Ramanujan J
DOI 10.1007/s11139-008-9143-z

Cranks—really, the final problem

Bruce C. Berndt · Heng Huat Chan · Song Heng Chan · Wen-Chin Liaw

Received: 29 August 2007 / Accepted: 5 September 2008

Abstract A survey of Ramanujan's work on cranks in his lost notebook is given. We give evidence that Ramanujan was concentrating on cranks when he died, that is to say, the final problem on which Ramanujan worked was *cranks—not mock theta functions*.

Keywords Crank · Partitions · Theta functions · Ramanujan's lost notebook · Rank

Mathematics Subject Classification (2000) Primary: 11P82 · Secondary: 11P83

Dedicated to our friend George Andrews on his 70th birthday.

B.C. Berndt's research is partially supported by grant H98230-07-1-0088. from the National Security Agency.
H.H. Chan's research is partially supported by National University of Singapore Academic Research Fund, Project Number R146000027112.
W.-C. Liaw's research is partially supported by grant NSC95-2115-M-194-012 from the National Science Council of the Republic of China.

B.C. Berndt (✉)
Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
e-mail: berndt@illinois.edu

H.H. Chan
Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543, Republic of Singapore
e-mail: matchh@nus.edu.sg

S.H. Chan
Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Republic of Singapore
e-mail: Chansh@ntu.edu.sg

W.-C. Liaw
Department of Mathematics, National Chung Cheng University, Min-Hsiung, Chia-Yi 62145, Taiwan, Republic of China
e-mail: wcliaw@math.ccu.edu.tw

Published online: 01 October 2009

1 Introduction

Recall that in his last letter to G.H. Hardy [17, pp. xxxi–xxxii, 6, pp. 220–223], Ramanujan announced a new class of functions, which he called mock theta functions, and gave several examples and theorems in illustration. This letter was written on January 20, 1920, only slightly more than three months before his death on April 26, 1920. When G.N. Watson wrote his last two papers [20, 21] on Ramanujan's work in 1936 and 1937, respectively, it was natural to entitle one of them [20, 7, pp. 325–347], *The final problem: an account of the mock theta functions*, which was also the title of his retiring Presidential address to the London Mathematical Society. In conversations with one of us (BCB) in December, 1983, Ramanujan's widow, Janaki, told us that her husband worked on mathematics, or in her words, that he was "doing his sums," until four days before he died, when his pain became too great and he could no longer concentrate on mathematics. The wide ranging content of the lost notebook [19] gives evidence that Ramanujan probably derived theorems in several topics during the last three months of his life. Of course, we can only speculate on Ramanujan's focus as his life withered away, but one of the goals in this short survey is to provide evidence that Ramanujan devoted his very last days to cranks (not to mock theta functions as suggested by Watson), although, of course, Ramanujan would not have used this terminology.

First, we briefly relate the origin of cranks *after* Ramanujan had studied them in his lost notebook. Secondly, we describe the extensive material in the lost notebook pertaining to cranks. Thirdly, we draw some conclusions from this material and ask several questions, because we do not know the motivation for most of Ramanujan's calculations on cranks. Thus, one purpose in writing this paper is to elicit reader insights into Ramanujan's motivations.

We offer no proofs in this paper, but we indicate where readers can find proofs of all the theorems discussed here.

2 Dyson, ranks, and cranks

Recall Ramanujan's celebrated congruences for the partition function $p(n)$, namely,

$$p(5n+4) \equiv 0 \pmod 5, \tag{2.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{2.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{2.3}$$

In attempting to find combinatorial interpretations for (2.1)–(2.3), in 1944, F.J. Dyson [9] defined the *rank of a partition*.

Definition 2.1 The rank of a partition is equal to the largest part minus the number of parts.

Let $N(m,n)$ denote the number of partitions of n with rank m, and let $N(m,t,n)$ denote the number of partitions of n with rank congruent to m modulo t. Then Dyson conjectured that

$$N(k,5,5n+4)=\frac{p(5n+4)}{5}, \quad 0\le k\le 4, \tag{2.4}$$

and

$$N(k,7,7n+5)=\frac{p(7n+5)}{7}, \quad 0\le k\le 6. \tag{2.5}$$

Thus, if (2.4) and (2.5) were true, the partitions counted by $p(5n+4)$ and $p(7n+5)$ fall into five and seven equinumerous classes, respectively, thus providing a partial answer to Dyson's query. Dyson also conjectured that the generating function for $N(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}N(m,n)a^m q^n=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(aq;q)_n(q/a;q)_n}, \quad |q|<1,\ |q|<|a|<1/|q|. \tag{2.6}$$

These conjectures were first proved by A.O.L. Atkin and H.P.F. Swinnerton-Dyer [3] in 1954. Although Ramanujan did not record any statement equivalent to Definition 2.1, he recorded theorems on its generating function (2.6) in his lost notebook; in particular, see [19, p. 20].

