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New analogues of Clausen's identities arising from the theory of modular forms

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Abstract

Around 1828, T. Clausen discovered that the square of certain hypergeometric ${}_2F_1$ function can be expressed as a hypergeometric ${}_3F_2$ function. Special cases of Clausen's identities were later used by S. Ramanujan in his derivation of 17 series for $1/\pi$. Since then, there were several attempts to find new analogues of Clausen's identities with the hope to derive new classes of series for $1/\pi$. Unfortunately, none were successful. In this article, we will present three new analogues of Clausen's identities. Their discovery is motivated by the study of relations between modular forms of weight 2 and modular functions associated with modular groups of genus 0.

Keywords: Clausen's identities; Modular forms of one variable

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1. Introduction

For $|x| < 1$, let

$$ {}_{p+1}F_p\left(\begin{matrix} a_0, a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_p)_n} \frac{x^n}{n!}, $$

where

$$ (a)_0 = 1 \quad \text{and} \quad (a)_n = a(a+1)\cdots(a+n-1) \quad \text{for } n \in \mathbb{Z}_{>0}. $$

In his pioneering paper "*Modular equations and approximations to* π" [16], S. Ramanujan constructed 17 series for $1/\pi$ and indicated briefly how he derived these series. In Section 13 of the paper, he cited a special case of Clausen's identities [16, Eq. (25)], namely,

$$ \left\{ {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix}; x\right)\right\}^2 = {}_3F_2\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1 \end{matrix}; 4x(1-x)\right), \tag{1.1} $$

and indicated its importance in the derivation of his series for $1/\pi$ such as

$$ \frac{4}{\pi} = \sum_{n=0}^{\infty} (6n+1) \frac{(\frac{1}{2})_n^3}{n!^3} \frac{1}{4^n}. $$

The complete proofs of Ramanujan's series for $1/\pi$, as well as many other generalizations, first appeared in the book "*Pi and the AGM*" [4] by J.M. Borwein and P.B. Borwein. Not surprisingly, Clausen's identities are essential in Borweins' proofs. Since then, several mathematicians (including some of us) have believed that in order to derive new classes of series for $1/\pi$, new analogues of Clausen's identities are needed. Of course, this belief is not well reflected in the literature (but see, for example, [3]), and it turns out to be incorrect. Around 2002, T. Sato [18] derived a series for $1/\pi$ associated with the Apéry numbers

$$ A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad n = 0, 1, 2, \ldots. \tag{1.2} $$

More precisely, Sato discovered that

$$ \frac{1}{\pi} \frac{\sqrt{15}}{6(4\sqrt{5}-9)} = \sum_{n=0}^{\infty} (20n + 10 - 3\sqrt{5}) A_n \left(\frac{\sqrt{5}-1}{2}\right)^{12n}. \tag{1.3} $$

Sato's series inspired H.H. Chan, S.H. Chan and Z.G. Liu [9] to construct Ramanujan-type series for $1/\pi$ without the use of Clausen's identities. Meanwhile, the search for new analogues of Clausen's identities appears to have ceased.

In this article, we will present three new identities similar to Clausen's identities associated with the Apéry-like integer sequences

$$r_n = \sum_{k=0}^{n} \binom{n}{k}^3, \qquad s_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k},$$

$$\text{and} \quad t_n = \sum_{k=0}^{n} \binom{n}{k} (-8)^{n-k} \sum_{j=0}^{k} \binom{k}{j}^3. \tag{1.4}$$

These are given by

$$\left\{ \sum_{n=0}^{\infty} r_n x^n \right\}^2 = \frac{1}{1+8x^2} \sum_{n=0}^{\infty} \binom{2n}{n} r_n \left(\frac{x(1+x)(1-8x)}{(1+8x^2)^2} \right)^n, \tag{1.5}$$

$$\left\{ \sum_{n=0}^{\infty} s_n x^n \right\}^2 = \frac{1}{1-9x^2} \sum_{n=0}^{\infty} \binom{2n}{n} s_n \left(\frac{x(1-9x)(1-x)}{(1-9x^2)^2} \right)^n, \tag{1.6}$$

and

$$\left\{ \sum_{n=0}^{\infty} t_n x^n \right\}^2 = \frac{1}{1-72x^2} \sum_{n=0}^{\infty} \binom{2n}{n} t_n \left(\frac{x(1+8x)(1+9x)}{(1-72x^2)^2} \right)^n. \tag{1.7}$$

It is clear that these identities are similar to (1.1) once we write (1.1) as

$$\left\{ \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(1)_n (1)_n} x^n \right\}^2 = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(1)_n (1)_n} \big(x(1-x)\big)^n. \tag{1.8}$$

Our proofs of (1.5)–(1.7) will be given based on the study of modular forms associated with subgroups of the normalizer of $\Gamma_0(6)$ in $SL_2(\mathbb{R})$.

