Journal of Number Theory 130 (2010) 1898–1913

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

New analogues of Ramanujan's partition identities

Heng Huat Chan, Pee Choon Toh *

National University of Singapore, Department of Mathematics, S17, 10 Lower Kent Ridge Road, Singapore 119076

ARTICLE INFO

Article history:
Received 21 June 2009
Communicated by David Goss

MSC:
11P83
11F03

Keywords:
Modular forms
Partitions
Congruences
Dedekind eta function

ABSTRACT

We establish several new analogues of Ramanujan's exact partition identities using the theory of modular functions.

1. Introduction

Let $p(n)$ denote the number of unrestricted partitions of the non-negative integer n, then

$$p\left(5^j n + \delta_{5,j}\right) \equiv 0 \pmod{5^j}, \tag{1.1}$$

$$p\left(7^j n + \delta_{7,j}\right) \equiv 0 \pmod{7^{\lfloor j/2 \rfloor + 1}}, \tag{1.2}$$

$$p\left(11^j n + \delta_{11,j}\right) \equiv 0 \pmod{11^j}, \tag{1.3}$$

where $\delta_{\ell,j} = 1/24 \pmod{\ell^j}$. These are commonly known as the Ramanujan congruences. The congruence (1.3) was proved by A.O.L. Atkin [1] while the proofs of (1.1) and (1.2) have traditionally been attributed to G.N. Watson [18], although Ramanujan had already outlined the proofs in an un-

* Corresponding author.
E-mail addresses: matchh@nus.edu.sg (H.H. Chan), mattpc@nus.edu.sg, peechoon@gmail.com (P.C. Toh).

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2010.02.017

published manuscript [4]. More details about this and other results concerning $p(n)$ can be found in [13, Chapter 5].

In [15], Ramanujan proved (1.1) and (1.2) for $j=1$ and 2 using the following two identities

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\prod_{n=1}^{\infty}\frac{(1-q^{5n})^5}{(1-q^n)^6}, \tag{1.4}$$

and

$$\sum_{n=0}^{\infty} p(7n+5)q^n = 7\prod_{n=1}^{\infty}\frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q\prod_{n=1}^{\infty}\frac{(1-q^{7n})^7}{(1-q^n)^8}. \tag{1.5}$$

Identity (1.4) was described by G.H. Hardy as Ramanujan's most beautiful identity [16, p. xxxv] and Ramanujan [15] provided an elegant proof using the following identity,

$$q^{-1/5}\prod_{n=1}^{\infty}\frac{(1-q^{n/5})}{(1-q^{5n})} = \frac{1}{R(q)} - 1 - R(q), \tag{1.6}$$

where $R(q)$ is the Rogers–Ramanujan continued fraction given by

$$R(q) = \frac{q^{1/5}}{1} \genfrac{}{}{0pt}{}{}{+} \frac{q}{1} \genfrac{}{}{0pt}{}{}{+} \frac{q^2}{1} \genfrac{}{}{0pt}{}{}{+} \cdots.$$

Recently, H.-C. Chan [5, Theorem 2] proved an analogue of (1.6) involving Ramanujan's cubic continued fraction and obtained an analogue of (1.4). He showed that if

$$\sum_{n=0}^{\infty} a(n)q^n = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)(1-q^{2n})},$$

then

$$\sum_{n=0}^{\infty} a(3n+2)q^n = 3\prod_{n=1}^{\infty}\frac{(1-q^{3n})^3(1-q^{6n})^3}{(1-q^n)^4(1-q^{2n})^4}. \tag{1.7}$$

H.-C. Chan and S. Cooper [7] subsequently obtained another analogue of (1.4), namely,

$$\sum_{n=0}^{\infty} c(2n+1)q^n = 2\prod_{n=1}^{\infty}\frac{(1-q^{2n})^4(1-q^{6n})^4}{(1-q^n)^6(1-q^{3n})^6}, \tag{1.8}$$

where the generating function of $c(n)$ is given by

$$\sum_{n=0}^{\infty} c(n)q^n = \prod_{n=1}^{\infty}\frac{1}{(1-q^n)^2(1-q^{3n})^2}.$$

Again (1.8) was derived via an identity analogous to (1.6). (See also [10].)

The existence of identities (1.4), (1.5), (1.7) and (1.8) can all be explained by the theory of modular functions. Furthermore, they can be established without first proving identities similar to (1.6). The first author and R.P. Lewis [9] used Hecke operators to provide a uniform proof of (1.4), (1.5), as well as an identity due to H.S. Zuckerman [19] involving the coefficients $p(13n+6)$. In the next section, we extend their technique to prove (1.7), (1.8) and several new identities. To state these identities, let t be a positive integer and define the infinite product

$$E(t)=\prod_{n=1}^{\infty}\left(1-q^{tn}\right). \tag{1.9}$$

A generalized partition π is an expression of the form

$$\pi=\left[\prod_{t} t^{r_t}\right], \tag{1.10}$$

where r_t are integers and only finitely many r_t are nonzero. For a fixed partition π, we define the associated infinite product

$$A_\pi(j)=\prod_{t}\prod_{n=1}^{\infty}\left(1-q^{jtn}\right)^{r_t}=\prod_{t}E(jt)^{r_t}, \tag{1.11}$$

and the coefficients $p_\pi(n)$ by

$$\sum_{n=0}^{\infty}p_\pi(n)q^n=\frac{1}{A_\pi(1)}. \tag{1.12}$$

In this notation, $a(n)=p_{[1^1 2^1]}(n)$ and $c(n)=p_{[1^2 3^2]}(n)$. We also adopt the convention that $p_\pi(n)=0$ whenever n is negative.

