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Two Dirichlet series evaluations found on page 196 of Ramanujan's Lost Notebook

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Abstract

On page 196 in his lost notebook, S. Ramanujan offers evaluations of two particular Dirichlet series. In this paper, we establish Ramanujan's evaluations and more general results by various approaches. The different evaluations arising from different methods yield intriguing, unsuspecting identities.

1. *Introduction*

On page 196 in his lost notebook [**8**, p. 196, equations (i), (ii)], Ramanujan recorded the identities

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n^2/a)}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \frac{r}{a}\left(1-\frac{r}{a}\right) \sin\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right) \tag{1·1}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n^2/a)}{n^2} = -\frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \frac{r}{a}\left(1-\frac{r}{a}\right) \cos\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right), \tag{1·2}$$

where a is an even positive integer. Note that when $a = 2$, (1·1) is equivalent to Euler's evaluation

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We also note that (1·1) and (1·2) are equivalent to the identity

$$\sum_{n=1}^{\infty} \frac{e^{\pi i n^2/a}}{n^2} = \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}} \sum_{r=1}^{a} \frac{r}{a}\left(1 - \frac{r}{a}\right) e^{i(\pi/4 - \pi r^2/a)}. \tag{1·3}$$

Motivated by the left-hand side of (1·3), we let

$$R_k(s) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n^2/k}}{n^s}, \qquad \operatorname{Re} s > 1, \tag{1·4}$$

where k is a positive integer. In this paper, we derive several identities, or evaluations, for $R_k(2m)$, when m is a positive integer.

To illustrate our work, we provide here three of our evaluations. The first identity expresses $R_k(2m)$, for $k \equiv 0 (\mathrm{mod}\ 4)$, in terms of the Bernoulli polynomials $B_n(t)$, $n \geqslant 0$, which are defined by

$$\frac{x e^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!} x^n, \qquad |x| < 2\pi, \tag{1·5}$$

and the Bernoulli numbers $B_n := B_n(0)$, $n \geqslant 0$.

THEOREM 1·1. *Let a be an even positive integer. Then*

$$R_{2a}(2m) = \frac{(-1)^{m+1}\pi^{2m}2^{2m-1}}{(2m)!} \times \left(B_{2m} + \frac{1}{\sqrt{a}} \sum_{\nu=1}^{a} \left(B_{2m}\left(\frac{\nu}{a}\right) - B_{2m} \right) e^{\pi i/4 - \pi i \nu^2/a} \right). \tag{1·6}$$

When $m = 1$, (1·6) reduces to (1·3).

To describe the second and third identities associated with $R_k(s)$, we recall the definitions of the Stirling numbers of the second kind $S(n, h)$ [**9**, p. 91] and the ballot numbers $c_{n,h}$ [**10**, p. 130]. For nonnegative integers n and h, they are defined by

$$S(n, h) = \frac{1}{h!} \sum_{j=0}^{h} (-1)^j \binom{h}{j} (h - j)^n$$

and

$$c_{n,h} = \frac{n - h + 1}{n + 1} \binom{n + h}{n}, \tag{1·7}$$

respectively, where in the last definition we also require that $h \leqslant n$. When $h = n$, the definition (1·7) reduces to the definition of the nth Catalan number [**10**, p. 101]. For a fixed positive integer k and for nonnegative integers u and v, let

$$T_{u,v} = \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \frac{\omega_r^v}{(\omega_r - 1)^u}, \tag{1·8}$$

where $\omega_r = e^{2\pi i r/k}$.

THEOREM 1·2. *Let $T_{u,v}$ be given by* (1·8). *Then*

$$R_k(2m) = \frac{(-1)^{m+1}}{2(2m)!}\left(\frac{2\pi}{k}\right)^{2m}\left(B_{2m} - 2m\sum_{s=1}^{2m-1}\sigma_{2m-1,s}\,T_{s+1,1}\right), \qquad (1·9)$$

where

$$\sigma_{n,h} = h!S(n,h).$$

THEOREM 1·3. *Let k be a positive integer. Then*

$$R_k(2m) = \frac{(-1)^{m+1}}{2\cdot(2m)!}\left(\frac{2\pi}{k}\right)^{2m}\left(B_{2m} - 2m\sum_{j=1}^{m}\alpha_{2m,\,j}T_{2j,\,j}\right), \qquad (1·10)$$

where

$$\alpha_{2m,1} = 1 \qquad (1·11)$$

and, for $j \geqslant 2$,

$$\alpha_{2m,\,j} = \sum_{s=0}^{j-2}(-1)^s c_{j-2,s}\,(j-s)!S(2m-1,\,j-s). \qquad (1·12)$$

We should note that the coefficients $\alpha_{2m,\,j}$ in the right-hand side of (1·10) are independent of k.

