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Rational analogues of Ramanujan's series for $1/\pi$ †

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Abstract

A general theorem is stated that unifies 93 rational Ramanujan-type series for $1/\pi$, 40 of which are believed to be new. Moreover, each series is shown to have a companion identity, thereby giving another 93 series, the majority of which are new.

1. *Introduction*

In 1914, Ramanujan [**39**] gave 17 series for $1/\pi$. One of his best known examples (see, e.g., [**3**], [**17**], [**27**]) is

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \binom{4k}{2k} \binom{2k}{k}^2 \frac{(1103 + 26390k)}{396^{4k}}. \tag{1}$$

It was used by R. W. Gosper in 1985 to compute 17,526,100 digits of π, then a world record. Ramanujan's formulas were not all proved until 1987 [**14**].

In [**23**] some new analogues of Ramanujan's series for $1/\pi$ were discovered, for example

$$\frac{1}{\pi} = \frac{3\sqrt{6}}{1225} \sum_{k=0}^{\infty} \binom{2k}{k} \left\{ \sum_{j=0}^{k} \binom{k}{j}^3 \right\} \frac{(53 + 561k)}{39200^k}. \tag{2}$$

The series (1) and (2) are each of the form

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \binom{2k}{k} s(k)(A + Bk)x^k$$

where $s(k)$ satisfies a recurrence relation of the type

$$(k+1)^2 s(k+1) = (ak^2 + ak + b)s(k) + ck^2 s(k-1). \tag{3}$$

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For Ramanujan's series (1) it is easy to check that $(a, b, c) = (64, 12, 0)$, so (3) is simply a two-term recurrence relation in this case. It is a classical result, but not obvious, that $(a, b, c) = (7, 2, 8)$ for the series (2).

The aim of this paper is to give a systematic classification of series such as (1) and (2). In the process, several new series will be given, for example

$$\frac{1}{\pi} = \frac{5\sqrt{47}}{7614} \sum_{k=0}^{\infty} \binom{2k}{k} \left\{ \sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j} \right\} (-1)^k \frac{(71 + 682k)}{15228^k}, \tag{4}$$

where the corresponding sequence (3) has parameters $(a, b, c) = (11, 3, 1)$.

There are 10 known instances of the parameters (a, b, c) for which the sequence $s(k)$ defined by (3) can be parameterized by modular forms. This includes the three examples discussed above, and all 10 cases are listed in Table 1. For these values, if

$$x = \frac{w(1 - aw - cw^2)}{(1 + cw^2)^2}$$

where w is a hauptmodul given in Table 1, then x satisfies an involution of the form

$$x\left(e^{-2\pi\sqrt{t/\ell}}\right) = x\left(e^{-2\pi/\sqrt{t\ell}}\right), \qquad t > 0.$$

In this paper, the positive integer ℓ coincides with the level of the modular function x. As such, we will refer to ℓ as the level. All of Ramanujan's examples in [**39**] correspond to $\ell = 1, 2, 3$ or 4 and in particular the series (1) belongs to the theory of level 2. The series (2) and (4) correspond to the levels 6 and 5, respectively. We will present some new series for level 6 and develop corresponding theories for levels 5, 8 and 9 which are new.

A further key to classifying identities such as (1), (2) and (4) is that the function z given by

$$z = \sum_{k=0}^{\infty} s(k) w^k \tag{5}$$

turns out to be a weight one modular form that satisfies the Clausen-type identity

$$(1 + cw^2) \left(\sum_{k=0}^{\infty} s(k) w^k \right)^2 = \sum_{k=0}^{\infty} \binom{2k}{k} s(k) \left(\frac{w(1 - aw - cw^2)}{(1 + cw^2)^2} \right)^k.$$

When $c = 0$ this reduces to a special case of Clausen's identity for the hypergeometric function. We will show further that each series such as (1), (2) and (4) has a companion series that can be deduced from the identity

$$(1 - aw - cw^2) \left(\sum_{k=0}^{\infty} s(k) w^k \right)^2 = \sum_{k=0}^{\infty} t(k) \left(\frac{w}{1 - aw - cw^2} \right)^k$$

where $t(k)$ satisfies the recurrence relation

$$(k+1)^3 t(k+1) = -(2k+1)(ak^2 + ak + a - 2b) t(k) - (4c + a^2) k^3 t(k-1)$$

and initial conditions $t(-1) = 0$, $t(0) = 1$. For example, the companion identity for (4) is

Takeshi Sato's example (we have corrected some misprints) [**42**]:

$$\frac{1}{\pi} = \frac{846}{\sqrt{5}} \sum_{k=0}^{\infty} t(k) \left(k + \frac{1}{2} - \frac{25\sqrt{5}}{141} \right) (-1)^k \left\{ \frac{1}{5\sqrt{5}} \left(\frac{\sqrt{5}-1}{2} \right)^{15} \right\}^{k+1/2} \tag{6}$$

where, as for (4), $(a, b, c) = (11, 3, 1)$.

The study of series for $1/\pi$ has a long and distinguished history, and the reader is referred to [**3**] or [**17**]. However it would be remiss not to mention [**4**, **9**, **11**, **13–15**, **21**, **22**, **26**, **39**] in connection with levels 1–4; [**30**] and [**42**] for levels 5 and 6; and [**19**, **23**, **24**, **41**] for level 6.

We end this introduction by mentioning that there are some series for $1/\pi$ that do not involve a weight one modular form such as (5) but are instead deduced directly from weight two modular forms without the use of a Clausen-type identity. See [**23**], [**31**] and [**46**] for some series for level 10 that were inspired by an example of Y. Yang, and see [**32**] for some series for levels 7 and 18.

2. *Statement of results*

Following [**23**], we shall say a series is a "rational series for $1/\pi$" if C/π can be expressed as a series of rational numbers for some algebraic number C. The next result gives 93 rational series for $1/\pi$ of 10 different types given by (13). Moreover, each series of type (13) has a companion series (which generally is not a rational series) given by (14).