The existence of new series for $1/\pi$ associated with the right-hand sides of (1.5)–(1.7) should not be surprising since the search of new series for $1/\pi$ is the main motivation behind finding new analogues of Clausen's identities. We will list a new series for $1/\pi$ associated with the sequences discussed in this article. Most of these series can be derived using ideas presented in [9] and we stress again that their proofs are independent of the Clausen-type identities (1.5)–(1.7). However the same modular parametrizations are needed for proving both the series and identities. We end this introduction with a preview of three of these new series for $1/\pi$. These are

$$\frac{25}{2\pi} = \sum_{n=0}^{\infty} (9n+2) \binom{2n}{n} r_n \left(\frac{1}{50} \right)^n,$$

$$\frac{9\sqrt{3}}{2\pi} = \sum_{n=0}^{\infty} (5n+1) \binom{2n}{n} s_n \left(\frac{1}{54} \right)^n,$$

and

$$\frac{8}{\sqrt{3}\pi} = \sum_{n=0}^{\infty} (10n+3) \binom{2n}{n} t_n \left(\frac{1}{64} \right)^n.$$

Note that there exists a different method for discovering and proving identities like (1.5)–(1.7), using computer algebra system such as MAPLE. We briefly outline this approach in Section 6 together with further identities of this kind; more details on the method can be found in the recent work [1] by G. Almkvist, D. van Straten and W. Zudilin. A drawback of this algebraic machinery is the lack of arithmetical insight; in particular, no series for $1/\pi$ can be obtained using this approach.

2. Some interesting binomial expressions and their relations with modular forms

We begin this section by describing the motivation behind our choices of modular forms and modular functions that lead us to the discoveries of (1.5)–(1.7). We will then concentrate on the derivation of (1.5) and its related identities (such as the series for $1/\pi$). The derivations of (1.6) and (1.7) are similar.

We first fix some notations. Let N be a positive integer and let

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \ (\mathrm{mod}\ N) \right\}.$$

If $e \| N$, we call the matrix

$$W_e = \begin{pmatrix} a\sqrt{e} & b/\sqrt{e} \\ cN/\sqrt{e} & d\sqrt{e} \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z},\ \det(W_e) = 1,$$

the Atkin–Lehner involution for $\Gamma_0(N)$. We let $\Gamma_0(N)_+$ to be the group generated by $\Gamma_0(N)$ and all the Atkin–Lehner involutions for $\Gamma_0(N)$. We will use $\Gamma_0(N)_{+e}$ to denote the group generated by $\Gamma_0(N)$ and W_e.

2.1. From binomial sums to modular forms

Our general approach is motivated by the following example. In light of Sato's series (1.3) for $1/\pi$, one may ask whether, for a given binomial expression analogous to (1.2), there exist $1/\pi$-series associated to it. As a test example, we consider the binomial expression

$$B_n = \sum_{k=0}^{n} \binom{n}{k}^4 \tag{2.1}$$

and its generating function

$$Z(X) = \sum_{n=0}^{\infty} B_n X^n.$$

Using the method of creative telescoping, one finds that $Z(X)$ satisfies the differential equation

$$\left(1 - 12X - 64X^2\right)\theta^3 Z - \left(18X + 192X^2\right)\theta^2 Z - \left(10X + 188X^2\right)\theta Z - \left(2X + 60X^2\right)Z = 0, \tag{2.2}$$

where θ denotes the differential operator Xd/dX. We can verify that this differential equation is the symmetric square of the differential equation

$$\left(1-12X-64X^2\right)\theta^2 Y-\left(6X+64X^2\right)\theta Y-\left(X+15X^2\right)Y=0. \tag{2.3}$$

(That is, $Z(X)^{1/2}$ is a solution of the latter second-order differential equation.) If its monodromy group is a congruence subgroup of $SL_2(\mathbb{R})$, then there exists a modular form Z of weight 2 and a modular function X such that Z, as function of X, satisfies (2.2).

To determine the monodromy group of (2.3), one can employ the method of [11] to compute the monodromy group approximately and hopefully one can read off the group from the numerical data. Here, however, the following indirect method works better.

Applying the Frobenius method to (2.2), we find a basis for the solution space near $X=0$ is given by

$$\begin{aligned}
F_0(X)&=1+2X+18X^2+164X^3+1810X^4+21\,252X^5+\cdots,\\
F_1(X)&=F_0(X)\big(\log X+g(X)\big),\\
F_2(X)&=F_0(X)\left(\frac{(\log X)^2}{2}+g(X)\log X+h(X)\right),
\end{aligned}$$

where

$$\begin{aligned}
g(X)&=4X+30X^2+\frac{760}{3}X^3+2695X^4+\frac{154\,704}{5}X^5+\cdots,\\
h(X)&=8X^2+120X^3+\frac{4390}{3}X^4+18\,380X^5+\frac{10\,651\,594}{45}X^6+\cdots.
\end{aligned}$$

(Note that $F_0(X)$ is just $Z(X)$.) Now if (2.2) is indeed the differential equation satisfied by a modular form $F_0(X(\tau))$ of weight 2 and a modular function $X(\tau)$, then we should have $2\pi i\tau=F_1(X)/F_0(X)$ and

$$e^{2\pi i\tau}=e^{F_1(X)/F_0(X)}=Xe^{g(X)}.$$

Setting $q=e^{2\pi i\tau}=Xe^{g(X)}$ and inverting the function, we get

$$X=q-4q^2-6q^3+56q^4-45q^5-360q^6+894q^7+960q^8+\cdots.$$

Substituting this into $Z(X)$, we obtain

$$Z=1+2q+10q^2+8q^3+26q^4+2q^5+40q^6+16q^7+58q^8+\cdots.$$

These should be the q-expansions of our hypothetical modular function and modular form. To identify what modular function X is, we observe that the orders of the local monodromy at the four singularities 0, $1/16$, $-1/4$, and ∞ are ∞, 2, 2, and 4, respectively. This means that X should be a modular function on a congruence subgroup with one cusp, two elliptic points of order 2, and one elliptic point of order 4. There are not many congruence subgroups of this signature. In fact, the only congruence subgroup we can think of is $\Gamma_0(10)_+$. Indeed, after some trial, we find that the function

