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# NEW REPRESENTATIONS FOR APÉRY-LIKE SEQUENCES

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*Abstract*. We prove algebraic transformations for the generating series of three Apéry-like sequences. As application, we provide new binomial representations for the sequences. We also illustrate a method that derives three new series for $1/\pi$ from a classical Ramanujan's series.

§1. *Introduction.* The main aim of this paper is to study three remarkable sequences: the Apéry numbers [**2**]

$$\alpha_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n}^2, \tag{1.1}$$

the (alternating version of the) Domb numbers [**7**]

$$\delta_n = (-1)^n \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2, \tag{1.2}$$

and the AZ numbers (abbreviation for the Almkvist–Zudilin numbers defined in [**1**])

$$\xi_n = \sum_{k=0}^{n} (-1)^{n-k} 3^{n-3k} \frac{(3k)!}{k!^3} \binom{n}{3k}\binom{n+k}{n}. \tag{1.3}$$

These three sequences satisfy similar difference equations of order two and degree three [**1**, §7] and a common feature is that their generating series,

$$F_\alpha(z) = \sum_{n=0}^{\infty} \alpha_n z^n, \qquad F_\delta(z) = \sum_{n=0}^{\infty} \delta_n z^n \quad \text{and} \quad F_\xi(z) = \sum_{n=0}^{\infty} \xi_n z^n, \tag{1.4}$$

admit modular parametrizations via the Hauptmoduls of the three subgroups of index two lying between $\Gamma_0(6)$ and its normalizer in $SL_2(\mathbb{R})$ [**8**]. The series converge in some neighborhoods of the origin; more precisely, from the analysis of the corresponding linear differential equations for (1.4), their convergence domains are

$$|z| < (\sqrt{2}-1)^4, \qquad |z| < \tfrac{1}{4} \quad \text{and} \quad |z| < \tfrac{1}{9},$$

respectively.

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Exploiting these modular parametrizations we prove new algebraic relations between the series (1.4) in §2 (Theorem 2.2) and also their expressions by means of certain ${}_3F_2$-hypergeometric series in §§3–5 (Theorems 3.2, 4.2, and 5.1). Recall that the generalized hypergeometric series is defined by

$$ {}_pF_q\left(\begin{matrix} a_1, & a_2, & \dots, & a_p \\ b_1, & b_2, & \dots, & b_q \end{matrix} \,\middle|\, z\right) = 1 + \sum_{n=1}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} $$

where $(a)_n = a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is Pochhammer's symbol.

Among the benefits we can get so far from the very nice algebraic relations between $F_\alpha(z)$, $F_\delta(z)$, and $F_\xi(z)$ are several rapidly convergent Ramanujan-type series for $1/\pi$, which we indicate in §6. We leave a hope that many other things (like explicit evaluation of three-variable Mahler measures [**10**]) can be achieved using the results and techniques of this paper.

§2. *Three subgroups of* $\Gamma_0(6)$ *and algebraic relations.* Let

$$ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = e^{2\pi i \tau}, \operatorname{Im} \tau > 0, $$

denote Dedekind's eta function. As already mentioned, there are precisely three subgroups of index two lying between

$$ \Gamma_0(6) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \ (\mathrm{mod}\ 6) \right\} $$

and its normalizer in $SL_2(\mathbb{R})$; these are $\Gamma_0(6)_{+k} = \langle \Gamma_0(6), w_k \rangle$ for $k = 6, 3, 2$, where

$$ w_6 = \frac{1}{\sqrt{6}} \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}, \qquad w_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -2 \\ 6 & -3 \end{pmatrix}, $$

and

$$ w_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}. $$

The corresponding Hauptmoduls of these groups are†

$$ X_{6,6}(\tau) = \left( \frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^{12}, \tag{2.1} $$

$$ X_{6,3}(\tau) = \left( \frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)} \right)^{6}, \tag{2.2} $$

and

$$ X_{6,2}(\tau) = \left( \frac{\eta(3\tau)\eta(6\tau)}{\eta(\tau)\eta(2\tau)} \right)^{4}, \tag{2.3} $$

† We will use $X_{n,k}$ to denote a Hauptmodul for $\Gamma_0(n)_{+k}$ with genus 0. Other related modular forms will be indexed using $(n, k)$.

with the choice

$$Z_{6,6}(\tau) = \frac{(\eta(2\tau)\eta(3\tau))^7}{(\eta(\tau)\eta(6\tau))^5}, \tag{2.4}$$

$$Z_{6,3}(\tau) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2}, \tag{2.5}$$

and

$$Z_{6,2}(\tau) = \frac{(\eta(\tau)\eta(2\tau))^3}{\eta(3\tau)\eta(6\tau)} \tag{2.6}$$

of modular forms of weight 2 on $\Gamma_0(6)_{+k}$. We have the following modular parametrizations.