Theorem 1. *The following identities hold.*

$$\sum_{n=0}^{\infty}p_{[1^1 7^1]}(2n-1)q^n=q\frac{E(2)^2E(14)^2}{E(1)^3E(7)^3}, \tag{1.13}$$

$$\sum_{n=0}^{\infty}p_{[1^4 5^4]}(2n-3)q^n=4q^2\frac{E(2)^4E(10)^4}{E(1)^8E(5)^8}+8q^3\frac{E(2)^8E(10)^8}{E(1)^{12}E(5)^{12}}, \tag{1.14}$$

$$\sum_{n=0}^{\infty}p_{[1^2 11^2]}(2n-3)q^n=2q^2\frac{E(2)^2E(22)^2}{E(1)^4E(11)^4}+2q^3\frac{E(2)^4E(22)^4}{E(1)^6E(11)^6}, \tag{1.15}$$

$$\sum_{n=0}^{\infty}p_{[1^1 5^1]}(3n-2)q^n=q\frac{E(3)E(15)}{E(1)^2E(5)^2}+3q^2\frac{E(3)^3E(15)^3}{E(1)^4E(5)^4}, \tag{1.16}$$

$$\sum_{n=0}^{\infty}p_{[1^1 23^1]}(2n-3)q^n=q^2\frac{E(2)E(46)}{E(1)^2E(23)^2}+q^3\frac{E(2)^2E(46)^2}{E(1)^3E(23)^3}, \tag{1.17}$$

$$\sum_{n=0}^{\infty}p_{[1^1 2^1]}(5n-3)q^n=3q\frac{E(5)E(10)}{E(1)^2E(2)^2}+25q^2\frac{E(5)^3E(10)^3}{E(1)^4E(2)^4}+125q^3\frac{E(5)^5E(10)^5}{E(1)^6E(2)^6}, \tag{1.18}$$

$$\sum_{n=0}^{\infty}p_{[1^1 11^1]}(3n-4)q^n=2q^2\frac{E(3)E(33)}{E(1)^2E(11)^2}+3q^3\frac{E(3)^2E(33)^2}{E(1)^3E(11)^3}+3q^4\frac{E(3)^3E(33)^3}{E(1)^4E(11)^4}, \tag{1.19}$$

$$\sum_{n=0}^{\infty} p_{[1^1 3^1]}(7n-8)q^n = 16q^2 \frac{E(7)E(21)}{E(1)^2E(3)^2} + 161q^3 \frac{E(7)^2E(21)^2}{E(1)^3E(3)^3} + 980q^4 \frac{E(7)^3E(21)^3}{E(1)^4E(3)^4} + 3773q^5 \frac{E(7)^4E(21)^4}{E(1)^5E(3)^5} + 9604q^6 \frac{E(7)^5E(21)^5}{E(1)^6E(3)^6} + 16\,807q^7 \frac{E(7)^6E(21)^6}{E(1)^7E(3)^7} + 16\,807q^8 \frac{E(7)^7E(21)^7}{E(1)^8E(3)^8}. \tag{1.20}$$

Besides proving (1.7) Chan [6] proved a higher power analogue of the Ramanujan congruence for $a(n)$. The corresponding analogue for $c(n)$ was subsequently established by Chan and Cooper [7]. The result for $a(n)$ is

Corollary 2. *(See H.-C. Chan [6].)*

$$a\big(3^j n + c_j\big) \equiv 0 \pmod{3^{2\lfloor j/2\rfloor+1}},$$

where $c_j = 1/8 \pmod{3^j}$.

In Section 3, we shall prove a different higher power congruence for $a(n)$, involving the prime 5, which leads to the following.

Corollary 3.

$$a\big(5^j n + d_j\big) \equiv 0 \pmod{5^{\lfloor \frac{j}{2}\rfloor}},$$

where $d_j = 1/8 \pmod{5^j}$.

Both Corollaries 2 and 3 were first discovered implicitly by P.C. Eggan [11].[1]

2. Proof of Theorem 1 and consequences

In this section, we shall illustrate how the identities in Theorem 1 may be constructed via the theory of modular functions. Most of the results that we need can be found in [2].

Let $\mathbb{H} = \{\tau \in \mathbb{C}\colon \operatorname{Im}\tau > 0\}$ be the complex upper half plane and Γ be any subgroup of $SL_2(\mathbb{Z})$. Elements of Γ act on $\mathbb{H}$ as linear fractional transformations and Γ has a connected fundamental domain. The points of compactification of the fundamental domain on the boundary of $\mathbb{H}$ are called cusps. Fix an integer k for a function $f(\tau)$ on $\mathbb{H}$. k is usually called the weight of $f(\tau)$, although in our treatment we are mainly interested in the case $k=0$. For any matrix $L = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$, we define the slash operator by

$$f \mid L = (ad-bc)^{k/2}(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right). \tag{2.1}$$

We say that $f(\tau)$ is a modular function (of weight 0) on Γ, if $f(\tau)$ is analytic in the interior of $\mathbb{H}$, meromorphic at each cusp of Γ and satisfies

$$f \mid V = f, \quad \text{for every } V \in \Gamma.$$

[1] We thank F.G. Garvan for this reference.

We define

$$\Gamma_0(m)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})\colon\ c\equiv 0 \pmod m\right\},$$

and

$$A_n=\begin{pmatrix} n & 0\\ 0 & 1\end{pmatrix},\qquad S=\begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix},\qquad W_e=\begin{pmatrix} ex & y\\ mz & ew\end{pmatrix},$$

where W_e is known as an Atkin–Lehner involution and satisfies $\det W_e=e$, $e\,\|\,m$. The action of W_e is independent of x, y, z and w in the definition and $W_e^2=I$ modulo $\Gamma_0(m)$. W_e is in the normalizer of $\Gamma_0(m)$ and we denote the group generated by $\Gamma_0(m)$ and W_e by $\Gamma_0(m)+W_e$. Finally, we define the Hecke operator U_ℓ^* for some prime ℓ by

$$f\;\big|\;U_\ell^*=\sum_{j=0}^{\ell-1} f\;\big|\;A_\ell^{-1}S^j. \tag{2.2}$$

Lemma 4. *Let ℓ be prime, $\ell\mid m$. If $f(\tau)$ is a modular function on $\Gamma_0(m)$ then $f\mid U_\ell^*$ is a modular function on $\Gamma_0(m)$. Furthermore, if $\ell^2\mid m$ then $f\mid U_\ell^*$ is a modular function on $\Gamma_0(m/\ell)$.*

Proof. [2, Lemma 7]. □

Lemma 5. *Let $e\,\|\,m$. If $f(\tau)$ is a modular function on $\Gamma_0(m)$ then $f\mid W_e$ is a modular function on $\Gamma_0(m)$.*