The corresponding analogue to (2.4) and (2.5) does not hold for $p(11n+6)\equiv 0 \pmod{11}$, and so Dyson conjectured the existence of a *crank* that would combinatorially explain this congruence. In his doctoral dissertation [10], F.G. Garvan defined vector partitions which became the forerunners of the crank. The *true crank* was discovered by Andrews and Garvan [1] during the afternoon of June 6, 1987 at Illinois Street Residence Hall, a student dormitory at the University of Illinois, following a meeting to commemorate the centenary of Ramanujan's birth.

Definition 2.2 For a partition π, let $\lambda(n)$ denote the largest part of π, let $\mu(\pi)$ denote the number of ones in π, and let $\nu(\pi)$ denote the number of parts of π larger than $\mu(\pi)$. The crank $c(\pi)$ is then defined to be [1]

$$c(\pi)=\begin{cases}\lambda(\pi), & \text{if } \mu(\pi)=0,\\ \nu(\pi)-\mu(\pi), & \text{if } \mu(\pi)>0.\end{cases}$$

For $n>1$, let $M(m,n)$ denote the number of partitions of n with crank m, while for $n\le 1$ we set

$$M(0,0)=1 \quad \text{and} \quad M(m,0)=0, \quad \text{otherwise,}$$
$$M(0,1)=-1, \qquad M(1,1)=M(-1,1)=1, \quad \text{and} \quad M(m,1)=0, \quad \text{otherwise.}$$

The generating function for $M(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}M(m,n)a^m q^n=\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty}. \tag{2.7}$$

The crank not only leads to a combinatorial interpretation of $p(11n+6) \equiv 0 \pmod{11}$, as desired by Dyson, but also to similar interpretations for $p(5n+4) \equiv 0 \pmod{5}$ and $p(7n+5) \equiv 0 \pmod{7}$.

Let $M(m,t,n)$ denote the number of partitions of n with crank congruent to m modulo t.

Theorem 2.3 *With $M(m,t,n)$ defined above,*

$$M(k,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \le k \le 4,$$
$$M(k,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \le k \le 6,$$
$$M(k,11,11n+6) = \frac{p(11n+6)}{11}, \quad 0 \le k \le 10.$$

An excellent introduction to cranks can be found in Garvan's survey paper [11].

Remarkably, the generating function (2.7) for cranks can be found in Ramanujan's lost notebook [19, p. 179] in the form

$$F(q) := F_a(q) := \frac{(q;q)_\infty}{(aq;q)_\infty (q/a;q)_\infty} =: \sum_{n=0}^{\infty} \lambda_n q^n, \tag{2.8}$$

and so from (2.7),

$$\lambda_n = \sum_{m=-\infty}^{\infty} M(m,n) a^m.$$

Note that when $a=1$, (2.7) and (2.8) reduce to the generating function for $p(n)$.

3 Dissections of the crank

Definition 3.1 If

$$P(q) := \sum_{n=0}^{\infty} a_n q^n$$

is any power series, then the m-dissection of $P(q)$ is given by

$$P(q) = \sum_{j=0}^{m-1} \sum_{n_j=0}^{\infty} a_{n_j m+j} q^{n_j m+j}.$$

We actually use below an extension of this definition to congruences in the ring of power series in the two variables a and q.