THEOREM 2·1. *Let* $\ell \in \{1, 2, 3, 4, 5, 6, 8, 9\}$. *Let* $w = w(q)$ *and* $(a, b, c) \in \mathbf{Z}^3$ *be as in Table 1, and note that there are three different examples that correspond to the level* $\ell = 6$. *Let* $s(k)$ *and* $t(k)$ *be the sequences defined by the recurrence relations*

$$(k+1)^2 s(k+1) = (ak^2 + ak + b)s(k) + ck^2 s(k-1) \tag{7}$$

and

$$(k+1)^3 t(k+1) = -(2k+1)(ak^2 + ak + a - 2b)t(k) - (4c + a^2)k^3 t(k-1) \tag{8}$$

and initial conditions

$$s(-1) = t(-1) = 0, \qquad s(0) = t(0) = 1. \tag{9}$$

Let

$$x = x(q) = \frac{w(1 - aw - cw^2)}{(1 + cw^2)^2} \tag{10}$$

and

$$y = y(q) = \frac{w}{1 - aw - cw^2}. \tag{11}$$

Let N *be a positive integer.*
Either: *let* ρ *and* q *take the particular values*

$$\rho = 2\pi\sqrt{N/\ell} \qquad \text{and} \qquad q = \exp(-\rho); \tag{12}$$

or: *let* ρ *and* q *take the values*

$$\rho = \begin{cases} 2\pi\sqrt{N/4\ell} & \text{if } \ell \equiv 1 \pmod 2, \\ 2\pi\sqrt{N/2\ell} & \text{if } \ell \equiv 2 \pmod 4, \\ 2\pi\sqrt{N/\ell} & \text{if } \ell \equiv 0 \pmod 4 \end{cases} \qquad \text{and} \qquad q = -\exp(-\rho).$$

Then the identity

$$\sqrt{1-4ax-16cx^2}\,\sum_{k=0}^{\infty}\binom{2k}{k}s(k)\,(k+\lambda)\,x^k \;=\; \frac{1}{\rho} \tag{13}$$

holds for the 93 *sets of values of ℓ, N, x and λ given in Tables* 3–12*. Moreover, for each series identity of the type* (13)*, the companion identity*

$$\sqrt{1+2ay+(4c+a^2)y^2}\,\sum_{k=0}^{\infty}t(k)\,(k+\tfrac{1}{2}+\mu)\,y^k \;=\; \frac{1}{\rho} \tag{14}$$

holds, where y and μ are given in terms of x and λ by

$$y=\frac{1-2ax-\sqrt{1-4ax-16cx^2}}{2x(4c+a^2)}$$

and

$$\mu=(\lambda-\tfrac{1}{2})\sqrt{1-4ax-16cx^2}.$$

The examples listed in the introduction are located in the tables as follows. Ramanujan's series (1) occurs in Table 4 with level $\ell=2$, degree $N=29$ and $q=\exp(-2\pi\sqrt{29/2})$. The series (2) occurs in Table 10 with level $\ell=6$, degree $N=17$ and $q=\exp(-2\pi\sqrt{17/6})$. The series (4) occurs in Table 7 with level $\ell=5$, degree $N=47$ and $q=-\exp(-\pi\sqrt{47/5})$. Sato's series (6) is the companion identity to (4).

As part of Tables 3–12, we have included the earliest known reference to the given series of types (13) and (14). Of the $93\times 2=186$ series represented by Tables 3–12, a total of 71 references are given. One of the series in Table 6 that corresponds to $\ell=4$, $N=4$ ought to be known, but we do not have a reference. The other 114 series, for which references are not given, are believed to be new.

In order to determine the convergence of (13) and (14), note that the characteristic equations for the recurrence relations (7) and (8) are

$$m^2-am-c=0 \qquad \text{and} \qquad m^2+2am+(a^2+4c)=0,$$

respectively. Therefore, by Poincaré's theorem on difference equations [**35**, p. 343]

$$\lim_{k\to\infty}|s(k)|^{1/k}\leqslant M_x:=\max\left|\frac{a\pm\sqrt{a^2+4c}}{2}\right|$$

and

$$\lim_{k\to\infty}|t(k)|^{1/k}\leqslant M_y:=\max\left|a\pm 2\sqrt{-c}\,\right|.$$

Hence, the series (13) and (14) converge for $|x|<R_x:=1/(4M_x)$ and $|y|<R_y:=1/M_y$, respectively. For reference, the values of R_x and R_y are given in Table 13.

3. *Background theory*

In this section and the next, we sketch a proof of Theorem 2·1. The theory for levels 1, 2, 3 and 4 has been analyzed in detail by by Chan, Chan and Liu [**19**], and theories for level 6 have been described in [**19**, **23**, **24**, **41**]. Our goal is to outline a single, systematic theory that applies to each level $\ell=1, 2, 3, 4, 5, 6$ (3 cases), 8 and 9. The emphasis will be on the commonality of the 10 cases. The theories for levels 5, 8 and 9 are new.

Ramanujan's Eisenstein series are defined by

$$P = P(q) = 1 - 24\sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}, \qquad Q = Q(q) = 1 + 240\sum_{j=1}^{\infty} \frac{j^3q^j}{1-q^j}$$

and

$$R = R(q) = 1 - 504\sum_{j=1}^{\infty} \frac{j^5q^j}{1-q^j}.$$

For any positive integer n let P_n and η_n be defined by

$$P_n = P(q^n) \qquad \text{and} \qquad \eta_n = q^{n/24}\prod_{j=1}^{\infty}(1-q^{nj}).$$

The first result we will need involves eta-quotients.

THEOREM 3·1. *For* $\ell \in \{2, 3, 4, 5, 6, 8, 9\}$ *and with* w *and* (a, b, c) *as given in Table 1, the function*

$$\frac{1}{w} - a - cw$$

is expressible as an eta-quotient as given by Table 1. For the level $\ell = 1$*, the corresponding result is not an eta-quotient but simply*

$$\frac{1}{w} - 432 = 432\left(\frac{Q^{3/2}+R}{Q^{3/2}-R}\right).$$

Proof. Identities such as the ones above for levels 5, 6, 8 and 9 hold because of properties of hauptmoduls associated with groups of genus 0. For more information, the reader is referred to [**20**] where many examples of such identities are given.