Proof. [2, Lemma 10]. □

Lemma 6. *Let ℓ be prime, $\ell\mid m$, $e\,\|\,m$ and $(\ell,e)=1$. If $f(\tau)$ is a modular function on $\Gamma_0(m)$ then*

$$\big(f\;\big|\;U_\ell^*\big)\;\big|\;W_e=(f\mid W_e)\;\big|\;U_\ell^*.$$

Proof. Following the proof of [2, Lemma 11], since $(\ell,e)=1$, we can choose

$$W_e\equiv\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\pmod{\ell}\quad\text{and}\quad M_j=\begin{pmatrix} \ell & -j\\ 0 & 1\end{pmatrix}.$$

By the definition (2.2), we have

$$f\;\big|\;U_\ell^*=\sum f\;\big|\;M_j^{-1},$$

thus it suffices to prove that each M_j^{-1} commutes with W_e. A straightforward computation reveals that $M_j^{-1}W_eM_j$ is a matrix with integral entries and is equal to some W_e'. Since these are involutions, $f\mid M_j^{-1}W_eM_jW_e^{-1}=f$, hence $f\mid M_j^{-1}W_e=f\mid W_eM_j^{-1}$. □

Lemma 7. *Let ℓ be prime, $\ell^2\mid m$, $e\,\|\,m$ and $(\ell,e)=1$. If $f(\tau)$ is a modular function on $\Gamma_0(m)+W_e$ then $f\mid U_\ell^*$ is a modular function on $\Gamma_0(m/\ell)+W_e$.*

Proof. By the hypothesis, f is also a modular function on $\Gamma_0(m)$, hence $f(\tau) \mid U_\ell^*$ is a modular function on $\Gamma_0(m/\ell)$. Thus it suffices to show that $f(\tau) \mid U_\ell^*$ is invariant under W_e which follows directly from Lemma 6. □

From this point onwards, we only consider modular functions that vanish at all cusps. If we let $q = e^{2\pi i\tau}$, the Fourier expansion of the modular function $f(\tau)$ is given by

$$f(\tau) = \sum_{n=1}^{\infty} a(n)q^n.$$

For a prime ℓ, the Hecke operator U_ℓ is defined by

$$f \mid U_\ell = \sum_{n=1}^{\infty} a(\ell n)q^n. \tag{2.3}$$

Lemma 8. *Let ℓ be prime, $\ell \mid m$. If $f(\tau)$ is a modular function (of weight 0) on $\Gamma_0(m)$ that vanishes at all cusps, then $\ell f \mid U_\ell = f \mid U_\ell^*$.*

Proof. [2, Lemma 14]. □

Lemma 9. *For a product of two Fourier expansions, we have*

$$\left(\sum_{n=1}^{\infty} a(n)q^{\ell n} \sum_{n=1}^{\infty} b(n)q^n\right) \Bigm| U_\ell = \left(\sum_{n=1}^{\infty} a(n)q^n\right)\left(\sum_{n=1}^{\infty} b(\ell n)q^n\right).$$

This result was known to Atkin and O'Brien [3, Eq. (28)], [12, p. 126] and plays a crucial role in our proof of the identities in Theorem 1. We have seen that the U_ℓ operator maps a modular function to another modular function on some Γ. Let $\mathbb{H}^*$ be the union of $\mathbb{H}$ and the set of cusps of Γ, if the genus of $\Gamma\backslash\mathbb{H}^*$ is zero, then we know the function field of $\Gamma\backslash\mathbb{H}^*$ over $\mathbb{C}$ is generated by a certain modular function h, known also as a hauptmodul. For brevity we simply say Γ has genus zero.

Proof of Theorem 1. Fix a prime ℓ and a partition π and define $A(j) = A_\pi(j)$ according to (1.11). ℓ and π are chosen such that

$$\phi = \phi(\tau) = q^\alpha \frac{A(\ell^2)}{A(1)} \quad \text{with } \alpha = \frac{\ell^2 - 1}{24} \sum_t t r_t, \tag{2.4}$$

is a modular function on $\Gamma_0(\ell^2 e) + W_e$, where $(\ell, e) = 1$. In addition, $\Gamma_0(\ell e) + W_e$ should have genus zero, with hauptmodul defined by

$$h = h(\tau) = q^\beta \frac{A(\ell)^\gamma}{A(1)^\gamma} \quad \text{where } \beta = \frac{(\ell - 1)\gamma}{24} \sum_t t r_t. \tag{2.5}$$

In the above, α, β and γ are all positive integers.

From Lemma 9,

$$\phi \mid U_\ell = \left(q^\alpha A\left(\ell^2\right) \sum_{n=0}^{\infty} p_\pi(n)q^n\right) \Bigm| U_\ell = A(\ell) \sum_{n=0}^{\infty} p_\pi(\ell n - \alpha)q^n. \tag{2.6}$$

On the other hand, by Lemma 7, $\phi \mid U_\ell$ is a modular function on $\Gamma_0(\ell e) + W_e$ which has genus zero. Thus, it must be generated by the hauptmodul h. For each of the identities in Theorem 1, as well as (1.7) and (1.8), a computation reveals that $\phi \mid U_\ell$ are all polynomials in h with integral coefficients, i.e.

$$\sum_{n=0}^{\infty} p_\pi(\ell n - \alpha) q^n = \frac{1}{A(\ell)} \sum_{j=1}^{N} a_j h^j, \quad a_j \in \mathbb{Z}. \qquad \square \tag{2.7}$$

We shall illustrate by proving identity (1.20) in detail. The Dedekind eta function is defined as

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} \left(1 - q^n\right) = q^{1/24} E(1), \quad \text{where } q = e^{2\pi i \tau}. \tag{2.8}$$