PROPOSITION 2.1. *In some neighborhoods of the* cusp $\tau = +i\infty$*,*

$$Z_{6,6}(\tau) = F_\alpha(X_{6,6}(\tau)), \tag{2.7}$$

$$Z_{6,3}(\tau) = F_\delta(X_{6,3}(\tau)), \tag{2.8}$$

*and*

$$Z_{6,2}(\tau) = F_\xi(X_{6,2}(\tau)). \tag{2.9}$$

*Proof.* See [6, 7], and [8, §3], respectively. □

THEOREM 2.2. *In a neighborhood of $y = 0$, for the series (1.4) we have*

$$\frac{1}{1+8y} F_\delta\left(\frac{y(1+9y)}{1+8y}\right) = \frac{1}{1+9y} F_\xi\left(\frac{y(1+8y)}{1+9y}\right), \tag{2.10}$$

$$\frac{1}{1-y} F_\alpha\left(\frac{y(1-9y)}{1-y}\right) = \frac{1}{1-9y} F_\xi\left(\frac{y(1-y)}{1-9y}\right), \tag{2.11}$$

$$\frac{1}{1+y} F_\alpha\left(\frac{y(1-8y)}{1+y}\right) = \frac{1}{1-8y} F_\delta\left(\frac{y(1+y)}{1-8y}\right). \tag{2.12}$$

To prove Theorem 2.2, we need a lemma.

LEMMA 2.3. *Let*

$$X_6(\tau) = \frac{\eta^9(6\tau)}{\eta^9(3\tau)} \frac{\eta^3(\tau)}{\eta^3(2\tau)}.$$

*Then we have*

$$X_{6,6}(\tau) = \frac{X_6(\tau)(1-8X_6(\tau))}{1+X_6(\tau)}, \tag{2.13}$$

$$X_{6,2}(\tau) = \frac{X_6(\tau)}{(1+X_6(\tau))(1-8X_6(\tau))} \tag{2.14}$$

*and*

$$X_{6,3}(\tau) = \frac{X_6(\tau)(1+X_6(\tau))}{1-8X_6(\tau)}. \tag{2.15}$$

*Proof.* The modular function $X_6(\tau)$ is a Hauptmodul with $X_6(0) = 1/8$, $X_6(i\infty) = 0$, $X_6(1/3) = \infty$, and $X_6(1/2) = -1$. Now $X_{6,6}(1/2) =$

$X_{6,6}(1/3) = \infty$. Hence the function

$$\frac{1 + X_6(\tau)}{X_6(\tau)} X_{6,6}(\tau)$$

is a modular function on $\Gamma_0(6)$ with no poles in the fundamental domain associated with $\Gamma_0(6)$ and must therefore be a polynomial in $X_6(\tau)$. By comparing the coefficients of the $q$-series expansions of the functions, we deduce that

$$\frac{1 + X_6(\tau)}{X_6(\tau)} X_{6,6}(\tau) = 1 - 8X_6(\tau).$$

Next, $X_{6,2}(0) = X_{6,2}(1/2) = \infty$ and so

$$(1 + X_6(\tau))(1 + 8X_6(\tau))X_{6,2}(\tau) = X_6(\tau).$$

Finally, $X_{6,3}(0) = X_{6,3}(1/3) = \infty$. Hence,

$$\frac{1 - 8X_6(\tau)}{X_6(\tau)} X_{6,3}(\tau) = 1 + X_6(\tau). \qquad \square$$

*Remark* 1. Identities (2.13)–(2.15) can also be proved by realizing that $X_{6,2}(\tau)$, $X_{6,3}(\tau)$, and $X_{6,6}(\tau)$ are rational functions of $X_6(\tau)$. By using computer algebra such as `Maple`, we can then derive these relations. We then multiply both sides by suitable modular forms of weight $r$ and compute the power series on both sides in $q$. If the power series of both sides agree up to $q^{r+1}$, then the relations hold. In general, we need to verify that both sides of the modular forms on $\Gamma_0(N)$ of weight $r$ agree up to $q^{m(N,r)}$ where

$$m(N, r) > \frac{rN}{12} \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

Our proofs of subsequent modular identities will be done using this approach.