The identity for level 5,

$$\frac{1}{w} - 11 - w = \frac{1}{q}\prod_{j=1}^{\infty}\frac{(1-q^j)^6}{(1-q^{5j})^6}$$

where

$$w = q\prod_{j=1}^{\infty}\frac{(1-q^{5j-4})^5(1-q^{5j-1})^5}{(1-q^{5j-3})^5(1-q^{5j-2})^5},$$

is classical and goes back at least to Ramanujan. See [**7**, pp. 161–164], [**10**], [**38**] and [**44**] for proofs that rely only on simple series manipulations. A different proof is outlined in the expository article by W. Duke [**34**]. Proofs of the identities for level 6 in cases B and C that use series manipulations have been given in [**30**]. The other identities can be proved in the same way, or by using the ideas in [**20**] or [**34**].

The next result gives formulas for the derivatives of the modular functions w.

THEOREM 3·2. *Let* $\ell \in \{1, 2, 3, 4, 5, 6, 8, 9\}$ *and let* w*,* z *and* (a, b, c) *be as given in Table 1, and note that there are three different examples that correspond to the level* $\ell = 6$.

Let

$$x = \frac{w(1 - aw - cw^2)}{(1 + cw^2)^2}. \tag{15}$$

Then

$$q\frac{d}{dq}\log w = (1 - aw - cw^2)z^2 \tag{16}$$

and

$$q\frac{d}{dq}\log x = \sqrt{1 - 4ax - 16cx^2}\,(1 + cw^2)z^2. \tag{17}$$

Proof. The identity (17) follows from (16) by using (15) and the chain rule. Therefore, it suffices to prove (16) and this may be achieved on a case by case basis. For levels 1, 2, 3 and 4, the results are equivalent to

$$q\frac{dx}{dq} = z^2 x(1 - x) \tag{18}$$

where x and z are parameters from Ramanujan's theories of elliptic functions to alternative bases for signatures 6, 4, 3 and 2, respectively; see [**5**, p. 124], [**6**, pp. 104, 148, 164] and [**7**, theorem 5·4·8]. In [**29**], the equation (18) was used as a starting point to develop the theories of elliptic functions for levels 1, 2, 3 and 4.

The result (16) for level 5 was given by Ramanujan [**40**, chapter 19, entry 9 (v)]. Explicitly, the identity is equivalent to

$$1 - 5\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1 - q^j} = \prod_{j=1}^{\infty}\frac{(1 - q^j)^5}{(1 - q^{5j})}$$

where $(j/5)$ is the Legendre symbol. For simple proofs, see [**33**] or [**37**]; for more information and references to other proofs, see [**5**, pp. 257–262]. For proofs of the result for level 6 in cases B and C see [**30**]. For proofs of the result for level 9, see [**12**], [**16**], [**25**, Corollary 3.3], [**28**] or [**36**, p. 307]. Similar proofs may be given for the identities for level 6 in case A and level 8.

The next result shows that each modular function z satisfies a second order linear differential equation in terms of w.

THEOREM 3·3. *Let* $\ell \in \{1, 2, 3, 4, 5, 6, 8, 9\}$. *Let* $w = w(q)$, $z = z(q)$ *and* $(a, b, c) \in \mathbf{Z}^3$ *be as in Table 1, and note that there are three different examples that correspond to the level* $\ell = 6$. *Then* z *satisfies the following second order linear differential equation with respect to* w:

$$\frac{d}{dw}\left(w(1 - aw - cw^2)\frac{dz}{dw}\right) = (b + cw)z.$$

Proof. For $\ell = 1, 2, 3$ or 4, the purported differential equation reduces to the special case of the hypergeometric differential equation given by

$$\frac{d}{dw}\left(w(1 - aw)\frac{dz}{dw}\right) = bz.$$

The result for level $\ell = 4$ is classical, while the results for levels 1, 2 and 3 were known to Ramanujan—see [**6**, chapter 33], [**8**] or [**29**, theorem 4·6].

A detailed analysis for the level $\ell = 5$ has been given by F. Beukers [**18**]; see also [**30**]. A proof for case C of level 6 has been given by Chan and Loo [**22**] and a proof for case B of level 6 has been given by Cooper [**30**].

The differential equations for all 6 sporadic cases, together with their modular parameterizations, were given by Zagier [**45**]. For more information, see the work of Verrill [**43**].

The differential equation in Theorem 3·3 has a regular singular point at $w = 0$, and hence there exists a solution in powers of w near $w = 0$. This is summarized in the next theorem.

THEOREM 3·4. *Let* $\ell \in \{1, 2, 3, 4, 5, 6, 8, 9\}$. *Let* $w = w(q)$, $z = z(q)$ *and* $(a, b, c) \in \mathbf{Z}^3$ *be as in Table 1, and note that there are three different examples that correspond to the level* $\ell = 6$. *Let* $s(k)$ *be the sequence defined by the recurrence relation*

$$(k+1)^2 s(k+1) = (ak^2 + ak + b)s(k) + ck^2 s(k-1) \tag{19}$$

and initial conditions $s(-1) = 0$, $s(0) = 1$. *Then*

$$z = \sum_{k=0}^{\infty} s(k) w^k.$$

Proof. This is an immediate consequence of the theory of series solutions for differential equations.

The next result contains two recent and exciting analogues of Clausen's formula given in [**23**] and [**1**], respectively.

THEOREM 3·5. *For any constants a, b and c let* $s(k)$ *and* $t(k)$ *be the sequences defined by the recurrence relations and initial conditions* (7), (8) *and* (9). *Then in a neighbourhood of* $w = 0$ *we have*

$$(1 + cw^2)\left(\sum_{k=0}^{\infty} s(k) w^k\right)^2 = \sum_{k=0}^{\infty} \binom{2k}{k} s(k) \left(\frac{w(1 - aw - cw^2)}{(1 + cw^2)^2}\right)^k \tag{20}$$

and

$$(1 - aw - cw^2)\left(\sum_{k=0}^{\infty} s(k) w^k\right)^2 = \sum_{k=0}^{\infty} t(k) \left(\frac{w}{1 - aw - cw^2}\right)^k. \tag{21}$$

Proof. These may be proved by showing that both sides satisfy the same third order linear differential equation and have the same initial conditions at $w = 0$. This is a routine but tedious exercise.

When $c = 0$, each of the identities (20) and (21) reduces to a special case of Clausen's identity for the square of a ${}_2F_1$ hypergeometric function [**2**, p. 116].