In his lost notebook [19], Ramanujan offers, in various guises, m-dissections for $F_a(q)$ for $m = 2,3,5,7,11$. In particular, on page 179 Ramanujan offers 2- and 3-dissections for $F_a(q)$ in the form of congruences. To give these dissections, we first need to define Ramanujan's theta function $f(a,b)$ by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{3.1}$$

Furthermore, Ramanujan sets

$$f(-q) := f(-q,-q^2) = (q;q)_\infty, \tag{3.2}$$

where the latter equality follows from the pentagonal number theorem. The two congruences are then given by

$$F(\sqrt{q}) \equiv \frac{f(-q^3,-q^5)}{(-q^2;q^2)_\infty} + \left(a - 1 + \frac{1}{a}\right)\sqrt{q}\frac{f(-q,-q^7)}{(-q^2;q^2)_\infty} \pmod{(a^2 + a^{-2})} \tag{3.3}$$

and

$$\begin{aligned} F(q^{1/3}) \equiv & \frac{f(-q^2,-q^7)f(-q^4,-q^5)}{(q^9;q^9)_\infty} \\ & + \left(a - 1 + \frac{1}{a}\right) q^{1/3} \frac{f(-q,-q^8)f(-q^4,-q^5)}{(q^9;q^9)_\infty} \\ & + \left(a^2 + \frac{1}{a^2}\right) q^{2/3} \frac{f(-q,-q^8)f(-q^2,-q^7)}{(q^9;q^9)_\infty} \pmod{(a^3 + 1 + a^{-3})}. \end{aligned} \tag{3.4}$$

Note that $\lambda_2 = a^2 + a^{-2}$, which trivially implies that $a^4 \equiv -1 \pmod{\lambda_2}$ and $a^8 \equiv 1 \pmod{\lambda_2}$. Thus, in (3.3), a behaves like a primitive 8th root of unity modulo λ_2. On the other hand, $\lambda_3 = a^3 + 1 + a^{-3}$, from which it follows that $a^9 \equiv -a^6 - a^3 \equiv 1 \pmod{\lambda_3}$. So in (3.4), a behaves like a primitive 9th root of unity modulo λ_3.

In contrast to (3.3) and (3.4), on page 20 in his lost notebook, Ramanujan gives a 5-dissection for $F_a(q)$ in terms of an *equality* instead of a congruence. For $a = e^{2\pi i n/5}$, with $n = 1$ or 2,

$$\begin{aligned} F_a(q^{1/5}) = & \frac{f(-q^2,-q^3)}{f^2(-q,-q^4)} f^2(-q^5) - 4\cos^2(2n\pi/5) q^{1/5} \frac{f^2(-q^5)}{f(-q,-q^4)} \\ & + 2\cos(4n\pi/5) q^{2/5} \frac{f^2(-q^5)}{f(-q^2,-q^3)} \\ & - 2\cos(2n\pi/5) q^{3/5} \frac{f(-q,-q^4)}{f^2(-q^2,-q^3)} f^2(-q^5). \end{aligned} \tag{3.5}$$

Observe that (3.5) contains no terms involving $q^{4/5}$, which is a reflection of the fact that $p(5n+4) \equiv 0 \pmod 5$. In fact [4, p. 105], one can replace (3.5) by an equivalent 5-dissection that is a congruence, and conversely (3.3) and (3.4) can be replaced by equalities involving 8th and 9th roots of unity, respectively. Indeed, one can always convert a dissection from one involving roots of unity to one involving a congruence, and conversely [4, pp. 118–119]. One advantage of formulations in terms of congruences is that one can let $a = 1$ in these congruences and obtain congruences originally found by Atkin and Swinnerton-Dyer [3].

Ramanujan also derived the 7- and 11-dissections for $F_a(q)$, but he does not explicitly state them. Instead, on pages 71 and 70, respectively, in his lost notebook he vaguely provides the quotients of theta functions that appear in these representations. These dissections immediately yield the congruences $p(7n+5) \equiv 0 \pmod 7$ and $p(11n+6) \equiv 0 \pmod{11}$, respectively. On pages 186 and 187 in the lost notebook [19], Ramanujan records modulo 11 the coefficients (in some cases up to 70) of the quotients of theta functions appearing in the 11-dissection of the crank generating function. Furthermore, on page 19 in his lost notebook, Ramanujan apparently intended to write down the 7-dissection of $F_a(q)$ when $a = e^{2\pi i n/7}$, but all he records is "$F_a(q)$." Did Ramanujan not record the 7-dissection for $F_a(q)$ because he was stymied by his illness? Or was the 7-dissection recounted on a *lost page* that is not contained in the extant lost notebook? If so, then possibly his seventh order mock theta functions, which are not found in the lost notebook, were also described on this possible missing page.