For a generalized partition π, we define the eta-quotient associated with π by

$$\eta_\pi(\tau) = \prod_t \eta(t\tau)^{r_t}. \tag{2.9}$$

Many examples of modular functions of weight 0 on $\Gamma_0(m) + W_e$ can be given in terms of eta-quotients. In particular, $\Gamma_0(21) + W_3$ has genus zero, and is generated by the hauptmodul

$$h = \frac{\eta(7\tau)\eta(21\tau)}{\eta(\tau)\eta(3\tau)} = q \frac{A(7)}{A(1)},$$

where $A(j) = E(j)E(3j)$ and $\pi = [1^1 3^1]$. Now using the results in [8], we can check that

$$\phi = \frac{\eta(49\tau)\eta(147\tau)}{\eta(\tau)\eta(3\tau)} = q^8 \frac{A(7^2)}{A(1)} = A\left(7^2\right) \sum_{n=0}^{\infty} p_{[1^1,3^1]}(n-8) q^n$$

is a modular function on $\Gamma_0(147) + W_3$. Thus

$$\phi \mid U_7 = A(7) \sum_{n=0}^{\infty} p_{[1^1,3^1]}(7n-8) q^n$$

is a modular function on $\Gamma_0(21) + W_3$. A direct calculation shows that it is equal to the following polynomial

$$p(h) = 16h^2 + 161h^3 + 980h^4 + 3773h^5 + 9604h^6 + 16\,807h^7 + 16\,807h^8, \tag{2.10}$$

which proves identity (1.20). The proofs of the other identities are similar, and can be deduced from Table 1.

The list of identities given in Theorem 1 is by no means exhaustive. In theory, there should be many modular functions where one can apply the U_ℓ operator to lower the level by ℓ. If the resulting congruence subgroup has genus zero, an identity in terms of the hauptmodul exists and can be easily calculated. But having genus zero is not a necessary condition. M.L. Lang has classified all congruence subgroups of level less than 300 that have genus zero and a list of these groups is given as an appendix in [8]. There are a total of 15 groups that explicitly meet the conditions set out in the proof of Theorem 1. The remaining 5 groups share the property that e in $\Gamma_0(m) + W_e$ is composite. Their associated identities can be deduced from Table 2.

Table 1

Identity	$\Gamma_0(\ell^2 e)+W_e$	ℓ	$\phi(\tau)=\eta_\pi$	π	$h=\eta_\pi$	$P(h)$
(1.7)	$\Gamma_0(18)+W_2$	3	$[\frac{9^1 18^1}{1^1 2^1}]$	$[1^1 2^1]$	$[\frac{3^4 6^4}{1^4 2^4}]$	$3h$
(1.8)	$\Gamma_0(12)+W_3$	2	$[\frac{4^2 12^2}{1^2 3^2}]$	$[1^2 3^2]$	$[\frac{2^6 6^6}{1^6 3^6}]$	$2h$
(1.13)	$\Gamma_0(28)+W_7$	2	$[\frac{4^1 28^1}{1^1 7^1}]$	$[1^1 7^1]$	$[\frac{2^3 14^3}{1^3 7^3}]$	h
(1.14)	$\Gamma_0(20)+W_5$	2	$[\frac{4^4 20^4}{1^4 5^4}]$	$[1^4 5^4]$	$[\frac{2^4 10^4}{1^4 5^4}]$	$4h^2+8h^3$
(1.15)	$\Gamma_0(44)+W_{11}$	2	$[\frac{4^2 44^2}{1^2 11^2}]$	$[1^2 11^2]$	$[\frac{2^2 22^2}{1^2 11^2}]$	$2h^2+2h^3$
(1.16)	$\Gamma_0(45)+W_5$	3	$[\frac{9^1 45^1}{1^1 5^1}]$	$[1^1 5^1]$	$[\frac{3^2 15^2}{1^2 5^2}]$	$h+3h^2$
(1.17)	$\Gamma_0(92)+W_{23}$	2	$[\frac{4^1 92^1}{1^1 23^1}]$	$[1^1 23^1]$	$[\frac{2^1 46^1}{1^1 23^1}]$	h^2+h^3
(1.18)	$\Gamma_0(50)+W_2$	5	$[\frac{25^1 50^1}{1^1 2^1}]$	$[1^1 2^1]$	$[\frac{5^2 10^2}{1^2 2^2}]$	$3h+25h^2+125h^3$
(1.19)	$\Gamma_0(99)+W_{11}$	3	$[\frac{9^1 99^1}{1^1 11^1}]$	$[1^1 11^1]$	$[\frac{3^1 33^1}{1^1 11^1}]$	$2h^2+3h^3+3h^4$
(1.20)	$\Gamma_0(147)+W_3$	7	$[\frac{49^1 147^1}{1^1 3^1}]$	$[1^1 3^1]$	$[\frac{7^1 21^1}{1^1 3^1}]$	$p(h)$ defined in (2.10)

Table 2

$\Gamma_0(\ell^2 e)+W_e$	ℓ	$\phi(\tau)=\eta_\pi$	π	$h=\eta_\pi$	$P(h)$
$\Gamma_0(36)+W_4$	3	$[\frac{2^1 9^1 36^1}{1^1 4^1 18^1}]$	$[1^1 2^{-1} 4^1]$	$[\frac{2^4 3^4 12^4}{1^4 4^4 6^4}]$	h
$\Gamma_0(36)+W_9$	2	$[\frac{3^2 4^3 36^3}{1^3 9^3 12^2}]$	$[1^3 3^{-2} 9^3]$	$[\frac{2^3 3^2 18^3}{1^3 6^2 9^3}]$	$3h^2+2h^3$
$\Gamma_0(60)+W_{15}$	2	$[\frac{3^1 4^2 5^1 60^2}{1^2 12^1 15^2 20^1}]$	$[1^2 3^{-1} 5^{-1} 15^2]$	$[\frac{2^2 3^1 5^1 30^2}{1^2 6^1 10^1 15^2}]$	$2h^2+h^3$
$\Gamma_0(72)+W_8$	3	$[\frac{2^1 4^1 9^2 72^2}{1^2 8^2 18^1 36^1}]$	$[1^2 2^{-1} 4^{-1} 8^2]$	$[\frac{2^1 3^2 4^1 24^2}{1^2 6^1 8^2 12^1}]$	$4h^2+6h^3+3h^4$
$\Gamma_0(100)+W_4$	5	$[\frac{2^2 25^2 100^2}{1^2 4^2 50^2}]$	$[1^2 2^{-2} 4^2]$	$[\frac{2^2 5^2 20^2}{1^2 4^2 10^2}]$	$11h^2+60h^3+175h^4+250h^5+125h^6$