COROLLARY 3·6. *For any constants a, b and c let* $t(k)$ *and* $s(k)$ *be the sequences defined by the recurrence relations and initial conditions* (7), (8) *and* (9). *For any variable* w *in a neighbourhood of* 0 *let* x *and* y *be defined by*

$$x = \frac{w(1 - aw - cw^2)}{(1 + cw^2)^2} \qquad \text{and} \qquad y = \frac{w}{1 - aw - cw^2}. \tag{22}$$

Then in a neighbourhood of $w = 0$ *we have*

$$\sum_{k=0}^{\infty} \binom{2k}{k} s(k) x^k = \sqrt{(1+ay)^2 + 4cy^2} \sum_{k=0}^{\infty} t(k) y^k, \tag{23}$$

$$\sum_{k=0}^{\infty} \binom{2k}{k} s(k) x^{k+1/2} = \sum_{k=0}^{\infty} t(k) y^{k+1/2} \tag{24}$$

and

$$\sqrt{1-4ax-16cx^2} \sum_{k=0}^{\infty} \binom{2k}{k} (k+1/2) s(k) x^k$$
$$= \sqrt{(1+ay)^2+4cy^2} \sum_{k=0}^{\infty} (k+1/2) t(k) y^k. \tag{25}$$

Proof. The identity (23) is an immediate consequence of (20) and (21) and the definition (22). The identity (24) follows from (23) by a simple rearrangement using (22). Finally, (25) follows from (24) by differentiation and the fact that

$$x = \frac{y}{(1+ay)^2 + 4cy^2}.$$

LEMMA 3·7. *Let* $\ell \in \{1, 2, 3, 4, 5, 6, 8, 9\}$. *Let* $w = w(q)$, $z = z(q)$ *and* $(a, b, c) \in \mathbf{Z}^3$ *be as in Table 1, and note that there are three different examples that correspond to the level* $\ell = 6$. *Let* $x = x(q)$ *be as defined by* (22). *Let*

$$Z = Z(q) = \left(1 + cw^2(q)\right) z^2(q). \tag{26}$$

Suppose $\mathrm{Re}(t) > 0$. *Then*

$$x\left(e^{-2\pi\sqrt{t/\ell}}\right) = x\left(e^{-2\pi/\sqrt{\ell t}}\right) \tag{27}$$

and

$$t\, Z\left(e^{-2\pi\sqrt{t/\ell}}\right) = \pm Z\left(e^{-2\pi/\sqrt{\ell t}}\right), \tag{28}$$

where the plus sign is used, unless $r = 9$ *in which case the minus sign is used.*

Proof. For $\ell \in \{1, 2, 3, 4\}$, these are the well-known transformation formulas satisfied by Eisenstein series and Dedekind's eta function, so it remains to prove the result for the six sporadic cases.

We will prove the result for $\ell = 5$ in detail. Suppose $\mathrm{Re}(t) > 0$ and define Dedekind's eta function by

$$\eta(it) = q^{1/24} \prod_{j=1}^{\infty} (1-q^j) \qquad \text{where } q = e^{-2\pi t}.$$

It satisfies the transformation formula

$$\eta(it) = \frac{1}{\sqrt{t}}\, \eta\left(\frac{i}{t}\right). \tag{29}$$

Let $w = w(q)$ and $z = z(q)$ be as in Table 1 for level $\ell = 5$ and let $Z = Z(q)$ be given by

(26). Let

$$\omega = w\left(\exp\left(-2\pi\sqrt{t/5}\right)\right) \qquad \text{and} \qquad \omega_* = w\left(\exp\left(-\frac{2\pi}{\sqrt{5t}}\right)\right).$$

It is well–known, e.g., see [**34**, (7·3)], that

$$\omega = \frac{\gamma^5 - \omega_*}{1 + \gamma^5\omega_*} \qquad \text{where} \qquad \gamma = \frac{\sqrt{5}-1}{2}. \tag{30}$$

It follows that

$$\frac{\omega}{1+\omega^2} = \frac{1}{5^{3/2}}\left(\frac{1 - 11\omega_* - \omega_*^2}{1+\omega_*^2}\right) \tag{31}$$

and

$$\frac{1 - 11\omega - \omega^2}{1+\omega^2} = 5^{3/2}\left(\frac{\omega_*}{1+\omega_*^2}\right). \tag{32}$$

If we multiply (31) and (32) and apply the definition (22), then we obtain (27).

Next, from the definitions of w, z and Z, we have

$$Z = (1+w^2)z^2 = \left(\frac{1+w^2}{w}\right)wz^2 = \left(\frac{1+w^2}{w}\right)\frac{\eta^5(5it)}{\eta(it)}. \tag{33}$$

Therefore, by (29), (31) and (33) we have

$$\begin{aligned} Z(e^{-2\pi\sqrt{t/5}}) &= \left(\frac{1+\omega^2}{\omega}\right) \times \frac{\eta^5(i\sqrt{5t})}{\eta(i\sqrt{t/5})} \\ &= 5^{3/2}\left(\frac{1+\omega_*^2}{1-11\omega_*-\omega_*^2}\right) \times \frac{1}{5^{3/2}t}\,\frac{\eta^5(i/\sqrt{5t})}{\eta(i\sqrt{5/t})} \\ &= \frac{1}{t}\left(\frac{1+\omega_*^2}{\omega_*}\right) \times \left(\frac{\omega_*}{1-11\omega_*-\omega_*^2}\right) \times \frac{\eta^5(i/\sqrt{5t})}{\eta(i\sqrt{5/t})}. \end{aligned} \tag{34}$$

Now apply Theorem 3·1 and (33) to (34) to finally obtain

$$Z(e^{-2\pi\sqrt{t/5}}) = \frac{1}{t}\left(\frac{1+\omega_*^2}{\omega_*}\right) \times \frac{\eta^6(i\sqrt{5/t})}{\eta^6(i/\sqrt{5t})} \times \frac{\eta^5(i/\sqrt{5t})}{\eta(i\sqrt{5/t})} = \frac{1}{t}Z(e^{-2\pi/\sqrt{5t}}).$$

This completes the proof for the level $\ell = 5$.