In versions involving roots of unity, Garvan first proved all five of Ramanujan's m-dissections [12, 13]. The present authors [4] devised uniform proofs of Ramanujan's dissections in the language of congruences in two distinct ways. One of the methods arises from obscure statements found on page 59 of the lost notebook. In the middle of that page are three (somewhat difficult to read) mathematical expressions, one of which is equal to $(q;q)_\infty$ times the generating function for the crank. Although not claimed by Ramanujan, the three expressions are equal, and it is one of these identities, namely,

$$\frac{(q;q)_\infty^2}{(aq;q)_\infty (q/a;q)_\infty} = \sum_{k=-\infty}^{\infty} (-1)^k \frac{(1-a)q^{k(k+1)/2}}{1-aq^k}, \tag{3.6}$$

that leads to one set of the aforementioned uniform proofs of Ramanujan's dissections [4]. See [5, pp. 94–96] for proofs and a history of earlier proofs.

Further references to other proofs of Ramanujan's five dissections for $F_a(q)$ and to proofs of certain other dissections of $F_a(q)$ can be found in [4]. The largest value of m for which a dissection of $F_a(q)$ has been given is $m = 11$.

On pages 179 and 180 in [19], Ramanujan recorded ten tables of values of n for which λ_n satisfies certain congruences. A rough preliminary version of this table can be found on page 61 in his lost notebook [19]. Although Ramanujan makes no claims about these tables, it is clear that he regarded them as complete. We provide below a list of the congruences and the number of values of n for which λ_n satisfies the given congruence. Following Ramanujan, we use the notation

$$a_n = a^n + a^{-n}. \tag{3.7}$$

For example, the 27 values of n with a_2 as a factor of λ_n are

2, 8, 9, 10, 11, 15, 19, 21, 22, 25, 26, 27, 28, 30, 31, 34, 40, 42, 45, 46, 47, 50, 55, 57, 58, 59, 62, 66, 70, 74, 75, 78, 79, 86, 94, 98, 106, 110, 122, 126, 130, 142, 154, 158, 170, 174, 206.

Table 1 Congruences for λ_n

$\lambda_n \equiv 0 \pmod{a_2}$	47	$\lambda_n \equiv 0 \pmod{a_1}$	3
$\lambda_n \equiv 1 \pmod{a_2}$	27	$\lambda_n \equiv 0 \pmod{a_1 - 1}$	19
$\lambda_n \equiv -1 \pmod{a_2}$	27	$\lambda_n \equiv 1 \pmod{a_1 - 1}$	26
$\lambda_n \equiv a_1 - 1 \pmod{a_2}$	22	$\lambda_n \equiv -1 \pmod{a_1 - 1}$	26
$\lambda_n \equiv -(a_1 - 1) \pmod{a_2}$	23	$\lambda_n \equiv 0 \pmod{a_1 + 1}$	2

For complete lists of all tables, see the authors' paper [5, pp. 85–88]. Remarkably, all ten tables are complete, except that in the third table Ramanujan missed one of the 27 values. For the first five tables, the appropriate values of n can be determined from examining the power series for the quotients of theta functions appearing in the 2-dissection (3.3) [5, pp. 85–86]. For example, consider the third congruence $\lambda_n \equiv -1 \pmod{a^2 + a^{-2}}$ listed in Table 1. To determine the values of n satisfying this congruence, from (3.3) with q replaced by q^2, we must determine the values of n for which the coefficient of q^n in

$$\frac{f(-q^6, -q^{10})}{(-q^4; q^4)_\infty} \tag{3.8}$$

is equal to -1 and simultaneously the coefficient of q^n in

$$q\frac{f(-q^2, -q^{14})}{(-q^4; q^4)_\infty} \tag{3.9}$$

is equal to 0. There are thus 27 such values.

For the remaining five congruences listed in Table 1, the appropriate values of n in Ramanujan's last five tables can be determined from congruences for the generating function (2.8) [5, pp. 86–88]. Thus, for the sixth congruence in Table 1, from the calculation

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \equiv \frac{(q;q)_\infty}{(-q^2;q^2)_\infty} = \frac{f(-q)f(-q^2)}{f(-q^4)} \pmod{a_1}, \tag{3.10}$$

where $f(-q)$ is defined by (3.2), we see that in his sixth table Ramanujan recorded the degree of q for the terms with zero coefficients in the power series expansion of $f(-q)f(-q^2)/f(-q^4)$ [5, p. 87].