We see from either Theorem 1·2 or Theorem 1·3 that we can represent $R_{2a}(2m)$ in terms of $e^{2\pi i r^2/k}\csc^{2j}(\pi r/k)$. More precisely, by Theorem 1·3,

$$R_k(2m) = \frac{(-1)^{m+1}}{2\cdot(2m)!}\left(\frac{2\pi}{k}\right)^{2m} \times\left(B_{2m} - 2m\sum_{j=1}^{m}\frac{(-1)^j\alpha_{2m,\,j}}{2^{2j}}\sum_{r=1}^{k-1}e^{2\pi i r^2/k}\csc^{2j}\left(\frac{\pi r}{k}\right)\right). \qquad (1·13)$$

In Section 4, we obtain a further representation for $R_k(2m)$ in terms of derivatives of odd order of the cotangent function. Such a formula inspires us to establish an explicit formula for odd-order derivatives of the cotangent, and we do so in Proposition 4·2. We are not aware of such a representation in the literature, and so we think that the formula (4·14) of Proposition 4·2 is of independent interest.

Lastly, in Section 5, we approach our original problem through the theory of periodic zeta functions.

2. *Representations in terms of Bernoulli polynomials*

In this section, we give a proof of Theorem 1·1. This proof is motivated by the observation that (1·6) is similar to Dirichlet's class number formula, which expresses a special value of a certain Dirichlet L-series as a finite sum of terms involving the Legendre symbol [**6**, p. 51]. In our considerations, values of the Hurwitz zeta function

$$\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^s}, \qquad \operatorname{Re} s > 1,$$

take the place of values of the Legendre symbol.

Proof of Theorem 1·1. We begin by writing $R_k(s)$, for Re $s > 1$, as

$$R_k(s) = \sum_{r=1}^{k} \sum_{n=0}^{\infty} \frac{e^{2\pi i (r+kn)^2/k}}{(r+kn)^s} = \frac{1}{k^s} \sum_{r=1}^{k} e^{2\pi i r^2/k} \zeta\left(s, \frac{r}{k}\right). \tag{2·1}$$

Since $\zeta(s, x)$ has an analytic continuation into the entire complex s-plane, the right-hand side of (2·1) gives the analytic continuation of $R_k(s)$ to the whole complex s-plane.

Using the functional equation of the Hurwitz zeta function [**1**, p. 261, theorem 12·8], we deduce that

$$\begin{aligned}\sum_{r=1}^{k} & e^{2\pi i r^2/k} \zeta\left(1-s, \frac{r}{k}\right) \\ &= \frac{\Gamma(s)}{(2\pi k)^s} \sum_{r=1}^{k} e^{2\pi i r^2/k} \sum_{\ell=1}^{k} \left(e^{-\pi i s/2} e^{2\pi i r\ell/k} + e^{\pi i s/2} e^{-2\pi i r\ell/k}\right) \zeta\left(s, \frac{\ell}{k}\right).\end{aligned} \tag{2·2}$$

Interchanging the summations on the right-hand side of (2·2), we deduce that

$$\begin{aligned}\sum_{r=1}^{k} & e^{2\pi i r^2/k} \zeta\left(1-s, \frac{r}{k}\right) \\ &= \frac{\Gamma(s)}{(2\pi k)^s} \sum_{\ell=1}^{k} \zeta\left(s, \frac{\ell}{k}\right) \left\{ e^{-\pi i s/2} \sum_{r=1}^{k} e^{2\pi i (r^2+r\ell)/k} + e^{\pi i s/2} \sum_{r=1}^{k} e^{2\pi i (r^2-r\ell)/k} \right\} \\ &= \frac{2\Gamma(s)}{(2\pi k)^s} \cos\left(\frac{\pi s}{2}\right) \sum_{\ell=1}^{k} \zeta\left(s, \frac{\ell}{k}\right) \left(\sum_{r=1}^{k} e^{2\pi i (r^2+r\ell)/k}\right).\end{aligned} \tag{2·3}$$