The results for $\ell = 6$ in cases B and C can be proved in the same way. The analogues of (30) for $\ell = 6$ in cases B and C are given, for example, in [**30**, theorem 3·2], and the analogue of (30) for $\ell = 8$ is given by

$$w(e^{-2\pi\sqrt{t/8}}) = \frac{\frac{1}{8} - w(e^{-2\pi/\sqrt{8t}})}{1 - 4w(e^{-2\pi/\sqrt{8t}})}.$$

The proofs for $\ell = 6$ in case A, and for $\ell = 9$, are simpler and can be achieved by just using the transformation formula for Dedekind's eta function, (29).

4. *Proof of Theorem* 2·1

In this section we shall indicate how to complete the proof of Theorem 2·1. The identity (14) follows immediately from (13) by using (23) and (25). It remains to prove (13), and this can be accomplished by using the following result of Chan, Chan and Liu [**19**, theorem 2·1]:

THEOREM 4·1 (H. H. Chan, S. H. Chan and Z.-G. Liu). *Suppose* $t > 0$. *Suppose* $x = x(q)$, $Z = Z(q)$ *and* $u = u(q)$ *satisfy the properties*:

$$tZ\left(e^{-2\pi\sqrt{t/\ell}}\right) = Z\left(e^{-2\pi/\sqrt{t\ell}}\right); \tag{35}$$

$$Z(q) = \sum_{k=0}^{\infty} h(k)x^k(q); \tag{36}$$

and

$$q\frac{d}{dq}\log x(q) = u(q)Z(q). \tag{37}$$

For any integer $N \geqslant 2$, *let*

$$M(q) = \frac{Z(q)}{Z(q^N)}. \tag{38}$$

Let λ, X *and* U *be defined by*

$$\lambda = \frac{x}{2N}\left.\frac{dM}{dx}\right|_{q=e^{-2\pi/\sqrt{N\ell}}}, \tag{39}$$

$$X = x(e^{-2\pi\sqrt{N/\ell}}), \tag{40}$$

$$U = u(e^{-2\pi\sqrt{N/\ell}}). \tag{41}$$

Then

$$\sqrt{\frac{\ell}{N}}\frac{1}{2\pi} = U\sum_{k=0}^{\infty} h(k)\,(k+\lambda)\,X^k. \tag{42}$$

We are now ready for

Proof of Theorem 2·1. We begin by assuming $q > 0$. With (a, b, c), w, z and $s(k)$ as given in Tables 1 and 2, let Z, x, u and $h(k)$ be defined by

$$Z = Z(q) = (1+cw^2)z^2,$$

$$x = x(q) = \frac{w(1-aw-cw^2)}{(1+cw^2)^2},$$

$$u = u(q) = \sqrt{1-4ax-16cx^2} \tag{43}$$

and

$$h(k) = \binom{2k}{k}s(k).$$

The hypotheses (35), (36) and (37) of Theorem 4·1 are satisfied because of Lemma 3·7 and Theorems 3·5 and 3·2, respectively. Thus, a series of the form (42) exists, and it remains to determine the values of X and λ. For this, we may use the modular equation of degree N that is satisfied by x. To illustrate, consider the level $\ell = 5$ and degree $N = 2$. Then $x = x(q)$ and $v = x(q^2)$ satisfy the modular equation

$$x^3+v^3-36(x^3v+xv^3)+324(x^3v^2+x^2v^3)+32(x^2v+xv^2)+495x^2v^2-xv = 0. \tag{44}$$

For the value $q = \exp(-2\pi/\sqrt{10})$ we have by (27) with $\ell = 5$ and $N = 2$

$$v = x(q^2) = x(\exp(-2\pi\sqrt{2/5})) = x(\exp(-2\pi/\sqrt{10})) = x,$$

so (44) simplifies to

$$x^2(72x - 1)(1 + 3x^2) = 0.$$

On comparing numerical values we deduce that

$$x(e^{-2\pi\sqrt{2/5}}) = x(e^{-2\pi/\sqrt{10}}) = \frac{1}{72}.$$

Next, by (37), (38) and (43) we have

$$M_2(q) = \frac{Z(q)}{Z(q^2)} = \frac{u(q^2)}{u(q)} \frac{q\frac{d}{dq}\log x(q)}{q^2\frac{d}{dq}\log x(q^2)}$$
$$= 2\frac{\sqrt{1 - 44v - 16v^2}}{\sqrt{1 - 44x - 16x^2}} \frac{v}{x} \frac{dx}{dv}.$$

The derivative dx/dv may be computed from (44) by implicit differentiation. Thus, we obtain $M_2(q)$ as an algebraic function of x and v. Then, taking the derivative with respect to x we find, after a computation, that

$$\left.\frac{dM_2}{dx}\right|_{q=\exp(-2\pi/\sqrt{10})} = \frac{288}{5}$$

and then from (39) we may deduce that $\lambda = 1/5$.

If $q < 0$ we use the fact

$$\prod_{j=1}^{\infty}(1 - (-q)^j) = \prod_{j=1}^{\infty} \frac{(1 - q^{2j})^3}{(1 - q^j)^2(1 - q^{4j})^2}.$$

If

$$z(q) = \sum_{n=0}^{\infty} s(n)(w(q))^n$$

where $z(q)$ and $w(q)$ are modular forms of level ℓ, then

$$z(-q) = \sum_{n=0}^{\infty}(-1)^n s(n)(-w(-q))^n$$

where $z(-q)$ and $-w(-q)$ are modular forms of level ℓ, 2ℓ or 4ℓ, depending on whether $\ell \equiv 0 \pmod 4$, $\ell \equiv 0 \pmod 2$ or ℓ is odd, respectively. Then Theorem 4·1 can be used to complete the proof.

In principle, all of the 93 series given by (13) and represented by Tables 3–12 can be proved in the same way. In practice, the results in the tables were discovered experimentally by searching for rational values of $1/X$ in (40) for various ℓ and N and then using the series (42) to numerically determine the value of λ. We conjecture that the tables are complete, that is, there are no other rational series for $1/\pi$ of these types.