Similarly [5, pp. 87–88],

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \equiv \frac{(q^2;q^2)_\infty}{(-q^3;q^3)_\infty} = \frac{f(-q^2)f(-q^3)}{f(-q^6)} \pmod{a_1 - 1} \tag{3.11}$$

and

$$\frac{(q;q)_\infty}{(aq;q)_\infty(q/a;q)_\infty} \equiv \frac{(q;q)^2_\infty}{(q^3;q^3)_\infty} = \frac{f^2(-q)}{f(-q^3)} \pmod{a_1 + 1}. \tag{3.12}$$

Thus, in his Tables 7–9, Ramanujan recorded the degree of q for the terms with coefficients 0, 1, and -1 respectively, in the theta quotient on the right side of (3.11), and in Table 10, he recorded the degree of q for the terms with zero coefficients in the theta quotient on the right-hand side of (3.12). In fact, the only values in this last table are $n = 14, 17$.

In order to demonstrate that Ramanujan's ten tables are complete, we need to examine the coefficients in the power series expansions of the quotients of theta functions in (3.8)–(3.12). In [5], using computer calculations, we conjectured that the coefficients in certain dissections of these products are monotonic for all n larger than certain specific values. If we could demonstrate that these conjectures are valid and then determine the values of λ_n below these certain thresholds, then we would have the means to show that Ramanujan's tables are complete. Using the Hardy–Ramanujan circle method, O.-Y. Chan [8] proved that the conjectured monotonicities did indeed hold. Upon checking the values of λ_n below the thresholds of monotonicity, Chan then completed the proof that all ten tables of values of n are complete (with the exception of one missing value noted earlier). It should be remarked that in the course of proving a conjecture of Andrews and R. Lewis [2] on ranks, D. Kane [14] also thereby proved the completeness of Ramanujan's Table 10.

The largest value of n found in Ramanujan's tables is $n = 302$ appearing in Table 3. Unless Ramanujan determined the values in his tables in a manner entirely different from how we have proceeded above, it seems that Ramanujan would have needed to calculate by hand the coefficients of five quotients of theta functions up to several hundred terms each. Even for Ramanujan, this appears to be a prodigious task, especially when he was dying.

Another question arises. Why did Ramanujan devote considerable energy in his last year to determining the values of λ_n satisfying these particular congruences in Table 1? Clearly, there was little hope of finding an infinite sequence of values of λ_n satisfying one of these congruences, like he discovered for $p(5n+4)$, for example. Why were these particular congruences chosen? Was his goal to discover something about $p(n)$?

We now present evidence that, as surmised in the penultimate paragraph, Ramanujan did calculate the coefficients in quotients of certain power series. First define

$$L_{p,r}(q) := \frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty (q^{p-r};q^p)_\infty} = (1-q^r)^{-1} F_{q^r}(q^p).$$

The functions $L_{11,r}(q)$, $1 \le r \le 5$, appear in the 11-dissection of $F_a(q)$ when $a = e^{2\pi i n/11}$. On pages 63 and 64 in his lost notebook [19], Ramanujan calculates the power series of $L_{11,1}(q)$ and $L_{11,2}(q)$, respectively, the first up to q^{100} and the second up to q^{33}. All of the calculated coefficients in these series are nonnegative, and most (but not all) are increasing. Liaw [15, 16], possibly in concordance with Ramanujan's observations illustrated by these two examples, proved the following theorem.

Theorem 3.2 *Let p and r be positive integers with $p \ge 2$ and $r < p$. Write*

$$L_{p,r}(q) = \sum_{n=0}^{\infty} b_{p,r}(n) q^n. \tag{3.13}$$

Then $b_{p,r}(n) \geq 0$ for all n. Moreover, we let

$$L_{p,r}(q) + q^p := \sum_{n=0}^{\infty} c_{p,r}(n)q^n := \Sigma_0 + \Sigma_1 + \cdots + \Sigma_{r-1}, \tag{3.14}$$

where the exponents in Σ_j are congruent to j modulo r, $0 \leq i \leq r-1$, i.e.,

$$\Sigma_i = \sum_{n=0}^{\infty} c_{p,r}(nr+i)q^{nr+i}.$$

Then for each i the coefficient sequence $\{c_{p,r}(nr+i)\}_{n=0}^{\infty}$ is non-decreasing.