*Proof of Theorem 2.2.* From the definition (2.3), (2.2), (2.6) and (2.5) of $X_{6,2}(\tau)$, $X_{6,3}(\tau)$, $Z_{6,2}(\tau)$ and $Z_{6,3}(\tau)$, we deduce that

$$\sqrt{X_{6,3}(\tau)}Z_{6,3}(\tau) = \sqrt{X_{6,2}(\tau)}Z_{6,2}(\tau).$$

Using (2.9), (2.8), (2.14) and (2.15), we find that

$$\sqrt{\frac{x(1+x)}{1-8x}} F_\delta\left(\frac{x(1+x)}{1-8x}\right) = \sqrt{\frac{x}{(1+x)(1-8x)}} F_\xi\left(\frac{x}{(1+x)(1-8x)}\right),$$

where $x = X_6(\tau)$. Letting $x = y/(1 + 8y)$, we obtain (2.10).

Identities (2.11) and (2.12) can be proved in a similar way. $\square$

§3. *Hypergeometric expressions for the Domb numbers.* In a recent paper, using identities associated with the ${}_5F_4$-hypergeometric series, Rogers [**10**, equation (3.4)] derived the following formula relating $F_\delta(u)$ to a ${}_3F_2$-hypergeometric series.

THEOREM 3.1 [10, Theorem 3.1]. *For $|u|$ sufficiently small,*

$$ {}_3F_2\left(\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1\end{matrix} \middle| \frac{108u^2}{(1-4u)^3}\right) = (1-4u)\sum_{n=0}^{\infty}(-1)^n\delta_n u^n. $$

Rogers then deduced several new series for $1/\pi$ associated with the Domb numbers†, one of which is

$$ \frac{9+5\sqrt{3}}{\pi} = \sum_{n=0}^{\infty}(-1)^n\delta_n(6n+3-\sqrt{3})\left(\frac{3\sqrt{3}-5}{4}\right)^n. $$

Using the transformation formula [10, equation (3.6)]

$$ \begin{aligned} (1-4u)\,{}_3F_2&\left(\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1\end{matrix} \middle| -\frac{108u}{(1-16u)^3}\right) \\ &= (1-16u)\,{}_3F_2\left(\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1\end{matrix} \middle| \frac{108u^2}{(1-4u)^3}\right), \end{aligned} \tag{3.1} $$

we observe that the following theorem is true.

THEOREM 3.2. *For $|u|$ sufficiently small,*

$$ F_\delta(u) = \sum_{n=0}^{\infty}\delta_n u^n = \frac{1}{1+16u}\cdot {}_3F_2\left(\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1\end{matrix} \middle| \frac{108u}{(1+16u)^3}\right). \tag{3.2} $$

In this section, we prove Theorem 3.2. We also derive (3.1) and, as indicated in [10], deduce Theorem 3.1 from Theorem 3.2 and (3.1).

In order to derive (3.2), we need the following lemma.

LEMMA 3.3 [7, pp. 405–406]. *Let*

$$ X_3(\tau) = \left(1+\frac{1}{27}\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12}\right)^{-1} \quad \textit{and} \quad Z_{3,3}(\tau) = \left(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}\right)^2. $$

*If $X_{3,3}(\tau) = 4X_3(\tau)(1-X_3(\tau))$, then*

$$ Z_{3,3}(\tau) = {}_3F_2\left(\begin{matrix}\frac{1}{3}, \frac{1}{2}, \frac{2}{3} \\ 1, 1\end{matrix} \middle| X_{3,3}(\tau)\right), \tag{3.3} $$

$$ X_{3,3}(\tau) = \frac{108X_{6,3}(\tau)}{(1+16X_{6,3}(\tau))^3} \quad \textit{and} \quad Z_{3,3}(\tau) = Z_{6,3}(\tau)(1+16X_{6,3}(\tau)). $$

*Proof of Theorem 3.2.* Combining Lemma 3.3 and Proposition 2.1, we deduce, for $|u|$ sufficiently small, the required identity (3.2). □

† Some of the series, proved by Rogers in [10], were conjectured in a paper by the first author and Verrill [8].