$$\left\{\left(\frac{\eta(\tau)\eta(10\tau)}{\eta(2\tau)\eta(5\tau)}\right)^6+\left(\frac{\eta(2\tau)\eta(5\tau)}{\eta(\tau)\eta(10\tau)}\right)^6-2\right\}^{-1} \tag{2.4}$$

has the same starting q-expansion as X, while the modular form

$$\frac{1}{12}\bigl(10P(10\tau)+5P(5\tau)-P(\tau)-2P(2\tau)\bigr) \tag{2.5}$$

has the same starting q-expansion as Z, where

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}\bigl(1-q^n\bigr)$$

is Dedekind's eta function and

$$P(\tau)=\frac{12}{\pi i}\frac{d\eta(\tau)/d\tau}{\eta(\tau)}=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$

its (normalized) logarithmic derivative. We then use the method of [23] to verify that this pair of modular function and modular form indeed satisfies the differential equation (2.2). Now to obtain a series for $1/\pi$ analogous to Sato's example (1.3), we may follow the procedure in [9]. For example, we find

$$\frac{18}{\sqrt{15}\pi}=\sum_{n=0}^{\infty}(4n+1)\sum_{k=0}^{n}\binom{n}{k}^4\frac{1}{36^n}; \tag{2.6}$$

here $1/36$ is the value of $X(\tau)$ at $\tau=i/\sqrt{30}$.

This example motivated us to search for other modular forms that are associated with elegant binomial expressions such as (2.1). To do so, one can start with simple binomial expression and attempt to find suitable modular forms that are related to the expression. One can, on the other hand, begin by studying various modular forms similar to the above example and hopefully derive binomial expressions. We will adopt the latter approach in this article. Identities (1.5)–(1.7) are discovered via this way.

Observe that the function in (2.4) is a Hauptmodul for $\Gamma_0(10)_+$ and the function in (2.5) is the logarithmic derivative of

$$\left(\frac{\eta(5\tau)\eta(10\tau)}{\eta(\tau)\eta(2\tau)}\right)^2,$$

which is a Hauptmodul for $\Gamma_0(10)_{+2}$. Thus, we introduce the following notations.

Assume that $\Gamma_0(N)_{+m}$ has genus 0 (see [5] for a complete list of such groups). Let $X_{N,m}$ denote a fixed Hauptmodul for $\Gamma_0(N)_{+m}$. Corresponding to this Hauptmodul, we fix a modular form of weight 2 and denote this form by $Z_{N,m}$. Next, we let

$$\hat{Z}_{N,m} = \vartheta\big(\ln(X_{N,m})\big),$$

where

$$\vartheta(F) = q\frac{dF}{dq} = \frac{1}{2\pi i}\frac{dF}{d\tau}.$$

Remark 2.1. Perhaps it would be more convenient to us to define $\hat{Z}_{N,m}$ as $Z_{\vartheta \ln(X_{N,m})}$, since $\hat{Z}_{N,m}$ is nothing else but $\vartheta \ln(X_{N,m})$. We keep the easier notation in order to make many formulas below more accessible.

With this choice of $\hat{Z}_{N,m}$, we fix a modular function and denote it by $\hat{X}_{N,m}$. Throughout the paper, we use the letter Z with subscripts for denoting modular forms of weight 2. With the above notation, the functions in (2.5) and (2.4) are written as $\hat{Z}_{10,2}$ and $\hat{X}_{10,2}$, respectively.

2.2. Binomial sums related to $\Gamma_0(6)_+$

A natural extension of the example above is to consider other subgroups of $\Gamma_0(10)_+$. We are, however, unable to derive sufficiently interesting results analogous to the example of Subsection 2.1. As such, we turn our attention to $\Gamma_0(6)_+$.

We let

$$X_{6,2} = \left(\frac{\eta(3\tau)\eta(6\tau)}{\eta(\tau)\eta(2\tau)}\right)^4; \tag{2.7}$$

this is our choice of a Hauptmodul for $\Gamma_0(6)_{+2}$. Its logarithmic derivative is

$$\hat{Z}_{6,2} = \frac{1}{6}\big(6P(6\tau) + 3P(3\tau) - 2P(2\tau) - P(\tau)\big). \tag{2.8}$$

We next set

$$\hat{X}_{6,2} = \frac{X_{6,6}}{(1 - X_{6,6})^2}, \tag{2.9}$$

where

$$X_{6,6} = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}. \tag{2.10}$$

Our first main result is

Theorem 2.1. *For sufficiently small values of $|\hat{X}_{6,2}|$, we have*

$$\hat{Z}_{6,2} = \sum_{n=0}^{\infty} R_n \hat{X}_{6,2}^n \quad \text{where } R_n = \binom{2n}{n} r_n = \binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^3. \tag{2.11}$$

The discovery of (2.11) is coincidental. Using the q-series expansions of $\hat{Z}_{6,2}$ and $\hat{X}_{6,2}$, we can first guess and then rigorously derive the recurrence satisfied by the sequence $\{R_n\}$, namely,

$$(n+1)^3 R_{n+1} + 2(2n+1)\bigl(7n^2+7n+2\bigr)R_n + 32n(2n+1)(2n-1)R_{n-1} = 0. \quad (2.12)$$

There is no efficient algorithm at present to determine R_n in terms of binomial expression from (2.12). The only way, as D. Zeilberger pointed out several years ago to the first author, is to search for this sequence in databases such as [20] or [2]. If a binomial expression of our sequence shows up in the database, then we verify that our sequence has the given form in the database by using creative telescoping (see [9,21,7] for such computations). If the sequence that we are interested does not appear in the database, then we are "out of luck". Such is the case in our attempt to find R_n. In the following sketch of the proof of Theorem 2.1, we will only discuss the derivation of R_n.