From the above identities, we see that only $a(n)$, $c(n)$, $p_{[1^4 5^4]}(n)$ and $p_{[1^2 11^2]}(n)$ satisfy

$$p_\pi(\ell n-\alpha)\equiv 0 \pmod{\ell}. \tag{2.11}$$

J. Sinick [17] has obtained some necessary conditions for such congruences.

We further note that identities (1.13), (1.16), (1.18), (1.19) and (1.20) can be written as the following congruences:

$$E(2)E(14)\sum_{n=0}^{\infty} p_{[1^1 7^1]}(2n-1)q^n \equiv \eta^3(\tau)\eta^3(7\tau) \pmod 2, \tag{2.12}$$

$$E(3)E(15)\sum_{n=0}^{\infty} p_{[1^1 5^1]}(3n-2)q^n \equiv \eta^4(\tau)\eta^4(5\tau) \pmod 3, \tag{2.13}$$

$$E(5)E(10)\sum_{n=0}^{\infty} p_{[1^1 2^1]}(5n-3)q^n \equiv 3\eta^8(\tau)\eta^8(2\tau) \pmod 5, \tag{2.14}$$

$$E(3)E(33)\sum_{n=0}^{\infty} p_{[1^1 11^1]}(3n-4)q^n \equiv 2\eta^4(\tau)\eta^4(11\tau) \pmod 3, \tag{2.15}$$

$$E(7)E(21)\sum_{n=0}^{\infty} p_{[1^1 3^1]}(7n-8)q^n \equiv 2\eta^{12}(\tau)\eta^{12}(3\tau) \pmod 7. \tag{2.16}$$

Recall that in each of the above, the expression on the left-hand side is a modular function on $\Gamma_0(\ell e) + W_e$ which is now congruent (modulo ℓ) to a modular form on $\Gamma_0(e)$, where the weight of this modular form is divisible by $\ell - 1$.

To state our results involving higher powers of the prime ℓ, we fix a partition $\pi = [\prod_t t^{r_t}]$, the associated infinite product $A_\pi(j)$, and the modular functions $\phi(\tau)$ and $h(\tau)$. (See (1.11), (2.4) and (2.5) for the respective definitions.) For positive integers j, define

$$L_{2j-1} = A_\pi(\ell) \sum_{n=0}^{\infty} p_\pi\left(\ell^{2j-1}n - \delta_{2j-1}\right)q^n,$$
$$L_{2j} = A_\pi(1) \sum_{n=0}^{\infty} p_\pi\left(\ell^{2j}n - \delta_{2j}\right)q^n, \tag{2.17}$$

where

$$\delta_{2j-1} = \delta_{2j} = \frac{1-\ell^{2j}}{1-\ell^2}\alpha.$$

(See (2.4) for a definition of α.)

Theorem 10. *In each of the fifteen cases listed in Tables* 1 *and* 2*, L_j is a polynomial in h, the hauptmodul, with integral coefficients.*

In terms of L_j, we can reformulate the results of [6] and [7] as the next two theorems.

Theorem 11. *Let $\pi = [1^1 2^1]$, $\ell = 3$ and $\alpha = 1$, then for each j,*

$$L_j \equiv c_j 3^{2\lfloor \frac{j}{2} \rfloor + 1} \eta^8(\tau)\eta^8(2\tau) \pmod{3^{2\lfloor \frac{j}{2} \rfloor + 2}},$$

where c_j is some integer relatively prime to 3.

Theorem 12. *Let $\pi = [1^2 3^2]$, $\ell = 2$ and $\alpha = 1$, then for each j,*

$$L_j \equiv d_j 2^{\lfloor \frac{j}{2} \rfloor + 1} \eta^6(\tau)\eta^6(3\tau) \pmod{2^{\lfloor \frac{j}{2} \rfloor + 2}},$$

where d_j is some integer relatively prime to 2.

We also have the following result.

Theorem 13. *Let $\pi = [1^1 2^1]$, $\ell = 5$ and $\alpha = 1$, then for each j,*

$$L_j \equiv e_j 5^{\lfloor \frac{j}{2} \rfloor} \eta^8(\tau)\eta^8(2\tau) \pmod{5^{\lfloor \frac{j}{2} \rfloor + 1}},$$

where e_j is some integer relatively prime to 5.

Corollaries 2 and 3 in Section 1 are immediate consequences of Theorems 11 and 13.

Remark 1. Theorems 11–13 are motivated by an observation of the first author. He found that for prime numbers ℓ, if l_n is the smallest positive integral solution to

$$24x \equiv 1 \pmod{\ell^n}$$

and

$$L_{n,\ell} = \begin{cases} \prod_{k=1}^{\infty}(1-q^{\ell k})\sum_{k=0}^{\infty} p(\ell^n k + l_n)q^{k+1}, & \text{if } n \text{ is odd,} \\ \prod_{k=1}^{\infty}(1-q^{k})\sum_{k=0}^{\infty} p(\ell^n k + l_n)q^{k+1}, & \text{if } n \text{ is even,} \end{cases}$$

then

$$L_{n,5} \equiv 5^n a_n \Delta \pmod{5^{n+1}}, \tag{2.18}$$

$$L_{n,7} \equiv 7^{\lfloor n/2 \rfloor+1} b_n \Delta \pmod{7^{\lfloor n/2 \rfloor+2}} \tag{2.19}$$

and

$$L_{n,11} \equiv 11^n c_n \Delta E_8 \pmod{11^{n+1}}, \tag{2.20}$$

where $(5, a_n) = (7, b_n) = (11, c_n) = 1$,

$$\Delta = q\prod_{k=1}^{\infty}(1-q^k)^{24}$$

and

$$E_8 = 1 + 480\sum_{k=1}^{\infty}\frac{k^7 q^k}{1-q^k}.$$

Note that (2.18) and (2.19) follow immediately from Watson's work (see [12]). The congruence (2.20), on the other hand, does not follow immediately from existing methods. Y.F. Yang, at the request of the first author, verified (2.20) for the first few values of n. It is very likely that one can obtain a rigorous proof of (2.20) using Atkin's method given in [1]. It is our hope that by writing the congruences in the form as in (2.18)–(2.20), new proofs which are independent of Atkin's identities and explicit use of modular equations such as (3.6) can be found.