*Proof of (3.1) and Theorem 3.1.* Note that

$$X_{3,3}(2\tau) = \frac{108X_{6,3}(\tau)^2}{(1+4X_{6,3}(\tau))^3}. \tag{3.4}$$

The left-hand side of (3.4) is a modular function on $\Gamma_0(6)_{+3}$ and it is therefore a rational function of $X_{6,3}(\tau)$. This identity can also be found in [**4**, (10.3)]. Next,

$$\frac{Z_{3,3}(\tau)}{Z_{3,3}(2\tau)} = \frac{1+16X_{6,3}(\tau)}{1+4X_{6,3}(\tau)}. \tag{3.5}$$

This can be found by computer algebra, since we know that (3.5) is a modular function on $\Gamma_0(6)_{+3}$ and hence a rational function of $X_{6,3}(\tau)$. To prove (3.5), we use the method mentioned in Remark 1.

Using (3.4), (3.5), and (3.3), we deduce immediately (3.1). Theorem 3.1 now follows immediately using Theorem 3.2 and (3.1). □

One advantage of our derivation of (3.2) is that we do not need to know the explicit form of $\delta_n$ given by (1.2). In fact, by comparing the coefficients of $u^n$ on both sides of (3.2), we obtain

$$\delta_n = \sum_{k=0}^{n}(-16)^{n-k}\binom{n}{k}\binom{2k}{k}^2\binom{n+2k}{n}.$$

COROLLARY 3.4. *We have the following binomial identity:*

$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}\binom{n}{k}^2 = \sum_{k=0}^{n}(-1)^k 2^{4(n-k)}\binom{n}{k}\binom{2k}{k}^2\binom{n+2k}{n}.$$

§4. *The AZ numbers.* The method we used to derive (3.2) can also be adopted to derive new representation of the series associated with analogues of the Domb numbers.

For the sequence (1.3), we need the following analogue of Lemma 3.3.

LEMMA 4.1. *Let*

$$X_2(\tau) = \left(1+\frac{1}{64}\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24}\right)^{-1} \quad \text{and} \quad Z_{2,2}(\tau) = \frac{\eta^8(\tau)+32\eta^8(4\tau)}{\eta^4(2\tau)}.$$

*If* $X_{2,2}(\tau) = 4X_2(1-X_2)$, *then*

$$Z_{2,2}(\tau) = {}_3F_2\left(\left.\begin{matrix}\frac14, \frac12, \frac34\\ 1, \ 1\end{matrix}\right| X_{2,2}(\tau)\right), \tag{4.1}$$

$$X_{2,2}(\tau) = \frac{256X_{6,2}(\tau)}{(1+27X_{6,2}(\tau))^4} \tag{4.2}$$

*and*

$$Z_{2,2}(\tau) = Z_{6,2}(\tau)(1+27X_{6,2}(\tau)). \tag{4.3}$$

*Proof.* Identity (4.1) can be found in [**5**, (4.11)]. To prove (4.2), we observe that $X_{2,2}(\tau)$ is a Hauptmodul for $\Gamma_0(2)_{+2}$ and hence a modular function for $\Gamma_0(6)_{+2}$. The function $X_{2,2}(\tau)$ is therefore a rational function of $X_{6,2}(\tau)$. Identity (4.3) is established in a similar way. □

Combining this lemma and Proposition 2.1, we arrive at the following theorem.