Proof of Theorem 2.1. Using the theory of modular forms, we can show that if

$$Z_{6,6} = \frac{1}{24}\bigl(-5P(\tau)+2P(2\tau)-3P(3\tau)+30P(6\tau)\bigr) = \frac{\eta^7(3\tau)\eta^7(2\tau)}{\eta^5(\tau)\eta^5(6\tau)}, \quad (2.13)$$

then

$$\hat{Z}_{6,2} = Z_{6,6}(1-X_{6,6}), \quad (2.14)$$

where $X_{6,6}$ is given by (2.10). It is known (see, for example, [7]) that

$$Z_{6,6} = \sum_{n=0}^{\infty} A_n X_{6,6}^n \quad (2.15)$$

where the sequence $\{A_n\}$ is the Apéry sequence (1.2). If we let R_n be such that

$$\hat{Z}_{6,2} = \sum_{n=0}^{\infty} R_n \hat{X}_{6,2}^n,$$

then by (2.9), (2.15) and (2.14) we find that

$$\sum_{k=0}^{\infty} R_k \frac{X^k}{(1-X)^{2k+1}} = \sum_{n=0}^{\infty} A_n X^n \quad (2.16)$$

where $X = X_{6,6}$. Now, the left-hand side of (2.16) can be written as

$$\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+2k}{m} R_k X^{m+k} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{R_k}{\binom{2k}{k}} X^n. \quad (2.17)$$

Comparing the coefficient of the resulting series in (2.17) with the corresponding one on the right-hand side of (2.16), we conclude that

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{R_k}{\binom{2k}{k}}. \tag{2.18}$$

In order to determine R_k from (2.18) we recall that the *Legendre transform* of a sequence $\{c_k\}$ is the sequence

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} c_k.$$

V. Strehl [21] and A. Schmidt [19] proved independently that the Legendre transform of the sequence

$$r_k = \sum_{j=0}^{k} \binom{k}{j}^3$$

is the Apéry sequence (1.2), namely,

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3. \tag{2.19}$$

Comparing (2.19) with (2.18) we conclude immediately that

$$R_k = \binom{2k}{k} r_k = \binom{2k}{k} \sum_{j=0}^{k} \binom{k}{j}^3. \tag{2.20}$$

This completes the sketch of our proof. □

2.3. Supercongruences

The following result shows that binomial expressions coming from differential equations satisfied by modular forms possess "strong" arithmetic properties.

Corollary 2.2. *For prime $p > 3$, we have*

$$R_{np} \equiv R_n \pmod{p^3}.$$

Proof. This follows immediately from the generalization [15]

$$\binom{2np}{np} \equiv \binom{2n}{n} \pmod{p^3}$$

of Wolstenholme's theorem and [10, Theorem 4.2],

$$\sum_{k=0}^{np} \binom{np}{k}^3 \equiv \sum_{k=0}^{n} \binom{n}{k}^3 \pmod{p^3}. \qquad \square$$

Corollary 2.2 was motivated by S. Chowla, J. Cowles and M. Cowles [12]. They conjectured that for the Apéry numbers (1.2) and primes $p > 3$,

$$A_p \equiv A_1 \pmod{p^3}.$$

I. Gessel [14] proved this conjecture by showing that $A_{np} \equiv A_n \pmod{p^3}$. Recently, H.H. Chan, S. Cooper and F. Sica [10] showed that the Domb sequence $\{D_n\}$, defined by

$$D_n = (-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k}, \tag{2.21}$$

satisfies the congruence $D_{np} \equiv D_n \pmod{p^3}$. They also indicated a possible source of sequences $\{F_n\}$ satisfying the congruence

$$F_{np} \equiv F_n \pmod{p^3}, \tag{2.22}$$

and it turns out that $\{R_n\}$ is one such example. It is therefore not surprising to see that Corollary 2.2 holds. However, the proof of Corollary 2.2 without the knowledge of the closed form of R_n is still missing.

The form $\hat{Z}_{6,2}$ and function $\hat{X}_{6,2}$ can also be written in terms of

$$Z_{6,3} = \frac{1}{6}\big(P(\tau) - 4P(2\tau) - 3P(3\tau) + 12P(6\tau)\big) = \frac{\eta^4(\tau)\eta^4(3\tau)}{\eta^2(2\tau)\eta^2(6\tau)} \tag{2.23}$$

and

$$X_{6,3} = \left(\frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)}\right)^6. \tag{2.24}$$

The analogues of (2.9) and (2.14) are then given by

$$\hat{X}_{6,2} = \frac{X_{6,3}}{(1 + 8X_{6,3})^2} \tag{2.25}$$

and

$$\hat{Z}_{6,2} = Z_{6,3}(1 + 8X_{6,3}). \tag{2.26}$$

It is known that (see [9,7] or [8])

$$Z_{6,3} = \sum_{n=0}^{\infty} D_n X_{6,3}^n,$$

where D_n is given by (2.21). Using (2.26) and (2.20) we conclude that

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-8)^{n-k} \sum_{j=0}^{k} \binom{k}{j}^3. \tag{2.27}$$