5. *Tables*

This section contains tables of values of values of parameters. Tables 1 and 2 contain the definitions of the modular forms for each level as well as the recurrence relations and their

Table 1. *Modular forms*

Level ℓ	(a,b,c)	w	$\frac{1}{w}-a-cw$	z	$q\frac{d}{dq}\log w$
1	$(432,60,0)$	$\frac{1}{864}\left(1-\frac{R}{Q^{3/2}}\right)$	$432\left(\frac{Q^{3/2}+R}{Q^{3/2}-R}\right)$	$Q^{1/4}$	$\frac{Q^{3/2}+R}{2Q}$
2	$(64,12,0)$	$\frac{\eta_2^{24}}{\eta_1^{24}+64\eta_2^{24}}$	$\frac{\eta_1^{24}}{\eta_2^{24}}$	$\left(\frac{\eta_1^{16}}{\eta_2^8}+\frac{64\eta_2^{16}}{\eta_1^8}\right)^{1/4}$	$\frac{\eta_1^8}{\eta_2^4}\left(1+\frac{64\eta_2^{24}}{\eta_1^{24}}\right)^{-1/2}$
3	$(27,6,0)$	$\frac{\eta_3^{12}}{\eta_1^{12}+27\eta_3^{12}}$	$\frac{\eta_1^{12}}{\eta_3^{12}}$	$\left(\frac{\eta_1^9}{\eta_3^3}+\frac{27\eta_3^9}{\eta_1^3}\right)^{1/3}$	$\frac{\eta_1^6}{\eta_3^2}\left(1+\frac{27\eta_3^{12}}{\eta_1^{12}}\right)^{-1/3}$
4	$(16,4,0)$	$\frac{\eta_1^8\eta_4^{16}}{\eta_2^{24}}$	$\frac{\eta_1^8}{\eta_4^8}$	$\frac{\eta_2^{10}}{\eta_1^4\eta_4^4}$	$\frac{\eta_1^8}{\eta_2^4}$
5	$(11,3,1)$	$q\frac{(q,q^4;q^5)_\infty^5}{(q^2,q^3;q^5)_\infty^5}$	$\frac{\eta_1^6}{\eta_5^6}$	$\frac{(q;q)_\infty^2}{(q,q^4;q^5)_\infty^5}$	$\frac{\eta_1^5}{\eta_5}$
6 (A)	$(-17,-6,-72)$	$\frac{\eta_2\eta_6^5}{\eta_1^5\eta_3}$	$\frac{\eta_2^{12}\eta_3^{12}}{\eta_1^{12}\eta_6^{12}}$	$\frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2}$	$\frac{\eta_2^7\eta_3^7}{\eta_1^5\eta_6^5}$
6 (B)	$(10,3,-9)$	$\frac{\eta_1^4\eta_6^8}{\eta_2^8\eta_3^4}$	$\frac{\eta_1^6\eta_3^6}{\eta_2^6\eta_6^6}$	$\frac{\eta_2^6\eta_3}{\eta_1^3\eta_6^2}$	$\frac{\eta_1^4\eta_3^4}{\eta_2^2\eta_6^2}$
6 (C)	$(7,2,8)$	$\frac{\eta_1^3\eta_6^9}{\eta_2^3\eta_3^9}$	$\frac{\eta_1^4\eta_2^4}{\eta_3^4\eta_6^4}$	$\frac{\eta_2\eta_3^6}{\eta_1^2\eta_6^3}$	$\frac{\eta_1^3\eta_2^3}{\eta_3\eta_6}$
8	$(12,4,-32)$	$\frac{\eta_1^4\eta_4^2\eta_8^4}{\eta_2^{10}}$	$\frac{\eta_1^8\eta_4^{16}}{\eta_2^{16}\eta_8^8}$	$\frac{\eta_2^{10}}{\eta_1^4\eta_4^4}$	$\frac{\eta_1^4\eta_4^{10}}{\eta_2^6\eta_8^4}$
9	$(-9,-3,-27)$	$\frac{\eta_9^3}{\eta_1^3}$	$\frac{\eta_3^{12}}{\eta_1^6\eta_9^6}$	$\frac{\eta_1^3}{\eta_3}$	$\frac{\eta_3^{10}}{\eta_1^3\eta_9^3}$

solutions in terms of binomial coefficients. Tables 3–12 contain the parameter values for the series for $1/\pi$. The tables are organized according to the level and include the earliest reference to each series. In the cases where no reference is included, the series is believed to be new.

Notation:

$$\eta_n = q^{n/24}\prod_{j=1}^{\infty}(1-q^{nj}), \qquad Q = 1+240\sum_{j=1}^{\infty}\frac{j^3q^j}{1-q^j}, \qquad R = 1-504\sum_{j=1}^{\infty}\frac{j^5q^j}{1-q^j},$$

$$(x;q)_\infty = \prod_{j=0}^{\infty}(1-xq^j), \qquad (x,y;q)_\infty = (x;q)_\infty(y;q)_\infty.$$

Table 2. *Solutions to recurrence relations*

Level ℓ	(a, b, c)	$s(k)$	$t(k)$
1	(432, 60, 0)	$\binom{6k}{3k}\binom{3k}{k}$	$\sum_j s(j)\binom{2j}{j}\binom{k+j}{k-j}(-a)^{k-j}$
2	(64, 12, 0)	$\binom{4k}{2k}\binom{2k}{k}$	$\sum_j s(j)\binom{2j}{j}\binom{k+j}{k-j}(-a)^{k-j}$
3	(27, 6, 0)	$\binom{3k}{k}\binom{2k}{k}$	$\sum_j s(j)\binom{2j}{j}\binom{k+j}{k-j}(-a)^{k-j}$
4	(16, 4, 0)	$\binom{2k}{k}^2$	$(-1)^k \sum_j \binom{2k-2j}{k-j}^2 \binom{2j}{j}^2$
5	(11, 3, 1)	$\sum_j \binom{k}{j}^2 \binom{k+j}{j}$	$\sum_j (-1)^{j+k}\binom{k}{j}^3\binom{4k-5j}{3k}$
6 (A)	(−17, −6, −72)	$\sum_{j,\ell}(-8)^{k-j}\binom{k}{j}\binom{j}{\ell}^3$	$\sum_j \binom{k}{j}^2\binom{k+j}{j}^2$
6 (B)	(10, 3, −9)	$\sum_j \binom{k}{j}^2\binom{2j}{j}$	$(-1)^k\sum_j \binom{k}{j}^2\binom{2j}{j}\binom{2k-2j}{k-j}$
6 (C)	(7, 2, 8)	$\sum_j \binom{k}{j}^3$	$\sum_j (-3)^{k-3j}\binom{k+j}{k}\binom{k}{3j}\binom{3j}{2j}\binom{2j}{j}$
8	(12, 4, −32)	$\sum_j 4^{k-2j}\binom{k}{2j}\binom{2j}{j}^2$	$(-1)^k\sum_j \binom{k}{j}^2\binom{2j}{k}^2$
9	(−9, −3, −27)	$\sum_j (-3)^{k-3j}\binom{k}{j}\binom{k-j}{j}\binom{k-2j}{j}$	$\sum_{j,\ell}\binom{k}{j}^2\binom{k}{\ell}\binom{j}{\ell}\binom{j+\ell}{k}$