THEOREM 4.2. *For $|u|$ sufficiently small,*

$$F_\xi(u) = \sum_{n=0}^{\infty} \xi_n u^n = \frac{1}{1+27u} \cdot {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix} \,\middle|\, \frac{256u}{(1+27u)^4}\right). \tag{4.4}$$

By comparing the coefficients of $u^n$ on both sides of (4.4), we deduce a new representation for the AZ numbers (cf. (1.3)):

$$\xi_n = \sum_{k=0}^{n} (-27)^{n-k} \frac{(4k)!}{k!^4} \binom{n+3k}{4k}.$$

COROLLARY 4.3. *We have the following binomial identity:*

$$\sum_{k=0}^{n} (-1)^k 3^{n-3k} \frac{(3k)!}{k!^3} \binom{n}{3k} \binom{n+k}{n} = \sum_{k=0}^{n} (-1)^k 3^{3(n-k)} \frac{(4k)!}{k!^4} \binom{n+3k}{4k}.$$

There is an analogue for (3.1) and it is given by

$$\begin{aligned}(1+3u)\,{}_3F_2&\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \\ 1, 1 \end{matrix} \,\middle|\, \frac{256u}{(1+27u)^4}\right) \\ &= (1+27u)\,{}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{3}{4}, \frac{1}{2} \\ 1, 1 \end{matrix} \,\middle|\, \frac{256u^3}{(1+3u)^4}\right).\end{aligned} \tag{4.5}$$

This identity can be proved using the two identities

$$X_{2,2}(3\tau) = \frac{256X_{2,2}(\tau)}{(1+3X_{2,2}(\tau))^4}$$

and

$$\frac{Z_{2,2}(\tau)}{Z_{2,2}(3\tau)} = \frac{1+27X_{6,2}(\tau)}{1+3X_{6,2}(\tau)}.$$

The above identities can be established by means of the same method as in the proofs of (3.4) and (3.5).

Using (4.5), we obtain another representation for $F_\xi$, namely,

$$F_\xi(u) = \sum_{n=0}^{\infty} \xi_n u^n = \frac{1}{1+3u} \cdot {}_3F_2\left(\begin{matrix} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ 1, 1 \end{matrix} \,\middle|\, \frac{256u^3}{(1+3u)^4}\right).$$

§5. *The Apéry numbers.* Our new representations of both the Domb numbers and the AZ numbers are consequences of identities such as (3.2) and (4.4). The studies of the Domb numbers and the AZ numbers are motivated by the well-known Apéry numbers (1.1). As such, a natural question is to ask for the existence of an identity similar to (3.2) and (4.4) for the Apéry numbers. An example of such identity was given by Yang (see [**13**, §6]). We will give here a similar but simpler formula. It is the following.

THEOREM 5.1. *For sufficiently small $|y|$,*

$$(1+y)^{-1}F_\alpha\left(\frac{y(1-8y)}{1+y}\right) = (1-8y)^{-3/2}\cdot {}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac12\\ 1,\ 1\end{matrix}\,\middle|\,-\frac{64y(1+y)^3}{(1-8y)^3}\right); \tag{5.1}$$

*this identity specialized at $y=\frac{1}{16}(1-u-\sqrt{1-34u+u^2})$ in a neighborhood of $u=0$ reads as*

$$F_\alpha(u)=\sum_{n=0}^{\infty}\alpha_n u^n = \frac{17-u-\sqrt{1-34u+u^2}}{4\sqrt{2}(1+u+\sqrt{1-34u+u^2})^{3/2}} \times {}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac12\\ 1,\ 1\end{matrix}\,\middle|\,-\frac{1024u}{(1-u+\sqrt{1-34u+u^2})^4}\right).$$

*Proof.* Taking the square of both sides of Kummer's quadratic transform [**11**, p. 49, equation (2.3.2.1)],

$${}_2F_1\left(\begin{matrix}\frac14,\ \frac14\\ 1\end{matrix}\,\middle|\,z\right)=(1-z)^{-1/4}\cdot {}_2F_1\left(\begin{matrix}\frac18,\ \frac38\\ 1\end{matrix}\,\middle|\,\frac{-4z}{(1-z)^2}\right), \tag{5.2}$$

and applying Clausen's identity [**11**, p. 75, equation (2.5.7)]

$${}_2F_1\left(\begin{matrix}a,\ b\\ a+b+\frac12\end{matrix}\,\middle|\,z\right)^2={}_3F_2\left(\begin{matrix}2a,\ 2b,\ a+b\\ a+b+\frac12,\ 2a+2b\end{matrix}\,\middle|\,z\right)$$

to (5.2), we obtain

$${}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac12\\ 1,\ 1\end{matrix}\,\middle|\,z\right)=(1-z)^{-1/2}\cdot {}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix}\,\middle|\,\frac{-4z}{(1-z)^2}\right). \tag{5.3}$$