The interesting fact is that if we define

$$\mathcal{A}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 x^{n-k},$$

then (2.19) and (2.27) imply that $\{\mathcal{A}_n(1)\}$ gives the Apéry sequence (1.2) while $\{\mathcal{A}_n(-8)\}$ gives the Domb sequence (2.21), and the two sequences satisfy the congruence (2.22). It does not appear that the sequences $\{\mathcal{A}_n(N)\}$ for integers $N \neq 1, -8$ satisfy (2.22) (this was checked for a modest range of N). We note that our function $\mathcal{A}_n(x)$ can be expressed as $x^n a(1/x)$ where $a(x)$ is Schmidt's function [19] defined by

$$a(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 x^k.$$

It may be interesting to study functions similar to $\mathcal{A}_n(x)$. We end this section with the observation that the Domb sequence and the Almkvist–Zudilin sequence (given in formula (3.2) below), a sequence that arises from $\Gamma_0(6)_{+2}$, can also be expressed as $\mathcal{B}_n(-16)$ and $\mathcal{C}_n(-27)$ (see [8]), where

$$\mathcal{B}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \binom{n+2k}{k} \binom{2k}{k} x^{n-k}$$

and

$$\mathcal{C}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \binom{n+2k}{k} \binom{n+3k}{k} x^{n-k}.$$

3. More identities arising from modular forms and functions on $\Gamma_0(6)_+$

In the previous section, we considered only the logarithmic derivative of $X_{6,2}$. We first discuss identities associated with the logarithmic derivative of $X_{6,3}$ defined by (2.24), followed by those associated with the logarithmic derivative of $X_{6,6}$ given in (2.10).

Theorem 3.1. *Let*

$$\hat{Z}_{6,3} = \frac{1}{4}\big(6P(6\tau) + 2P(2\tau) - P(\tau) - 3P(3\tau)\big)$$

and

$$\hat{X}_{6,3}=\frac{X_{6,2}}{(1+9X_{6,2})^2}=\frac{X_{6,6}}{(1+X_{6,6})^2}.$$

Then for sufficiently small $|\hat{X}_{6,3}|$,

$$\hat{Z}_{6,3}=\sum_{n=0}^{\infty}S_n\hat{X}_{6,3}^n,$$

where

$$S_n=\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}. \tag{3.1}$$

In this case, the sequence $\{S_n\}$ matches nicely with a sequence in [20]. We then verified that the binomial expression in the database is indeed S_n using the method illustrated in [9]. With the additional identities

$$\hat{Z}_{6,3}=Z_{6,6}(1+X_{6,6})=Z_{6,2}(1+9X_{6,2})$$

and identities proved in [7], we derive the identities

$$\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}\sum_{j=0}^{k}\binom{k}{j}^2\binom{2j}{j}$$

and

$$\sum_{k=0}^{n}\frac{(4k)!}{(k!)^4}\binom{n+3k}{4k}(-27)^{n-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-9)^{n-k}\sum_{j=0}^{k}\binom{k}{j}^2\binom{2j}{j}. \tag{3.2}$$

These are clearly analogues of (2.19) and (2.27).

Theorem 3.2. *Let*

$$\hat{Z}_{6,6}=\frac{1}{2}\big(6P(6\tau)+P(\tau)-2P(2\tau)-3P(3\tau)\big)$$

and

$$\hat{X}_{6,6}=\frac{X_{6,2}}{(1-9X_{6,2})^2}=\frac{X_{6,3}}{(1-8X_{6,3})^2}.$$

Then

$$\hat{Z}_{6,6} = \sum_{n=0}^{\infty} T_n \hat{X}_{6,6}^n,$$

where

$$T_n = \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} (-8)^{n-k} \sum_{j=0}^{k} \binom{k}{j}^3. \tag{3.3}$$

The sequence $\{T_n\}$ does not match with any sequence in [20]. However, motivated by the expressions in (2.20) and (3.1), we consider the sequence $t_n = T_n/\binom{2n}{n}$ that matches with Verrill's sequence [22]. Once again, using the method in [9], we verify that T_n is given by (3.3).

Using the additional identities

$$\hat{Z}_{6,6} = Z_{6,2}(1 - 9X_{6,2}) = Z_{6,3}(1 - 8X_{6,3})$$

and the ones from [7], we derive the identities

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} 8^{n-k} \sum_{j=0}^{k} \binom{k}{j} (-8)^{k-j} \sum_{i=0}^{j} \binom{j}{i}^3$$

and

$$\sum_{k=0}^{n} \frac{(4k)!}{(k!)^4} \binom{n+3k}{4k} (-27)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} 9^{n-k} \sum_{j=0}^{k} \binom{k}{j} (-8)^{k-j} \sum_{i=0}^{j} \binom{j}{i}^3.$$

These are again analogues of (2.19) and (2.27).

4. Proofs of the new analogues of Clausen's identities

The motivation behind the derivations of (1.5)–(1.7) comes from the presence of the factor $\binom{2n}{n}$ in (2.20), (3.1) and (3.3). In view of (1.8), one naturally expects identities similar to Clausen's identities to hold for sequences $\{R_n\}$, $\{S_n\}$ and $\{T_n\}$.

We now discuss the proof of (1.5). The proofs of (1.6) and (1.7) are similar.

Proof of (1.5). In [22], Verrill showed that if

$$z_{6,-2} = \frac{\eta^6(3\tau)\eta(2\tau)}{\eta^2(\tau)\eta^3(6\tau)} \tag{4.1}$$

and

$$X_{6,-2} = \frac{\eta^3(\tau)\eta^9(6\tau)}{\eta^3(2\tau)\eta^9(3\tau)}, \tag{4.2}$$

then

$$z_{6,-2} = \sum_{n=0}^{\infty} r_n X_{6,-2}^n. \tag{4.3}$$

(We use the small letter z for modular forms of weight 1.)