Notation:

$$\begin{aligned}(k+1)^2 s(k+1) &= (ak^2+ak+b)s(k)+ck^2 s(k-1),\\ (k+1)^3 t(k+1) &= -(2k+1)(ak^2+ak+a-2b)t(k)-(4c+a^2)k^3 t(k-1);\end{aligned}$$

$$s(-1)=t(-1)=0, \qquad s(0)=t(0)=1.$$

Table 3. *Series for level* $\ell = 1$, $(a, b, c) = (432, 60, 0)$

q	N	x	λ	Reference for (13)
$e^{-2\pi\sqrt{N}}$	2	$\frac{1}{20^3}$	$\frac{3}{28}$	[**14**, p. 187]
	3	$\frac{4}{60^3}$	$\frac{1}{11}$	[**39**, (33)]
	4	$\frac{1}{66^3}$	$\frac{5}{63}$	[**14**, p. 187]
	7	$\frac{1}{255^3}$	$\frac{8}{133}$	[**39**, (34)]
$-e^{-\pi\sqrt{N}}$	7	$\frac{-1}{15^3}$	$\frac{8}{63}$	[**26**, (1·4)]
	11	$\frac{-1}{32^3}$	$\frac{15}{154}$	[**26**, (1·4)]
	19	$\frac{-1}{96^3}$	$\frac{25}{342}$	[**26**, (1·4)]
	27	$\frac{-9}{480^3}$	$\frac{31}{506}$	[**13**, p. 371]
	43	$\frac{-1}{960^3}$	$\frac{263}{5418}$	[**26**, (1·4)]
	67	$\frac{-1}{5280^3}$	$\frac{10177}{261702}$	[**26**, (1·4)]
	163	$\frac{-1}{640320^3}$	$\frac{13591409}{545140134}$	[**26**, (1·4)]

Table 4. *Series for level* $\ell = 2$, $(a, b, c) = (64, 12, 0)$

q	N	x	λ	Reference for (13)
$e^{-2\pi\sqrt{N/2}}$	2	$\frac{2}{6^4}$	$\frac{1}{7}$	[**11**, (4·6)]
	3	$\frac{1}{48^2}$	$\frac{1}{8}$	[**39**, (40)]
	5	$\frac{1}{12^4}$	$\frac{1}{10}$	[**39**, (41)]
	9	$\frac{1}{28^4}$	$\frac{3}{40}$	[**39**, (42)]
	11	$\frac{1}{1584^2}$	$\frac{19}{280}$	[**39**, (43)]
	29	$\frac{1}{396^4}$	$\frac{1103}{26390}$	[**39**, (44)]
$-e^{-\pi\sqrt{N}}$	5	$\frac{-1}{2^{10}}$	$\frac{3}{20}$	[**39**, (35)]
	7	$\frac{-1}{63^2}$	$\frac{8}{65}$	[**11**, (4·8)]
	9	$\frac{-3}{192^2}$	$\frac{3}{28}$	[**39**, (36)]
	13	$\frac{-1}{288^2}$	$\frac{23}{260}$	[**39**, (37)]
	25	$\frac{-5}{5760^2}$	$\frac{41}{644}$	[**39**, (38)]
	37	$\frac{-1}{14112^2}$	$\frac{1123}{21460}$	[**39**, (39)]

Table 5. *Series for level* $\ell = 3$, $(a, b, c) = (27, 6, 0)$

q	N	x	λ	Reference for (13)
$e^{-2\pi\sqrt{N/3}}$	2	$\frac{1}{6^3}$	$\frac{1}{6}$	[**14**, p. 190]
	4	$\frac{4}{18^3}$	$\frac{2}{15}$	[**39**, (31)]
	5	$\frac{1}{15^3}$	$\frac{4}{33}$	[**39**, (32)]
$-e^{-\pi\sqrt{N/3}}$	9	$\frac{-1}{192}$	$\frac{1}{5}$	[**21**, (1·14)]
	17	$\frac{-1}{12^3}$	$\frac{7}{51}$	[**21**, (1·15)]
	25	$\frac{-25}{60^3}$	$\frac{1}{9}$	[**21**, (1·16)]
	41	$\frac{-1}{48^3}$	$\frac{53}{615}$	[**21**, (1·17)]
	49	$\frac{-49}{252^3}$	$\frac{13}{165}$	[**21**, (1·18)]
	89	$\frac{-1}{300^3}$	$\frac{827}{14151}$	[**21**, (1·19)]

Table 6. *Series for level* $\ell = 4$, $(a, b, c) = (16, 4, 0)$

q	N	x	λ	Reference for (13)
$e^{-\pi\sqrt{N}}$	3	$\frac{1}{256}$	$\frac{1}{6}$	[**39**, (28)]
	7	$\frac{1}{4096}$	$\frac{5}{42}$	[**39**, (29)]
$-e^{-\pi\sqrt{N}}$	2	$\frac{-1}{64}$	$\frac{1}{4}$	[**4**]
	4	$\frac{-1}{512}$	$\frac{1}{6}$	

Table 7. *Series for level* $\ell = 5$, $(a, b, c) = (11, 3, 1)$

q	N	x	λ	Reference for (14)
$e^{-2\pi\sqrt{N/5}}$	2	$\frac{1}{72}$	$\frac{1}{5}$	
	3	$\frac{1}{147}$	$\frac{2}{11}$	
$-e^{-\pi\sqrt{N/5}}$	23	$\frac{-1}{828}$	$\frac{29}{190}$	
	47	$\frac{-1}{15228}$	$\frac{71}{682}$	[**42**]