Choosing $z=-64x/((1-x)(1-9x)^3)$ in (5.3) and $u=x(1-x)/(1-9x)$ in (4.4) and applying (2.11) we obtain

$$\begin{aligned}\frac{1}{1-x}\,F_\alpha\left(\frac{x(1-9x)}{1-x}\right) &= \frac{1}{1-9x}\,F_\xi\left(\frac{x(1-x)}{1-9x}\right)\\ &= \frac{1}{1+18x-27x^2}\cdot {}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix}\,\middle|\,\frac{256x(1-x)(1-9x)^3}{(1+18x-27x^2)^4}\right)\\ &= \frac{1}{(1-x)^{1/2}(1-9x)^{3/2}}\cdot {}_3F_2\left(\begin{matrix}\frac12,\ \frac12,\ \frac12\\ 1,\ 1\end{matrix}\,\middle|\,-\frac{64x}{(1-x)(1-9x)^3}\right).\end{aligned}$$

Substituting $x=y/(1+y)$ in the latter identity results in (5.1). □

By comparing the coefficients of $y^n$ on both sides of (5.1), we deduce the following corollary.

COROLLARY 5.2. *For each $n = 0, 1, 2, \dots,$ we have the following binomial identity:*

$$\sum_{k,l} \alpha_k (-1)^{n-k} 2^{3l} \binom{k}{l} \binom{n-l}{k}$$
$$= \sum_{k,l} (-1)^k 2^l \binom{2k}{k}^3 \binom{3k}{n-k-l} \frac{(3k)!(6k+2l+1)!}{(6k+1)!l!(3k+l)!},$$

*where the Apéry numbers $\alpha_k$ are defined in (1.1).*

§6. *Applications to Ramanujan-type formulas for* $1/\pi$. Let us show how the transformations obtained in §§2–5 and the techniques of [**10, 12**] can be used for derivation of Ramanujan-type series for $1/\pi$ [**9**]. Our examples below are based on the *simplest* Ramanujan's formula

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^3}{n!^3} (4n+1)(-1)^n = \frac{2}{\pi}, \tag{6.1}$$

which was first proved by Bauer [**3**] in 1859.

THEOREM 6.1. *The following Ramanujan-type formulas for* $1/\pi$, *involving the Apéry (1.1), Domb (1.2), and AZ (1.3) numbers, are valid:*

$$\frac{\sqrt{2}}{8\pi} = \sum_{n=0}^{\infty} \left(2n+1-\frac{\sqrt{6}}{4}\right) \alpha_n (\sqrt{3}-\sqrt{2})^{4n+2}, \tag{6.2}$$

$$\frac{2(\sqrt{3}+\sqrt{2})}{5\pi} = \sum_{n=0}^{\infty} \left(2n+1-\frac{\sqrt{6}}{5}\right) \delta_n \left(\frac{(\sqrt{3}-\sqrt{2})^2}{8}\right)^n, \tag{6.3}$$

$$\frac{\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} (4n+1) \xi_n \frac{1}{3^{4n+1}}. \tag{6.4}$$

Note that the identity (6.4) is proved in [**8**] by a different method, while (6.2) and (6.3) are two conjectural entries from Tables 2 and 3 there.

*Proof.* Denote the ${}_3F_2$ series on the right-hand side in (5.1) by $G(z)$ and write the identities (2.11) with $y$ replaced by $y/(1+y)$, (2.12) and (5.1) as

$$\frac{1}{(1-8y)^{3/2}} G\left(-\frac{64y(1+y)^3}{(1-8y)^3}\right) = \frac{1}{1+y} F_\alpha\left(\frac{y(1-8y)}{1+y}\right)$$
$$= \frac{1}{1-8y} F_\delta\left(\frac{y(1+y)}{1-8y}\right) = \frac{1}{(1+y)(1-8y)} F_\xi\left(\frac{y}{(1+y)(1-8y)}\right). \tag{6.5}$$