Squaring (4.1) we find that

$$z_{6,-2}^2 = Z_{6,6} \frac{\eta^5(3\tau)\eta(\tau)}{\eta^5(2\tau)\eta(6\tau)}, \tag{4.4}$$

where we have used the product representation of $Z_{6,6}$ in (2.13). Now, we have

$$\frac{\eta^5(3\tau)\eta(\tau)}{\eta^5(2\tau)\eta(6\tau)} = \frac{1}{1 + X_{6,-2}}. \tag{4.5}$$

Substituting (4.5) and (2.14) into (4.4) we conclude that

$$z_{6,-2}^2 = \frac{\hat{Z}_{6,2}}{(1 - X_{6,6})(1 + X_{6,-2})}. \tag{4.6}$$

On the other hand, from [8] we find that

$$X_{6,6} = \frac{X_{6,-2}(1 - 8X_{6,-2})}{1 + X_{6,-2}}. \tag{4.7}$$

The parametrization (4.7) allows us to conclude that

$$1 - X_{6,6} = \frac{1 + 8X_{6,-2}^2}{1 + X_{6,-2}} \tag{4.8}$$

and to write (2.9) as

$$\hat{X}_{6,2} = \frac{X_{6,-2}(1 + X_{6,-2})(1 - 8X_{6,-2})}{(1 + 8X_{6,-2}^2)^2}. \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.6) and using (2.11) and (4.3) we conclude that for sufficiently small $|x|$, the identity (1.5) holds. □

The proofs of (1.6) and (1.7) are similar. In the case of (1.6) we use the fact (see [22]) that if

$$z_{6,-3} = \frac{\eta^6(2\tau)\eta(3\tau)}{\eta^2(6\tau)\eta^3(\tau)}$$

and

$$X_{6,-3} = \frac{\eta^4(\tau)\eta^8(6\tau)}{\eta^4(3\tau)\eta^8(2\tau)},$$

then

$$z_{6,-3} = \sum_{n=0}^{\infty} s_n X_{6,-3}^n.$$

Then one applies several algebraic relations between $X_{6,-3}$ and other modular functions such as $X_{6,2}$; the details are left to the reader.

The proof of (1.7) requires the fact (see [22]) that if

$$z_{6,-6} = \frac{\eta^6(\tau)\eta(6\tau)}{\eta^3(2\tau)\eta^2(3\tau)}$$

and

$$X_{6,-6} = \frac{\eta(2\tau)\eta^5(6\tau)}{\eta(3\tau)\eta^5(\tau)},$$

then

$$z_{6,-6} = \sum_{n=0}^{\infty} t_n X_{6,-6}^n.$$

Once again, the relations between $X_{6,-6}$ and other modular functions such as $X_{6,2}$ are needed.

5. Series for $1/\pi$

As mentioned in the introduction, our discovery of the new analogues of Clausen-type identities is motivated by numerical identities like (1.3) and (2.6). In [9], a general method is given to derive new classes of series for $1/\pi$ if one has the relation

$$Z = \sum_{k=0}^{\infty} a_n X^n$$

for some modular form Z of weight 2 and modular function X and provided that Z satisfies a certain transformation formula. In this section, we will list the main identities and the "rational" series for $1/\pi$ associated with (Z, X) when

$$(Z, X) = (\hat{Z}_{6,2}, \hat{X}_{6,2}), \quad (\hat{Z}_{6,3}, \hat{X}_{6,3}), \quad \text{and} \quad (\hat{Z}_{6,6}, \hat{X}_{6,6}).$$

By "rational" series for $1/\pi$, we mean that C/π, for some algebraic number C, can be expressed as a series with rational terms. The integer N in the left column of the corresponding formula indicates the degree of modular equation we use in the derivation of the series for $1/\pi$. To prove each series, one is required to derive a modular equation of degree N for the corresponding X. For more details the reader should consult [9].

5.1. Series associated with (1.5)

Let the sequence R_n be given by (2.20). The identities we need in order to prove the following series for $1/\pi$ are

$$q\frac{d\hat{X}_{6,2}}{dq} = \hat{Z}_{6,2}\hat{X}_{6,2}\sqrt{1-28\hat{X}_{6,2}-128\hat{X}_{6,2}^2}$$

and

$$\hat{Z}_{6,2}\big(e^{-2\pi\sqrt{1/(6m)}}\big) = m\hat{Z}_{6,2}\big(e^{-2\pi\sqrt{m/6}}\big).$$

For each value of N, we would also need a modular equation of degree N associated with $\hat{X}_{6,2}$.