3. Identities involving higher prime powers

We shall illustrate how Theorem 10 may be proved by induction on j, with the identities of Theorem 1 forming the respective base cases, L_1. This method is due to Atkin [1], [12, Chapters 7 and 8]. We first establish the following general lemmas.

Lemma 14. *For a positive integer j, we have*

$$L_{2j-1} \mid U_\ell = L_{2j},$$

$$(\phi L_{2j}) \mid U_\ell = L_{2j+1}.$$

Proof. We shall only prove the second equation.

$$
\begin{aligned}
\phi L_{2j} &= q^{\alpha}\frac{A(\ell^2)}{A(1)}\cdot A(1)\sum_{n=0}^{\infty} p_\pi\left(\ell^{2j}n-\delta_{2j}\right)q^n \\
&= A\left(\ell^2\right)\sum_{n=0}^{\infty} p_\pi\left(\ell^{2j}(n-\alpha)-\left(1+\ell^2+\cdots+\ell^{2j-2}\right)\alpha\right)q^n \\
&= A\left(\ell^2\right)\sum_{n=0}^{\infty} p_\pi\left(\ell^{2j}n-\delta_{2j+1}\right)q^n.
\end{aligned}
$$

The result then follows from Lemma 9. □

We need to study the effect of applying the Hecke operator U_ℓ, on the modular functions ϕ and h. For a positive integer r, define

$$
\begin{aligned}
S_r &= \sum_{j=0}^{\ell-1}\phi^r\left(\frac{\tau+j}{\ell}\right) \\
&= \sum_{j=0}^{\ell-1}\phi^r(\tau)\ \Big|\ A_\ell^{-1}S^j \\
&= \phi^r(\tau)\ \Big|\ U_\ell^* \\
&= \ell\phi^r(\tau)\ \Big|\ U_\ell.
\end{aligned} \tag{3.1}
$$

Lemma 15. *For a positive integer p, we have*

$$
\begin{aligned}
h^p\ \Big|\ U_\ell &= \frac{S_{p\gamma}}{\ell h^p}, \\
\left(\phi h^p\right)\ \Big|\ U_\ell &= \frac{S_{p\gamma+1}}{\ell h^p}.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
h^p &= q^{p\beta}\frac{A(\ell)^{p\gamma}}{A(1)^{p\gamma}} \\
&= \frac{q^{p\beta}}{q^{p\beta(\ell+1)}}\frac{A(\ell)^{p\gamma}}{A(\ell^2)^{p\gamma}}\cdot q^{\alpha p\gamma}\frac{A(\ell^2)^{p\gamma}}{A(1)^{p\gamma}} \\
&= \frac{1}{q^{p\beta\ell}}\frac{A(\ell)^{p\gamma}}{A(\ell^2)^{p\gamma}}\phi^{p\gamma}.
\end{aligned} \tag{3.2}
$$

In the above, we used the fact that $\alpha\gamma = \beta(\ell+1)$. By applying U_ℓ and Lemma 9,

$$
\begin{aligned}
h^p\ \Big|\ U_\ell &= \frac{1}{q^{p\beta}}\frac{A(1)^{p\gamma}}{A(\ell)^{p\gamma}}\left(\phi^{p\gamma}\right)\ \Big|\ U_\ell \qquad (3.3) \\
&= \frac{1}{h^p}\cdot\frac{1}{\ell}S_{p\gamma}.
\end{aligned}
$$

The proof of the second equation is similar. □

Next, we have to show that $\phi(\frac{\tau+j}{\ell})$ are distinct roots of a polynomial with coefficients in terms of h. By Newton's formula for symmetric power sum of roots [12, p. 118], each S_r satisfies a recurrence relation and can be written as a polynomial in terms of h. Thus if L_j is a polynomial in h, multiplying L_j by ϕ if necessary and applying U_ℓ, we see that L_{j+1} can be written in terms of a sum of S_r which in turn can be written as a polynomial in h.

At this stage, the proof depends crucially on a modular equation satisfied by the corresponding ϕ and h and we shall provide the details for $\ell = 5$ and the partition $\pi = [1^1 2^1]$. Hence $A_\pi(j) = A(j) = E(j)E(2j)$ and $p_\pi(n) = a(n)$. We have

$$\phi(\tau) = \frac{\eta(25\tau)\eta(50\tau)}{\eta(\tau)\eta(2\tau)} = q^3\frac{A(25)}{A(1)}, \tag{3.4}$$

and

$$h(\tau) = \frac{\eta^2(5\tau)\eta^2(10\tau)}{\eta^2(\tau)\eta^2(2\tau)} = q\frac{A(5)^2}{A(1)^2}. \tag{3.5}$$

$\phi(\tau)$ is a modular function on $\Gamma_0(50) + W_2$ and $h(\tau)$ is the hauptmodul of $\Gamma_0(10) + W_2$. Furthermore $h(5\tau)$ is also a modular function on $\Gamma_0(50) + W_2$ and thus $\phi(\tau)$ and $h_5(\tau) = h(5\tau)$ belong to the same congruence subgroup and as a result, satisfy a polynomial equation of degree 5 and degree 3 respectively [14, p. 110], i.e.

$$\begin{aligned}\phi^5(\tau) &= \big(15h_5(\tau) + 5^3h_5^2(\tau) + 5^4h_5^3(\tau)\big)\phi^4(\tau) - \big(5^2h_5^2(\tau) + 5^3h_5^3(\tau)\big)\phi^3(\tau) \\ &\quad + \big(5h_5^2(\tau) + 5^2h_5^3(\tau)\big)\phi^2(\tau) - 5h_5^3(\tau)\phi(\tau) + h_5^3(\tau).\end{aligned} \tag{3.6}$$

The above equation holds with $h_5(\tau) = h(5\tau)$ invariant when we replace τ by $\tau + j/5$ for j from 1 to 4, thus we conclude that the following polynomial

$$\begin{aligned}Q(u) &= u^5 - \big(15h + 5^3h^2 + 5^4h^3\big)u^4 + \big(5^2h^2 + 5^3h^3\big)u^3 \\ &\quad - \big(5h^2 + 5^2h^3\big)u^2 + 5h^3u - h^3\end{aligned} \tag{3.7}$$

has five distinct roots $\phi(\frac{\tau+j}{5})$, for j from 0 to 4. Now by Newton's theorem, S_r for $r \geqslant 5$, satisfies

$$\begin{aligned}S_r &= \big(15h + 5^3h^2 + 5^4h^3\big)S_{r-1} - \big(5^2h^2 + 5^3h^3\big)S_{r-2} \\ &\quad + \big(5h^2 + 5^2h^3\big)S_{r-3} - 5h^3S_{r-4} + h^3S_{r-5}.\end{aligned} \tag{3.8}$$