Applying the differential operator $\theta = y(d/dy)$ to the four sides we obtain

$$\frac{12y}{(1-8y)^{5/2}} G\left(-\frac{64y(1+y)^3}{(1-8y)^3}\right)$$
$$+ \frac{1+20y-8y^2}{(1-8y)^{5/2}(1+y)} (\theta G)\left(-\frac{64y(1+y)^3}{(1-8y)^3}\right)$$

$$
\begin{aligned}
&= -\frac{y}{(1+y)^2} F_\alpha\left(\frac{y(1-8y)}{1+y}\right) + \frac{1-16y-8y^2}{(1+y)^2(1-8y)}\,(\theta F_\alpha)\left(\frac{y(1-8y)}{1+y}\right)\\
&= \frac{8y}{(1-8y)^2}\,F_\delta\left(\frac{y(1+y)}{1-8y}\right) + \frac{1+2y-8y^2}{(1-8y)^2(1+y)}\,(\theta F_\delta)\left(\frac{y(1+y)}{1-8y}\right).\\
&= \frac{y(7+16y)}{(1+y)^2(1-8y)^2}\,F_\xi\left(\frac{y}{(1+y)(1-8y)}\right)\\
&\quad + \frac{1+8y^2}{(1+y)^2(1-8y)^2}\,(\theta F_\xi)\left(\frac{y}{(1+y)(1-8y)}\right). \qquad (6.6)
\end{aligned}
$$

Now we specialize $y = (-22+9\sqrt{6})/4$; in the notation $\zeta = \sqrt{3}-\sqrt{2}$ we have

$$
\begin{gathered}
y = \frac{\zeta^3}{2\sqrt{2}}, \qquad 1-8y = 9\zeta^2, \qquad 1+y = \frac{9\zeta}{2\sqrt{2}},\\
1+20y-8y^2 = 27\sqrt{2}\zeta^3,\\
1-16y-8y^2 = 18\sqrt{2}\zeta^3, \qquad 1+2y-8y^2 = \frac{45\zeta^3}{\sqrt{2}},\\
1+8y^2 = 18(3+\sqrt{6})\zeta^4 \quad \text{and} \quad 7+16y = 9(3+2\sqrt{6})\zeta^2.
\end{gathered}
$$

Thus, the identities in (6.5) and (6.6) become

$$
\begin{aligned}
&\frac{1}{27\zeta^3}G(-1) = \frac{2\sqrt{2}}{9\zeta}F_\alpha(\zeta^4) = \frac{1}{9\zeta^2}F_\delta\left(\frac{\zeta^2}{8}\right) = \frac{2\sqrt{2}}{81\zeta^3}F_\xi\left(\frac{1}{81}\right),\\
&\frac{\sqrt{2}}{81\zeta^2}G(-1) + \frac{4}{81\zeta^3}(\theta G)(-1)\\
&\quad = -\frac{2\sqrt{2}\zeta}{81}F_\alpha(\zeta^4) + \frac{16\sqrt{2}}{81\zeta}(\theta F_\alpha)(\zeta^4)\\
&\quad = \frac{2\sqrt{2}}{81\zeta}F_\delta\left(\frac{\zeta^2}{8}\right) + \frac{10}{81\zeta^2}(\theta F_\delta)\left(\frac{\zeta^2}{8}\right)\\
&\quad = \frac{2\sqrt{2}(3+2\sqrt{6})}{729\zeta}F_\xi\left(\frac{1}{81}\right) + \frac{16(3+\sqrt{6})}{729\zeta^2}(\theta F_\xi)\left(\frac{1}{81}\right).
\end{aligned}
$$

Multiplying the first quadruple equality by $27\zeta^3(1-\sqrt{2}\zeta)$, the second one by $81\zeta^3$ and summing up, we deduce that

$$
\begin{aligned}
G(-1) + 4(\theta G)(-1) &= 2\sqrt{2}(4-\sqrt{6})\zeta^2 F_\alpha(\zeta^4) + 16\sqrt{2}\zeta^2(\theta F_\alpha)(\zeta^4)\\
&= (5-\sqrt{6})\zeta F_\delta\left(\frac{\zeta^2}{8}\right) + 10\zeta(\theta F_\delta)\left(\frac{\zeta^2}{8}\right)\\
&= \frac{4\sqrt{3}}{9}F_\xi\left(\frac{1}{81}\right) + \frac{16\sqrt{3}}{9}(\theta F_\xi)\left(\frac{1}{81}\right).
\end{aligned}
$$

Identity (6.1) tells us that the left-hand side is $2/\pi$; this leads to all three required formulas. □

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