In these settings we have

$$N=2:\quad \frac{25}{2\pi} = \sum_{n=0}^{\infty}(9n+2)R_n\left(\frac{1}{50}\right)^n,$$

$$N=3:\quad \frac{3\sqrt{2}}{\pi} = \sum_{n=0}^{\infty}(5n+1)R_n\left(\frac{1}{96}\right)^n,$$

$$N=5:\quad \frac{8\sqrt{5}}{\sqrt{27}\pi} = \sum_{n=0}^{\infty}(6n+1)R_n\left(\frac{1}{320}\right)^n,$$

$$N=7:\quad \frac{16\sqrt{7}}{\pi} = \sum_{n=0}^{\infty}(90n+13)R_n\left(\frac{1}{896}\right)^n,$$

$$N=13:\quad \frac{50\sqrt{39}}{\pi} = \sum_{n=0}^{\infty}(918n+99)R_n\left(\frac{1}{10400}\right)^n,$$

$$N=17:\quad \frac{1225\sqrt{6}}{\pi} = \sum_{n=0}^{\infty}(10098n+954)R_n\left(\frac{1}{39200}\right)^n.$$

5.2. Series associated with (1.6)

Let the sequence S_n be given by (3.1). The identities we need in order to prove the following series for $1/\pi$ are

$$q\frac{d\hat{X}_{6,3}}{dq} = \hat{Z}_{6,3}\hat{X}_{6,3}\sqrt{1-40\hat{X}_{6,3}+144\hat{X}_{6,3}^2}$$

and

$$\hat{Z}_{6,3}\big(e^{-2\pi\sqrt{1/(6m)}}\big) = m\hat{Z}_{6,3}\big(e^{-2\pi\sqrt{m/6}}\big).$$

For each value of N, we would also need a modular equation of degree N associated with $\hat{X}_{6,3}$.

Then

$$N=2:\quad \frac{9\sqrt{3}}{2\pi}=\sum_{n=0}^{\infty}(5n+1)S_n\left(\frac{1}{54}\right)^n,$$

$$N=3:\quad \frac{25}{\sqrt{3}\pi}=\sum_{n=0}^{\infty}(16n+3)S_n\left(\frac{1}{100}\right)^n,$$

$$N=5:\quad \frac{3^{7/2}}{\pi}=\sum_{n=0}^{\infty}(80n+13)S_n\left(\frac{1}{324}\right)^n,$$

$$N=7:\quad \frac{75}{2^{9/2}\pi}=\sum_{n=0}^{\infty}(7n+1)S_n\left(\frac{1}{900}\right)^n,$$

$$N=13:\quad \frac{17^2\sqrt{6}}{2^5\pi}=\sum_{n=0}^{\infty}(65n+7)S_n\left(\frac{1}{10\,404}\right)^n,$$

$$N=17:\quad \frac{3^3\cdot 11^2\cdot\sqrt{3}}{2\pi}=\sum_{n=0}^{\infty}(9520n+899)S_n\left(\frac{1}{39\,204}\right)^n.$$

5.3. *Series associated with (1.7)*

Finally, let the sequence T_n be given by (3.3). The required identities for the series below are

$$q\frac{d\hat{X}_{6,6}}{dq}=\hat{Z}_{6,6}\hat{X}_{6,6}\sqrt{1+68\hat{X}_{6,6}+1152\hat{X}_{6,6}^2}$$

and

$$\hat{Z}_{6,6}\left(e^{-2\pi\sqrt{1/(6m)}}\right)=-m\hat{Z}_{6,6}\left(e^{-2\pi\sqrt{m/6}}\right).$$

For each value of N, we would also need a modular equation of degree N associated with $\hat{X}_{6,6}$.

Then

$$N=3:\quad \frac{8}{\sqrt{3}\pi}=\sum_{n=0}^{\infty}(10n+3)T_n\left(\frac{1}{64}\right)^n,$$

$$N=5:\quad \frac{\sqrt{6}}{\pi}=\sum_{n=0}^{\infty}(5n+1)T_n\left(\frac{1}{288}\right)^n,$$

$$N=7:\quad \frac{18\sqrt{3}}{\pi}=\sum_{n=0}^{\infty}(70n+11)T_n\left(\frac{1}{864}\right)^n,$$

$$N=13:\quad \frac{432\sqrt{3}}{\pi}=\sum_{n=0}^{\infty}(2210n+241)T_n\left(\frac{1}{10\,368}\right)^n,$$

$$N = 17: \quad \frac{32\sqrt{51}}{\pi} = \sum_{n=0}^{\infty} (770n + 73) T_n \left(\frac{1}{39\,168} \right)^n.$$

6. Concluding remarks

Remark 6.1. In [6], while investigating Ramanujan's cubic continued fraction, Chan and K.P. Loo attempted to derive a series for $1/\pi$ associated with the square of the series

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}^3 G^n,$$

with

$$G = G(q) = \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \cdots.$$

The final result is

$$\frac{1}{\pi} = \frac{3\sqrt{3}(3 - 2\sqrt{2})}{2} \sum_{n=0}^{\infty} \left(n + 1 - \frac{2}{3}\sqrt{2} \right) \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \binom{k}{j}^3 \sum_{i=0}^{n-k} \binom{n-k}{i}^3 \right\} \left(\frac{3\sqrt{2} - 4}{4} \right)^n.$$

We now know that the authors should have used the parametrization

$$\mathcal{G} = G^3 \frac{(1 + G^3)(1 - 8G^3)}{(1 + 8G^6)^2}. \tag{6.1}$$

When $q = e^{-2\pi/\sqrt{6}}$, we have $G = -1 + \sqrt{6}/2$ and from (6.1), we find that $\mathcal{G} = 1/96$. The corresponding "correct" series should therefore be the second one (for $N = 3$) in Subsection 5.1.

Remark 6.2. The sequences

$$r_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad \text{and} \quad s_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}$$

used in this article can be found in a manuscript written by D. Zagier more than ten years ago. This article is now published in [24].