4 Factoring the crank

On page 58 in his lost notebook [19], Ramanujan wrote out completely the first 22 terms of the power series representation (2.8) for the crank. Recalling that a_n is defined in (3.7), we provide only a few of the terms

$$\begin{aligned}
&1 + q(a_1 - 1) + q^2 a_2 + q^3(a_3 + 1) + q^4(a_4 + a_2 + 1) + q^5(a_5 + a_3 + a_1 + 1) + \cdots \\
&\quad + q^{20}(a_2 - a_1 + 1)(a_3 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\quad \times (a_{10} + a_6 + a_4 + a_3 + 2a_2 + 2a_1 + 3) \\
&\quad + q^{21} a_1 a_2 (a_3 + 1)(a_2 - a_1 + 1)(a_5 + a_4 + a_3 + a_2 + a_1 + 1) \\
&\quad \times (a_8 - a_6 + a_4 + a_1 + 2) + \cdots .
\end{aligned}$$

It should be noted that Ramanujan wrote the coefficients λ_n in factored form. On p. 59, Ramanujan wrote out some (but not all) of the factors of the coefficients of q^n, $13 \leq n \leq 26$. He gave no hint of his reasons for either recording the coefficients in detail, or for factoring them, or for examining only certain factors. It appears that Ramanujan was attempting to find some patterns in the coefficients of the crank generating function. Perhaps, on setting $a = 1$ he had then hoped to draw conclusions about the factors of $p(n)$.

On p. 181 in his lost notebook [19], Ramanujan returns to the coefficients λ_n in the generating function (2.8) of the crank. He factors λ_n, $1 \leq n \leq 21$, as before, but singles out nine particular factors by giving them special notation. The criterion that Ramanujan apparently uses is that of multiple occurrence, i.e., each of these nine factors appears more than once in the 21 factorizations, while other factors not favorably designated appear only once. The factors designated by Ramanujan are

$\rho_1 = a_1 - 1$	$\rho_5 = a_4 + a_2 + 1$
$\rho = a_2 - a_1 + 1$	$\rho_7 = a_3 + a_2 + a_1 + 1$
$\rho_2 = a_2$	$\rho_9 = (a_2 + 1)(a_3 + 1)$
$\rho_3 = a_3 + 1,$	$\rho_{11} = a_5 + a_4 + a_3 + a_2 + a_1 + 1$
$\rho_4 = a_1 a_2$	

At first glance, there does not appear to be any reasoning behind the choice of subscripts; note that there is no subscript for the second value. However, observe that in each case if we set $a = 1$, then the subscript n is equal to the right-hand side. The reason ρ does not have a subscript is that the value of n in this case would be $3 - 2 = 1$, which has been reserved for the first factor. In the table below, we record the content of page 181.

$$
\begin{aligned}
p(1) &= 1, & \lambda_1 &= \rho_1,\\
p(2) &= 2, & \lambda_2 &= \rho_2,\\
p(3) &= 3, & \lambda_3 &= \rho_3,\\
p(4) &= 5, & \lambda_4 &= \rho_5,\\
p(5) &= 7, & \lambda_5 &= \rho_7\rho,\\
p(6) &= 11, & \lambda_6 &= \rho_1\rho_{11},\\
p(7) &= 15, & \lambda_7 &= \rho_3\rho_5,\\
p(8) &= 22, & \lambda_8 &= \rho_1\rho_2\rho_{11},\\
p(9) &= 30, & \lambda_9 &= \rho_2\rho_3\rho_5,\\
p(10) &= 42, & \lambda_{10} &= \rho\rho_2\rho_3\rho_7,\\
p(11) &= 56, & \lambda_{11} &= \rho_4\rho_7(a_5 - a_4 + a_2),\\
p(12) &= 77, & \lambda_{12} &= \rho_7\rho_{11}(a_4 - 2a_3 + 2a_2 - a_1 + 1),\\
p(13) &= 101, & \lambda_{13} &= \rho\rho_1\,(a_{10} + 2a_9 + 2a_8 + 2a_7 + 3a_6\\
& & &\quad + 4a_5 + 6a_4 + 8a_3 + 9a_2 + 9a_1 + 9)\,,\\
p(14) &= 135, & \lambda_{14} &= \rho_5\rho_9(a_5 - a_3 + a_1 + 1),\\
p(15) &= 176, & \lambda_{15} &= \rho_4\rho_{11}(a_7 - a_6 + a_4 + a_1),\\
p(16) &= 231, & \lambda_{16} &= \rho_3\rho_7\rho_{11}(a_5 - 2a_4 + 2a_3 - 2a_2 + 3a_1 - 3),\\
p(17) &= 297, & \lambda_{17} &= \rho_9\rho_{11}(a_7 - a_6 + a_3 + a_1 - 1),\\
p(18) &= 385, & \lambda_{18} &= \rho_5\rho_7\rho_{11}(a_6 - 2a_5 + a_4 + a_3 - a_2 + 1),\\
p(19) &= 490, & \lambda_{19} &= \rho_1\rho_2\rho_5\rho_7(a_9 - a_7 + a_4 + 2a_3 + a_2 - 1),\\
p(20) &= 627, & \lambda_{20} &= \rho\rho_3\rho_{11}(a_{10} + a_6 + a_4 + a_3 + 2a_2 + 2a_1 + 3),\\
p(21) &= 792, & \lambda_{21} &= \rho\rho_3\rho_4\rho_{11}(a_8 - a_6 + a_4 + a_1 + 2).
\end{aligned}
$$