Letting s tend to $1-2m$, where m is a positive integer, we find, using the residue of $\Gamma(s)$ at $s = 1-2m$ [**1**, p. 250], that

$$\begin{aligned}\lim_{s\to 1-2m} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) &= \lim_{s\to 1-2m} (s-(1-2m))\Gamma(s) \frac{\cos\left(\frac{\pi s}{2}\right)}{s-(1-2m)} \\ &= \frac{(-1)^m \pi}{2(2m-1)!}.\end{aligned} \tag{2·4}$$

From [**1**, p. 264, theorem 12·13], we find that

$$\zeta\left(1-2m, \frac{\ell}{k}\right) = -\frac{1}{2m} B_{2m}\left(\frac{\ell}{k}\right). \tag{2·5}$$

Using (2·4) and (2·5) in (2·3), and then (2·3) in (2·1), with s replaced by $1-2m$, we deduce that

$$R_k(2m) = \frac{\pi(-1)^{m+1}}{(2\pi)^{1-2m} k \cdot (2m)!} S_{2m}, \tag{2·6}$$

where

$$S_n = \sum_{\ell=1}^{k} B_n\left(\frac{\ell}{k}\right) \sum_{r=1}^{k} e^{2\pi i (r^2+r\ell)/k}. \tag{2·7}$$

Next, let $k=2a$, where a is even. Then

$$\begin{aligned}\sum_{r=1}^{2a} e^{2\pi i(r^2+r\ell)/(2a)} &= \sum_{r=1}^{a} e^{2\pi i(r^2+r\ell)/(2a)} + \sum_{r=1}^{a} e^{2\pi i((a+r)^2+(a+r)\ell)/(2a)}\\ &= (1+(-1)^\ell)\sum_{r=1}^{a} e^{\pi i(r^2+r\ell)/a}. \end{aligned} \tag{2·8}$$

Hence, the left-hand side of (2·8) vanishes when ℓ is odd. Let $\ell=2\nu$. Then, we may write (2·8) as

$$\begin{aligned}\sum_{r=1}^{2a} e^{2\pi i(r^2+r\ell)/(2a)} &= 2\sum_{r=1}^{a} e^{\pi i((r+\nu)^2-\nu^2)/a}\\ &= 2e^{-\pi i\nu^2/a}\sum_{s=1}^{a} e^{\pi i s^2/a}\\ &= \sqrt{2a}(1+i)e^{-\pi i\nu^2/a}\\ &= 2\sqrt{a}e^{\pi i/4-\pi i\nu^2/a}, \end{aligned} \tag{2·9}$$

where we have used the fact (see [**1**, p. 195, equation (30)] or [**4**, p. 15, corollary 1·2·3])

$$\sum_{r=1}^{4c} e^{2\pi i r^2/(4c)} = (1+i)\sqrt{4c},$$

where c is any positive integer. By (2·7)–(2·9), we deduce that

$$\begin{aligned} S_{2m} &= B_{2m}\sum_{r=1}^{2a}\sum_{\ell=1}^{2a} e^{2\pi i(r^2+r\ell)/(2a)}\\ &\quad + \sum_{\ell=1}^{2a}\left(B_{2m}\left(\frac{\ell}{2a}\right)-B_{2m}\right)\sum_{r=1}^{2a} e^{2\pi i(r^2+r\ell)/(2a)}\\ &= 2a\cdot B_{2m} + 2\sqrt{a}\sum_{\nu=1}^{a}\left(B_{2m}\left(\frac{\nu}{a}\right)-B_{2m}\right)e^{\pi i/4-\pi i\nu^2/a}. \end{aligned}$$

Substituting the last equality into (2·6), we conclude the proof of (1·6).

3. *Representations in terms of cosecant functions*

In this section, we give proofs of Theorems 1·2 and 1·3. We first establish a lemma. The polynomials U_n in Lemma 3·1 below are related to the Eulerian polynomials [**3**, equation (0·1)].