We also have

$$\begin{aligned}S_1 &= 15h + 125h^2 + 625h^3, \\ S_2 &= 175h^2 + 3500h^3 + 34\,375h^4 + 156\,250h^5 + 390\,625h^6, \\ S_3 &= 15h^2 + 2325h^3 + 69\,375h^4 + 1\,031\,250h^5 + 8\,750\,000h^6 + 46\,875\,000h^7 \\ &\quad + 146\,484\,375h^8 + 244\,140\,625h^9, \\ S_4 &= 280h^3 + 33\,375h^4 + 1\,237\,500h^5 + 24\,312\,500h^6 + 295\,312\,500h^7 + 2\,412\,109\,375h^8 \\ &\quad + 13\,476\,562\,500h^9 + 51\,269\,531\,250h^{10} + 122\,070\,312\,500h^{11} \\ &\quad + 152\,587\,890\,625h^{12}.\end{aligned} \tag{3.9}$$

Hence if we write

$$S_r = \sum a_{r,p} h^p, \tag{3.10}$$

we obtain the recurrence relation

$$\begin{aligned} a_{r,p} = {} & 15a_{r-1,p-1} + 125a_{r-1,p-2} + 625a_{r-1,p-3} - 25a_{r-2,p-2} - 125a_{r-2,p-3} \\ & + 5a_{r-3,p-2} + 25a_{r-3,p-3} - 5a_{r-4,p-3} + a_{r-5,p-3}, \end{aligned} \tag{3.11}$$

for $r \geqslant 5$. Moreover, an induction argument shows that

$$a_{r,p} \neq 0 \quad \text{if and only if} \quad \left\lfloor \frac{3r+4}{5} \right\rfloor \leqslant p \leqslant 3r. \tag{3.12}$$

If we now define $\nu(a_{r,p})$ as the highest power of 5 that divides $a_{r,p}$ and $\nu(0) = \infty$, we can show by induction that

$$\nu(a_{r,p}) \geqslant \left\lfloor \frac{5p-3r+4}{4} \right\rfloor. \tag{3.13}$$

We are now ready to prove Theorem 10 for ϕ and h defined in (3.4) and (3.5). Eq. (1.18) is our base case and can be rewritten as $L_1 = 3h + 25h^2 + 125h^3$. Suppose that we have

$$L_{2j-1} = \sum b_{2j-1,r} h^r, \tag{3.14}$$

where only finitely many $b_{2j-1,r}$ are nonzero and each is an integer. Applying U_5,

$$\begin{aligned} L_{2j-1} \mid U_5 = L_{2j} &= \sum b_{2j-1,r} h^r \,\Big|\, U_5 \\ &= \sum b_{2j-1,r} \sum_{p=\mu}^{6r} \frac{a_{2r,p} h^p}{5h^r}. \end{aligned} \tag{3.15}$$

It follows from (3.12) that for every p in the second sum,

$$p \geqslant \mu = \left\lfloor \frac{6r+4}{5} \right\rfloor \geqslant r + \left\lfloor \frac{r+4}{5} \right\rfloor \geqslant r+1. \tag{3.16}$$

Furthermore,

$$\nu(a_{2r,p}) \geqslant \left\lfloor \frac{5p-6r+4}{4} \right\rfloor \geqslant \left\lfloor \frac{6r+4-1-6r+4}{4} \right\rfloor \geqslant 1. \tag{3.17}$$

Thus L_{2j} is also a polynomial in h with integral coefficients. The inductive case from L_{2j} to L_{2j+1} is similar. The other fourteen cases for Theorem 10 can be dealt with in the same way.

We will proceed to prove Theorem 13. We first note that

$$h(\tau) = \frac{\eta^2(5\tau)\eta^2(10\tau)}{\eta^2(\tau)\eta^2(2\tau)} \equiv \eta^8(\tau)\eta^8(2\tau) \pmod 5. \tag{3.18}$$

If we set

$$L_j = \sum b_{j,r} h^r, \tag{3.19}$$

then Theorem 13 follows from the next lemma.

Lemma 16.

$$\begin{aligned} \nu(b_{2j-1,1}) &= j-1, \\ \nu(b_{2j-1,r}) &\geqslant j-1+\left\lfloor \frac{5r-4}{4} \right\rfloor, \quad \textit{if } r>1, \\ \nu(b_{2j,1}) &= j, \\ \nu(b_{2j,r}) &\geqslant j+\left\lfloor \frac{5r-5}{4} \right\rfloor, \quad \textit{if } r>1. \end{aligned}$$

Proof. We shall prove this by induction. Since $L_1 = 3h + 25h^2 + 125h^3$, $\nu(b_{1,1}) = 0$, $\nu(b_{1,2}) = 2$ and $\nu(b_{1,3}) = 3$.

Now from (3.15),

$$L_{2j} = \sum_{s \geqslant 1} b_{2j,s} h^s = \sum_{r \geqslant 1} b_{2j-1,r} \left(\sum_{p=\mu}^{6r} \frac{a_{2r,p} h^{p-r}}{5} \right). \tag{3.20}$$

Equating coefficients, we have

$$b_{2j,s} = \sum_{r \geqslant 1} b_{2j-1,r} \left(\frac{a_{2r,s+r}}{5} \right). \tag{3.21}$$

Hence, for $s > 1$, we have

$$\begin{aligned} \nu(b_{2j,s}) &\geqslant -1 + \min_{r \geqslant 1} \left\{ j-1+\left\lfloor \frac{5r-4}{4} \right\rfloor + \left\lfloor \frac{5(s+r)-3(2r)+4}{4} \right\rfloor \right\} \\ &\geqslant j-1+\min_{r \geqslant 1} \left\{ \left\lfloor \frac{5r-4}{4} \right\rfloor + \left\lfloor \frac{5s-r}{4} \right\rfloor \right\} \\ &\geqslant j-1+\left\lfloor \frac{5s-1}{4} \right\rfloor \\ &\geqslant j+\left\lfloor \frac{5s-5}{4} \right\rfloor. \end{aligned}$$