These factors lead to the rapid calculation of values for $p(n)$. For example, since $\lambda_{10} = \rho\rho_2\rho_3\rho_7$, then $p(10) = 1 \cdot 2 \cdot 3 \cdot 7 = 42$.

Ramanujan evidently was searching for some general principles or theorems on the factorization of λ_n so that he could not only compute $p(n)$ but make deductions about the divisibility of $p(n)$. No theorems are stated by Ramanujan. Is it possible to determine that certain factors appear in some precisely described infinite family of values of λ_n? What are the motivations that led Ramanujan to make these factorizations?

5 The evidence: summary

In the foregoing discussions of the crank, pp. 20, 58–59, 61, 63–64, 70–71, and 179–181 in the lost notebook were cited. Ramanujan's 5-dissection for the crank is given on p. 20. *All* of the material on the remaining ten pages is devoted to the crank. In fact, all of pp. 58–89 contain either incipient material or scratch work. Needless to say, it is very difficult to determine what Ramanujan was contemplating in most of his scratch work, but it is likely that all these pages are related to cranks. Pages 65 (which is the same as p. 73), 66, 72, 77, 80–81, and 83–85 are almost certainly related to crank calculations, while we are unable to determine conclusively if pp. 60, 62, 67–69, 74–76, 78–79, 82, and 86–89 pertain to cranks.

6 Ramanujan's final days

In December, 1983, one of the present authors (BCB) asked Janaki about the extent of the papers left behind by her late husband. In particular, Berndt mentioned to her that Ramanujan's lost notebook contains 138 pages. She claimed that Ramanujan had many more than 138 pages in his possession at his death. She related that as her husband "did his sums," he would deposit his papers in a large leather trunk beneath his bed. She told Berndt that during her husband's funeral, certain people, whom she named but whom we do not name here, came to her home and stole most of her husband's papers and never returned them. She later donated the un-stolen papers of Ramanujan to the University of Madras. These papers certainly contain or possibly entirely comprise the lost notebook.

It is our contention that Ramanujan kept at least two stacks of papers while doing mathematics in his last year. In one stack, he put primarily the pages he wanted to save, i.e., those containing the statements of his theorems, and in another stack or stacks he put papers containing his calculations and proofs. The one stack of papers containing the lost notebook was likely in a different place and missed by those taking his other papers. Of course, it is certainly possible that more than one stack of papers contained statements of results that Ramanujan wanted to save. Undoubtedly, Ramanujan produced scores of pages containing calculations, scratch work, and proofs, but, remarkably, the extant scratch work belongs almost exclusively to one topic, ***cranks***. Our guess is that when Ramanujan ceased research four days prior to his death, he was thinking about cranks. His power series expansions, factorizations, preliminary tables, and scratch work were part of his deliberations and had not yet been put in a secondary pile of papers. Thus, these sheets were found with the papers that were set aside containing his final theorems and thus unofficially became part of his lost notebook. In particular, pp. 58–89 likely include some pages that Ramanujan intended to keep in his principal stack, but most of this work probably would have been relegated to a secondary stack if Ramanujan had lived longer.

