad \text{and} \quad h(\tau) = 1 - 5\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{ne^{2\pi i n\tau}}{1-e^{2\pi i n\tau}},$$

then (4.1) may be written as

$$g(\tau) = \frac{-1}{5^2\sqrt{5}\,\tau^2} h\left(\frac{-1}{5\tau}\right). \tag{4.2}$$

Before proceeding further, we state the following results.

LEMMA 4.1 (Functional Equation for $\zeta(s)$). *For all complex numbers s,*

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\frac{s\pi}{2}\Gamma(s)\zeta(s).$$

Proof. See [1, p. 259, Theorem 12.7].

LEMMA 4.2 (Functional Equation for $L(s,\chi)$ with χ a Primitive Character mod k). *For all complex numbers s,*

$$L(1-s,\chi) = (2\pi)^{-s}\Gamma(s)k^{s-1}\left(e^{-\pi is/2} + \chi(-1)e^{\pi is/2}\right)G(\chi)L(s,\chi),$$

where $G(\chi) = \sum_{h=1}^{k-1}\chi(h)e^{2\pi ih/k}$.

Proof. See [1, p. 263, Theorem 12.11].

LEMMA 4.3. *Suppose that for some positive constant c,* $a_n,\ b_n = O(n^c)$. *For* $\operatorname{Im} z > 0$, *let*

$$u(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi in\tau}, \qquad v(\tau) = \sum_{n=0}^{\infty} b_n e^{2\pi in\tau}.$$

For $\sigma > c+1$, *set*

$$\phi(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad \psi(s) = \sum_{n=1}^{\infty} b_n n^{-s}.$$

Suppose $\phi(s)$ *and* $\psi(s)$ *have analytic continuations into the entire complex plane. For all s, define*

$$\Phi(s) = (2\pi)^{-s}\Gamma(s)\phi(s) \qquad \text{and} \qquad \Psi(s) = (2\pi)^{-s}\Gamma(s)\psi(s).$$

Then the following are equivalent:

(I) *For certain positive numbers A and k and for some complex number C, the function* $\Phi(s) + A^{-s/2}(a_0/s + Cb_0/(k-s))$ *is entire and bounded in every vertical strip, and*

$$\Phi(s) = CA^{k/2-s}\Psi(k-s).$$

(II) $u(\tau) = CA^{k/2}(A\tau/i)^{-k}v(-1/A\tau)$.

Proof. See [7, V-6–V-7].

We are now ready to prove (4.2). The Dirichlet series associated to $g(\tau)$ is $L(s,\chi)\zeta(s-1)$, and the Dirichlet series associated to $h(\tau)$ is $-5L(s-1,\chi)\zeta(s)$, where $\chi = (\cdot/5)$. So we may let

$$\phi(s) = L(s,\chi)\zeta(s-1) \qquad \text{and} \qquad \psi(s) = -5L(s-1,\chi)\zeta(s).$$

Therefore,

$$\Phi(s) = (2\pi)^{-s}\Gamma(s)L(s,\chi)\zeta(s-1) \quad \text{and}$$
$$\Psi(s) = -(2\pi)^{-s}\Gamma(s)5L(s-1,\chi)\zeta(s).$$

Applying Lemma 4.2 with $\chi = (\cdot/5)$, and $G(\chi) = \sqrt{5}$ [6, pp. 197–201], we have

$$L(1-s,\chi) = (2\pi)^{-s}\Gamma(s)5^{s-1}2\cos\frac{\pi s}{2}\sqrt{5}\,L(s,\chi). \tag{4.3}$$

Now, by Lemma 4.1 and (4.3),

$$\begin{aligned}\Psi(2-s) &= -5(2\pi)^{-(2-s)}\Gamma(2-s)L(1-s,\chi)\zeta(2-s)\\ &= -5(2\pi)^{-(2-s)}\Gamma(2-s)(2\pi)^{-s}\Gamma(s)5^{s-1}2\cos\frac{\pi s}{2}\sqrt{5}\,L(s,\chi)\\ &\quad\times 2^{2-s}\pi^{1-s}\cos\left(\frac{(s-1)\pi}{2}\right)\Gamma(s-1)\zeta(s)\\ &= 5^s\sqrt{5}\,\Phi(s),\end{aligned}$$

since $\Gamma(z)\Gamma(1-z) = \pi/\sin\pi z$ [1, p. 250].

Now, $\Phi(s)$ has a pole at $s=2$ with residue

$$(2\pi)^{-2}\Gamma(2)L(2,\chi) = \frac{1}{5^2\sqrt{5}},$$

since [12, p. 31, Theorem 4.2]

$$L(2,\chi) = \frac{4\pi}{5^2\sqrt{5}}.$$

Therefore, $P(s) := \Phi(s) + 5^{-s/2}/(5\sqrt{5}(2-s))$ is entire. The fact that $P(s)$ is bounded in every vertical strip follows from the bounds of $\zeta(s)$, $L(s,\chi)$ [1, pp. 270–272] and $\Gamma(s)$ [4, p. 223] in vertical strips. More precisely, if we let $s = \sigma + it$, then given $a \le \sigma \le b$, there exist A_1, A_2, A_3, α, β, and γ depending on a and b such that for $|t| \ge 1$,