We made use of the fact that the minimum occurs at $r = 1$. Next for the case $s = 1$, we have

$$\begin{aligned} 5b_{2j,1} &= \sum_{r \geqslant 1} b_{2j-1,r} (a_{2r,1+r}) \\ &= b_{2j-1,1}(a_{2,2}) + \sum_{r \geqslant 2} b_{2j-1,r}(a_{2r,1+r}) \\ &= 175 b_{2j-1,1} + \sum_{r \geqslant 2} b_{2j-1,r}(a_{2r,1+r}). \end{aligned} \tag{3.22}$$

Thus,

$$\begin{aligned}\nu(b_{2j,1}) &= -1 + \min\left\{2 + j - 1, \min_{r \geqslant 2}\left\{j - 1 + \left\lfloor \frac{5r-4}{4} \right\rfloor + \left\lfloor \frac{5-r+4}{4} \right\rfloor\right\}\right\} \\ &= -1 + (j+1) = j.\end{aligned}$$

Next, we have

$$\begin{aligned}L_{2j+1} &= \sum b_{2j+1,s} h^s = (\phi L_{2j}) \mid U_5 \\ &= \sum b_{2j,r} \sum_p \frac{a_{2r+1,p} h^{p-r}}{5}. \qquad (3.23)\end{aligned}$$

Equating coefficients, we have

$$b_{2j+1,s} = \sum_{r \geqslant 1} b_{2j,r} \left(\frac{a_{2r+1,s+r}}{5}\right). \qquad (3.24)$$

Thus, for $s > 1$, we have

$$\begin{aligned}\nu(b_{2j+1,s}) &\geqslant -1 + \min_{r \geqslant 1}\left\{j + \left\lfloor \frac{5r-5}{4} \right\rfloor + \left\lfloor \frac{5(s+r) - 3(2r+1) + 4}{4} \right\rfloor\right\} \\ &\geqslant j - 1 + \min_{r \geqslant 1}\left\{\left\lfloor \frac{5r-5}{4} \right\rfloor + \left\lfloor \frac{5s-r+1}{4} \right\rfloor\right\} \\ &\geqslant j - 1 + \left\lfloor \frac{5s}{4} \right\rfloor \\ &\geqslant j + \left\lfloor \frac{5s-4}{4} \right\rfloor.\end{aligned}$$

When $s = 1$,

$$\begin{aligned}5b_{2j+1,1} &= b_{2j,1}(a_{3,2}) + \sum_{r \geqslant 2} b_{2j,r}(a_{2r+1,1+r}) \\ &= 15 b_{2j,1} + \sum_{r \geqslant 2} b_{2j,r}(a_{2r+1,1+r}).\end{aligned}$$

Finally,

$$\begin{aligned}\nu(b_{2j+1,1}) &= -1 + \min\left\{1 + j, \min_{r \geqslant 2}\left\{j + \left\lfloor \frac{5r-5}{4} \right\rfloor + \left\lfloor \frac{5-r+1}{4} \right\rfloor\right\}\right\} \\ &= -1 + (1+j) = j,\end{aligned}$$

which completes the proof of Lemma 16. □

References

[1] A.O.L. Atkin, Proof of a conjecture of Ramanujan, Glasg. Math. J. 8 (1967) 14–32.
[2] A.O.L. Atkin, J. Lehner, Hecke operators of $\Gamma_0(m)$, Math. Ann. 185 (1970) 134–160.
[3] A.O.L. Atkin, J.N. O'Brien, Some properties of $p(n)$ and $c(n)$ modulo powers of 13, Trans. Amer. Math. Soc. 126 (1967) 442–459.
[4] B.C. Berndt, K. Ono, Ramanujan's unpublished manuscript on the partition and tau functions, in: D. Foata, G.-N. Han (Eds.), The Andrews Festschrift, Springer-Verlag, Berlin, 2001, pp. 39–110.
[5] H.-C. Chan, Ramanujan's cubic continued fraction and a generalization of his "most beautiful identity", Int. J. Number Theory, in press.
[6] H.-C. Chan, Ramanujan's cubic continued fraction and Ramanujan type congruences for a certain partition function, Int. J. Number Theory, in press.
[7] H.-C. Chan, S. Cooper, Congruences modulo powers of 2 for a certain partition function, Ramanujan J. 22 (2010) 101–117.
[8] H.H. Chan, M.L. Lang, Ramanujan's modular equations and Atkin–Lehner involutions, Israel J. Math. 103 (1998) 1–16.
[9] H.H. Chan, R.P. Lewis, Partition identities and congruences associated with the Fourier coefficients of the Euler products, J. Comput. Appl. Math. 160 (2003) 69–75.
[10] S. Cooper, Series and iterations for $1/\pi$, Acta Arith. 141 (2010) 33–58.
[11] P.C. Eggan, Linear congruences for power products of Dedekind η-functions, J. Lond. Math. Soc. 39 (2) (1989) 231–245.
[12] M.I. Knopp, Modular Functions in Analytic Number Theory, second ed., Chelsea Publishing Company, New York, 1993.
[13] K. Ono, The Web of Modularity, American Mathematical Society, Providence, 2003.
[14] R.A. Rankin, Modular Forms and Functions, Cambridge University Press, Cambridge, 1977.
[15] S. Ramanujan, Some properties of $p(n)$, the number of partitions of n, Proc. Cambridge Philos. Soc. 19 (1919) 207–210.
[16] S. Ramanujan, Collected Papers of Srinivasa Ramanujan, American Mathematical Society Chelsea Publisher, Providence, 2000.
[17] J. Sinick, Ramanujan congruences for a class of eta quotients, Int. J. Number Theory, in press.
[18] G.N. Watson, Ramanujan's vermutung über zerfällungsanzahlen, J. Reine Angew. Math. 179 (1938) 97–128.
[19] H.S. Zuckerman, Identities analogous to Ramanujan's identities involving the partition function, Duke Math. J. 5 (1939) 88–110.