7 Did Ramanujan think combinatorially?

Since Definitions 2.1 and 2.2 of rank and crank, respectively, are important combinatorial concepts, it is natural to ask if Ramanujan knew the combinatorial interpretations of the generating functions (2.6) for the rank and (2.8) for the crank. His notebooks [18] have little discourse, and his lost notebook [19] contains almost no words at all. Since Ramanujan had so many ideas and so little time in which to record his findings, doubtless, he was not going to use the small amount of valuable time that remained in the last year of his life to record definitions or combinatorial interpretations. To him, such interpretations might have been obvious with no need to record them, and especially since both the "ordinary" notebooks and lost notebook were never intended for publication but were simply his own personal compilation of what he had discovered, the need for Ramanujan to record such interpretations was probably close to nil. Some of his published papers are testimony that Ramanujan was an excellent combinatorial thinker. Thus, in conclusion, one can only speculate about how much or how little Ramanujan thought combinatorially, especially about ranks and cranks, but Ramanujan probably discovered more combinatorially than he has been given credit for.

Acknowledgements The authors gratefully acknowledge perspicacious observations about dissections supplied by Frank Garvan.

References

1. Andrews, G.E., Garvan, F.G.: Dyson's crank of a partition. Bull. Am. Math. Soc. (N.S.) **18**, 167–171 (1988)
2. Andrews, G.E., Lewis, R.: The ranks and cranks of partitions moduli 2, 3, and 4. J. Number Theory **85**, 74–84 (2000)
3. Atkin, A.O.L., Swinnerton-Dyer, H.P.F.: Some properties of partitions. Proc. Lond. Math. Soc. (3) **4**, 84–106 (1954)
4. Berndt, B.C., Chan, H.H., Chan, S.H., Liaw, W.-C.: Cranks and dissections in Ramanujan's lost notebook. J. Comb. Theory (A) **109**, 91–120 (2005)
5. Berndt, B.C., Chan, H.H., Chan, S.H., Liaw, W.-C.: Ramanujan and cranks. In: Ismail, M.E.H., Koelink, E. (eds.) Theory and Applications of Special Functions. A Volume Dedicated to Mizan Rahman, pp. 77–98. Springer, Dordrecht (2005)
6. Berndt, B.C., Rankin, R.A.: Ramanujan: Letters and Commentary. American Mathematical Society, Providence (1995). London Mathematical Society, London (1995)
7. Berndt, B.C., Rankin, R.A.: Ramanujan: Essays and Surveys. American Mathematical Society, Providence (2001). London Mathematical Society, London (2001)
8. Chan, O.-Y.: Some asymptotics for cranks. Acta Arith. **120**, 107–143 (2005)
9. Dyson, F.J.: Some guesses in the theory of partitions. Eureka (Cambridge) **8**, 10–15 (1944)
10. Garvan, F.G.: Generalizations of Dyson's rank. Ph. D. Thesis, Pennsylvania State University, University Park, PA (1986)
11. Garvan, F.G.: Combinatorial interpretations of Ramanujan's partition congruences. In: Andrews, G.E., Askey, R.A., Berndt, B.C., Ramanathan, K.G., Rankin, R.A. (eds.) Ramanujan Revisited, pp. 29–45. Academic Press, Boston (1988)
12. Garvan, F.G.: New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11. Trans. Am. Math. Soc. **305**, 47–77 (1988)
13. Garvan, F.G.: The crank of partitions mod 8, 9 and 10. Trans. Am. Math. Soc. **322**, 79–94 (1990)
14. Kane, D.M.: Resolution of a conjecture of Andrews and Lewis involving cranks of partitions. Proc. Am. Math. Soc. **132**, 2247–2256 (2004)

15. Liaw, W.-C.: Contributions to Ramanujan's theories of modular equations, Ramanujan-type series for $1/\pi$, and partitions. Ph. D. Thesis, University of Illinois at Urbana-Champaign, Urbana (1999)
16. Liaw, W.-C.: Note on the monotonicity of coefficients for some q-series arising from Ramanujan's lost notebook. Ramanujan J. **3**, 385–388 (1999)
17. Ramanujan, S.: Collected Papers. Cambridge University Press, Cambridge (1927). reprinted by Chelsea, New York (1962); reprinted by the American Mathematical Society, Providence (2000)
18. Ramanujan, S.: Notebooks. Tata Institute of Fundamental Research, Bombay (1957) (2 volumes)
19. Ramanujan, S.: The Lost Notebook and Other Unpublished Papers. Narosa, New Delhi (1988)
20. Watson, G.N.: The final problem: an account of the mock theta functions. J. Lond. Math. Soc. **11**, 55–80 (1936)
21. Watson, G.N.: The mock theta functions (2). Proc. Lond. Math. Soc. **42**, 274–304 (1937)