Table 8. *Series for level* $\ell = 6$*; Case A,* $(a, b, c) = (-17, -6, -72)$

q	N	x	λ	Reference for (13)	Reference for (14)
$e^{-2\pi\sqrt{N/6}}$	3	$\frac{1}{64}$	$\frac{3}{10}$	[**23**]	[**24**]
	5	$\frac{1}{288}$	$\frac{1}{5}$	[**23**]	[**42**]
	7	$\frac{1}{864}$	$\frac{11}{70}$	[**23**]	[**42**]
	13	$\frac{1}{10368}$	$\frac{241}{2210}$	[**23**]	[**42**]
	17	$\frac{1}{39168}$	$\frac{73}{770}$	[**23**]	[**42**]
$-e^{-\pi\sqrt{N/3}}$	3	$\frac{-1}{48}$	0		
	5	$\frac{-1}{81}$	$\frac{4}{35}$		
	7	$\frac{-1}{144}$	$\frac{1}{7}$		
	11	$\frac{-1}{432}$	$\frac{8}{55}$		
	19	$\frac{-1}{2736}$	$\frac{8}{65}$		
	31	$\frac{-1}{24336}$	$\frac{107}{1085}$		
	59	$\frac{-1}{1123632}$	$\frac{25808}{359605}$		

Table 9. *Series for level $\ell = 6$; Case B, $(a, b, c) = (10, 3, -9)$*

q	N	x	λ	Reference for (13)	Reference for (14)
$e^{-2\pi\sqrt{N/6}}$	2	$\frac{1}{54}$	$\frac{1}{5}$	[**23**]	[**41**]
	3	$\frac{1}{100}$	$\frac{3}{16}$	[**23**]	[**24**]
	5	$\frac{1}{324}$	$\frac{13}{80}$	[**23**]	[**24**]
	7	$\frac{1}{900}$	$\frac{1}{7}$	[**23**]	[**24**]
	13	$\frac{1}{10404}$	$\frac{7}{65}$	[**23**]	[**24**]
	17	$\frac{1}{39204}$	$\frac{899}{9520}$	[**23**]	[**24**]
$-e^{-\pi\sqrt{N/3}}$	5	$\frac{-1}{45}$	$\frac{2}{7}$		[**19**]
	7	$\frac{-1}{108}$	$\frac{13}{56}$		
	11	$\frac{-1}{396}$	$\frac{7}{40}$		
	19	$\frac{-1}{2700}$	$\frac{253}{1976}$		
	31	$\frac{-1}{24300}$	$\frac{2239}{22568}$		
	59	$\frac{-1}{1123596}$	$\frac{2587}{36040}$		

Table 10. *Series for level* $\ell = 6$*; Case C,* $(a, b, c) = (7, 2, 8)$

q	N	x	λ	Reference for (13)	Reference for (14)
$e^{-2\pi\sqrt{N/6}}$	2	$\frac{1}{50}$	$\frac{2}{9}$	[**23**]	[**24**]
	3	$\frac{1}{96}$	$\frac{1}{5}$	[**23**]	[**24**]
	5	$\frac{1}{320}$	$\frac{1}{6}$	[**23**]	[**24**]
	7	$\frac{1}{896}$	$\frac{13}{90}$	[**23**]	[**24**]
	13	$\frac{1}{10400}$	$\frac{11}{102}$	[**23**]	[**24**]
	17	$\frac{1}{39200}$	$\frac{53}{561}$	[**23**]	[**24**]
$-e^{-\pi\sqrt{N/3}}$	5	$\frac{-1}{49}$	$\frac{4}{15}$		
	7	$\frac{-1}{112}$	$\frac{2}{9}$		
	11	$\frac{-1}{400}$	$\frac{17}{99}$		
	19	$\frac{-1}{2704}$	$\frac{109}{855}$		
	31	$\frac{-1}{24304}$	$\frac{58}{585}$		
	59	$\frac{-1}{1123600}$	$\frac{14903}{207621}$		

Table 11. *Series for level $\ell = 8$, $(a, b, c) = (12, 4, -32)$*

q	N	x	λ
$e^{-2\pi\sqrt{N/8}}$	3	$\frac{1}{64}$	$\frac{1}{6}$
	5	$\frac{1}{160}$	$\frac{1}{6}$
	9	$\frac{1}{800}$	$\frac{1}{7}$
	11	$\frac{1}{1600}$	$\frac{61}{462}$
	29	$\frac{1}{156832}$	$\frac{193}{2310}$
$-e^{-2\pi\sqrt{N/8}}$	5	$\frac{-1}{128}$	$\frac{7}{30}$
	9	$\frac{-1}{768}$	$\frac{11}{70}$
	11	$\frac{-1}{1568}$	$\frac{23}{165}$
	29	$\frac{-1}{156800}$	$\frac{2081}{24882}$

Table 12. *Series for level* $\ell = 9$, $(a, b, c) = (-9, -3, -27)$

q	N	x	λ
$e^{-2\pi\sqrt{N/9}}$	4	$\frac{1}{54}$	$\frac{2}{7}$
	7	$\frac{1}{243}$	$\frac{26}{133}$
$-e^{-\pi\sqrt{N/9}}$	7	$\frac{-1}{27}$	$\frac{2}{7}$
	11	$\frac{-1}{44}$	$\frac{3}{14}$
	19	$\frac{-1}{108}$	$\frac{7}{38}$
	43	$\frac{-1}{972}$	$\frac{85}{602}$
	67	$\frac{-1}{5292}$	$\frac{481}{4154}$
	163	$\frac{-1}{640332}$	$\frac{58831}{786638}$

Table 13. *Radii of convergence. The series* (13) *and* (14) *converge for* $|x| < R_x$ *and* $|y| < R_y$*, respectively*

ℓ	1	2	3	4	5	6A	6B	6C	8	9
R_x	$\frac{1}{1728}$	$\frac{1}{256}$	$\frac{1}{108}$	$\frac{1}{64}$	$\frac{5\sqrt{5}-11}{8}$	$\frac{1}{36}$	$\frac{1}{36}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{\sqrt{3}}{36}$
R_y	$\frac{1}{432}$	$\frac{1}{64}$	$\frac{1}{27}$	$\frac{1}{16}$	$\frac{\sqrt{5}}{25}$	$17-12\sqrt{2}$	$\frac{1}{16}$	$\frac{1}{9}$	$\frac{3-2\sqrt{2}}{4}$	$\frac{2\sqrt{3}-3}{9}$

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