$$|\zeta(s-1)| \le A_1|t|^{\alpha},$$
$$|L(s,\chi)| \le A_2|t|^{\beta},$$

and

$$|\Gamma(s)| \le A_3|t|^{\gamma}e^{-\pi|t|/2}.$$

Thus, $P(s)$ is uniformly bounded for $|t| \geq 1$ and $a \leq \sigma \leq b$. $P(s)$ is clearly bounded in the remaining region where $|t| \leq 1$ since it is an entire function. Now, since condition (I) of Lemma 4.3 is satisfied with $a_0 = 0$, $b_0 = 1$, $A = 5$, $k = 2$, and $C = 1/5\sqrt{5}$, we conclude that

$$\begin{aligned} g(\tau) &= \frac{1}{5\sqrt{5}} 5\left(\frac{5\pi}{i}\right)^{-2} h\left(\frac{-1}{5\tau}\right) \\ &= \frac{-1}{5^2\sqrt{5}\,\tau^2} h\left(\frac{-1}{5\tau}\right), \end{aligned}$$

which is (4.2).

5. EQUIVALENCE OF TWO OTHER RAMANUJAN IDENTITIES

In the aforementioned manuscript, Ramanujan [9, p. 145] wrote down the two identities

$$\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) q^n \frac{(1+q^n)}{(1-q^n)^3} = qf^3(-q)f^3(-q^7) + 8q^2 \frac{f^7(-q^7)}{f(-q)} \tag{5.1}$$

and

$$8 - 7\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{n^2 q^n}{1-q^n} = 49qf^3(-q)f^3(-q^7) + 8\frac{f^7(-q)}{f(-q^7)}. \tag{5.2}$$

We will apply the ideas illustrated in the previous two sections to show the equivalence of (5.1) and (5.2). By Lemma 3.1, we observe that

$$\begin{aligned} &qf^3(-q)f^3(-q^7) + 8q^2 \frac{f^7(-q^7)}{f(-q)} \\ &\qquad = -iq_1 \frac{f^3(-q_1)f^3(-q_1^7)}{7\sqrt{7}\,\tau^3} - 8i \frac{f^7(-q_1)}{7^3\sqrt{7}\,\tau^3 f(-q_1^7)}, \end{aligned}$$

where $q_1 = e^{-2\pi i/7\tau}$. Hence, to show the equivalence of these two identities, it suffices to show that

$$\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) q^n \frac{(1+q^n)}{(1-q^n)^3} = \frac{-i}{7^3\sqrt{7}\,\tau^3} \left(8 - 7\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{n^2 q_1^n}{1-q_1^n}\right).$$

If we write

$$g_1(\tau) = \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) e^{2\pi i n\tau} \frac{(1 - e^{2\pi i n\tau})}{(1 - e^{2\pi i n\tau})^3}$$

and

$$h_1(\tau) = 8 - 7\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) n^2 \frac{e^{2\pi i n\tau}}{1 - e^{2\pi i n\tau}},$$

then, by (5.3), we need to show that

$$g_1(\tau) = \frac{-i}{7^3\sqrt{7}\,\tau^3} h_1\left(\frac{-1}{7\tau}\right). \tag{5.4}$$

The associated Dirichlet series for $g_1(\tau)$ is $\phi_1(s) = L(s, \chi_1)\zeta(s-2)$, while the associated Dirichlet series for $h_1(\tau)$ is $\psi_1(s) = -7L(s-2, \chi_1)\zeta(s)$, where χ_1 is the Legendre symbol $(\cdot/7)$. Therefore,

$$\Phi_1(s) = (2\pi)^{-s}\Gamma(s)L(s, \chi_1)\zeta(s-2)$$

and

$$\Psi_1(s) = (2\pi)^{-s}\Gamma(s)(-7)L(s-2, \chi_1)\zeta(s).$$

By Lemmas 4.1 and 4.2 and the fact that $G(\chi_1) = i\sqrt{7}$ [6, pp. 197–201], we easily verify that

$$\Psi_1(3-s) = 7^s\sqrt{7}\,\Phi_1(s).$$

Now, the function $\Phi_1(s)$ has a pole at $s = 3$, with residue

$$(2\pi)^{-3}\Gamma(3)L(3, \chi_1).$$

Since [12, p. 31, Theorem 4.2]

$$L(3, \chi_1) = \frac{32\pi^3}{7^3\sqrt{7}},$$

we see that $P_1(s) := \Phi_1(s) + 7^{-s/2}(8/7^2(3-s))$ is entire. The function $P_1(s)$ is also bounded in every vertical strip (the argument is similar to that given in Section 5). Thus, (I) of Lemma 4.3 is satisfied with $a_0 = 0$, $b_0 = 8$, $k = 3$, $A = 7$, and $C = 1/7^2$. Therefore, we conclude that

$$g_1(\tau) = \frac{1}{7} 7^{3/2} \left(\frac{7\tau}{i} \right)^{-3} h_1\left(\frac{-1}{7\tau} \right)$$
$$= \frac{-i}{7^3 \sqrt{7}\, \tau^3} h_1\left(\frac{-1}{7\tau} \right),$$

which is (5.4).

REMARK. Garvan has recently given a short and elegant proof of (5.1). See [2].

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