Remark 6.3. The first series for $1/\pi$ associated with $S_n = \binom{2n}{n} s_n$ appears in M. Rogers' work [17, Eq. (3.12)]. The series,

$$\frac{2(64 + 29\sqrt{3})}{\pi} = \sum_{n=0}^{\infty} (520n + 159 - 48\sqrt{3}) S_n \left(\frac{80\sqrt{3} - 139}{484} \right)^n,$$

was discovered as a consequence of his study of Mahler's measure and a hypergeometric ${}_5F_4$ function.

Remark 6.4. Our choices of N are obtained directly from the tables in [7], where series for $1/\pi$ corresponding to the Apéry numbers, the Domb numbers and the Almkvist–Zudilin numbers are given. With the exception of three "rational" series for $1/\pi$, the series in [7] involve powers of radicals in contrast with the ones given in Section 5.

Remark 6.5. There is another "uniform" way of proving identities (1.5)–(1.7) which avoids using the modular parametrizations and is in the spirit of the method given in [1]. The series

$$z(x) = \sum_{n=0}^{\infty} a_n x^n$$

on the left-hand side of (1.5)–(1.7), where a_n is one of the sequences in (1.4), satisfy the second-order differential equation

$$\bigl(\theta^2 - x\bigl(a\theta^2 + a\theta + b\bigr) + cx^2(\theta+1)^2\bigr)z = 0 \quad \text{where } \theta = x\frac{d}{dx},$$

with

$$(a, b, c) = (7, 2, -8), \quad (10, 3, 9), \quad \text{and} \quad (-17, -6, 72),$$

respectively (cf. [1]). This is equivalent to saying that

$$(n+1)^2 a_{n+1} - \bigl(an^2 + an + b\bigr)a_n + cn^2 a_{n-1} = 0 \quad \text{for } n = 0, 1, 2, \ldots.$$

One can then show that

$$Z(x) = \sum_{n=0}^{\infty} \binom{2n}{n} a_n x^n$$

satisfies the third-order differential equation

$$\bigl(\theta^3 - 2x(2\theta+1)\bigl(a\theta^2 + a\theta + b\bigr) + 4cx^2(\theta+1)(2\theta+1)(2\theta+3)\bigr)Z = 0.$$

Finally, with the method given in [1], one finds that

$$z^2(x) = \frac{1}{1 - cx^2} \cdot Z\left(\frac{x(1 - ax + cx^2)}{(1 - cx^2)^2}\right).$$

Applying the above idea to the remaining three cases of [1, Eq. (28)] with

$$(a, b, c) = (11, 3, -1), \quad (12, 4, 32), \quad \text{and} \quad (9, 3, 27),$$

G. Almkvist and the fourth author (who also happens to be a coauthor of [1]) obtain three further analogues of Clausen's identities:

$$\left\{\sum_{n=0}^{\infty} u_n x^n\right\}^2 = \frac{1}{1+x^2}\sum_{n=0}^{\infty}\binom{2n}{n}u_n\left(\frac{x(1-11x-x^2)}{(1+x^2)^2}\right)^n, \tag{6.2}$$

$$\left\{\sum_{n=0}^{\infty} v_n x^n\right\}^2 = \frac{1}{1-32x^2}\sum_{n=0}^{\infty}\binom{2n}{n}v_n\left(\frac{x(1-4x)(1-8x)}{(1-32x^2)^2}\right)^n, \tag{6.3}$$

and

$$\left\{\sum_{n=0}^{\infty} w_n x^n\right\}^2 = \frac{1}{1-27x^2}\sum_{n=0}^{\infty}\binom{2n}{n}w_n\left(\frac{x(1-9x+27x^2)}{(1-27x^2)^2}\right)^n, \tag{6.4}$$

where

$$u_n = \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{n}, \qquad v_n = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k},$$
$$\text{and} \quad w_n = \sum_{k=0}^{n}(-1)^k 3^{n-3k}\binom{n}{3k}\frac{(3k)!}{(k!)^3}.$$

Identity (6.2) was first discovered by S. Cooper [13] when he was searching for identities similar to those presented in this article and corresponding to this example, he derived two new series for $1/\pi$.

The disadvantage of using the method of [1] is that the identities do not shed any light on its connection with modular forms, in particular, are of no use in proving Ramanujan-type series for $1/\pi$. To make the connection, one needs the algorithm in Section 2 to search for the corresponding modular forms based on the binomial expressions. We end this paper by stating the respective modular forms and functions associated with (6.2)–(6.4). In Table 1 we use the notation $[a]^b := \eta^b(a\tau)$, and the relation between Z and X is given by

$$Z = \sum_{n=0}^{\infty}\binom{2n}{n}a_n X^n.$$

Table 1

$\Gamma_0(N)$	a_n	Z	$1/X$
$\Gamma_0(5)$	u_n	$6q\frac{d\ln(\frac{[5]}{[1]})}{dq}$	$\frac{[1]^6}{[5]^6}+22+125\frac{[5]^6}{[1]^6}$
$\Gamma_0(8)$	v_n	$8q\frac{d\ln(\frac{[8][2]^2}{[1][4]^2})}{dq}$	$\frac{[4]^8[2]^8}{[1]^8[8]^8}+8+16\frac{[1]^8[8]^8}{[2]^8[4]^8}$
$\Gamma_0(36)$	w_n	$6q\frac{d\ln(\frac{[2]^3[18]^3[3]^2[12]^2}{[6]^6[9][1][4][36]})}{dq}$	$(\frac{[6]^6[9][1][4][36]}{[2]^3[18]^3[3]^2[12]^2})^6+18-27(\frac{[2]^3[18]^3[3]^2[12]^2}{[6]^6[9][1][4][36]})^6$

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