LEMMA 3·1. *Suppose* $\alpha \neq 1$ *and*

$$\frac{xe^x}{e^x-\alpha} =: \sum_{n=1}^{\infty}\frac{U_n}{n!}x^n. \tag{3·1}$$

Then

$$U_1 = u \ \ \textit{and}\ \ U_n = n(1-u)\sum_{h=1}^{n-1}\sigma_{n-1,h}u^h(-1)^{h-1}, \quad n \geqslant 2,$$

where

$$u = \frac{1}{1-\alpha} \qquad \textit{and} \qquad \sigma_{n,h} = h!S(n,h). \tag{3·2}$$

Proof. From (3·1), we find that

$$\begin{aligned} xe^x &= (e^x - \alpha)\left(\sum_{n=1}^{\infty} \frac{U_n}{n!}x^n\right) \\ &= \left(1 - \alpha + \sum_{n=1}^{\infty} \frac{x^n}{n!}\right)\left(\sum_{n=1}^{\infty} \frac{U_n}{n!}x^n\right). \end{aligned}$$

Comparing the coefficients of x^n for $n \geqslant 1$, we find that

$$U_1 = \frac{1}{1-\alpha} = u$$

and

$$n = (1-\alpha)U_n + \sum_{j=1}^{n-1} \binom{n}{j} U_{n-j}, \qquad n \geqslant 2.$$

Using the value $U_1 = u$, we see that U_n, $n \geqslant 2$, must satisfy the recurrence relation

$$U_n = nu(1-u) - u\sum_{j=2}^{n-1} \binom{n}{j} U_j. \tag{3·3}$$

Let

$$U_j = ju(1-u)V_j.$$

Note that $V_1 = 1/(1-u)$ and $V_2 = 1$. By (3·3), V_j, $j \geqslant 2$, satisfies the recurrence relation

$$V_n = 1 - u\sum_{j=2}^{n-1} \binom{n-1}{j-1} V_j.$$

Define

$$W_1 = \frac{1}{1-u}$$

and

$$W_n = \sum_{h=1}^{n-1} \sigma_{n-1,h} u^{h-1}(-1)^{h-1}, \qquad n \geqslant 2.$$

Clearly,

$$W_1 = V_1 \quad \text{and} \quad W_2 = S(1,1) = 1 = V_2.$$

Thus, in order to prove Lemma 3·1, it suffices to show that W_n satisfies the same recurrence relation as V_n, namely,

$$W_n = 1 - u\sum_{j=2}^{n-1} \binom{n-1}{j-1} W_j, \qquad n \geqslant 2. \tag{3·4}$$

Now, we observe that

$$1 - u\sum_{j=2}^{n-1}\binom{n-1}{j-1}W_j = 1 + \sum_{j=2}^{n-1}\binom{n-1}{j-1}\sum_{h=1}^{j-1}\sigma_{j-1,h}u^h(-1)^h$$
$$= 1 + \sum_{h=1}^{n-2}u^h(-1)^h\sum_{j=h+1}^{n-1}\binom{n-1}{j-1}\sigma_{j-1,h}.$$

In order to show that W_n satisfies (3·4), it suffices to prove that

$$\sigma_{n-1,h+1} = \sum_{j=h+1}^{n-1}\binom{n-1}{j-1}\sigma_{j-1,h}, \tag{3·5}$$

because $S(n-1, 1) = 1$.

Now, using (3·2), we may rewrite (3·5) as

$$(h+1)S(n-1, h+1) = \sum_{j=h}^{n-2}\binom{n-1}{j}S(j, h),$$

or, with n replaced by $n+1$,

$$(h+1)S(n, h+1) = \sum_{j=h}^{n-1}\binom{n}{j}S(j, h), \tag{3·6}$$

where $h \leqslant n-1$. Adding the term $S(n, h)$ to both sides of (3·6), we conclude that

$$S(n, h) + (h+1)S(n, h+1) = \sum_{j=h}^{n}\binom{n}{j}S(j, h). \tag{3·7}$$

It is known that [**9**, p. 43, equation 14(b)] the right-hand side of (3·7) equals $S(n+1, h+1)$. Hence, (3·7) is equivalent to

$$S(n+1, h+1) = S(n, h) + (h+1)S(n, h+1). \tag{3·8}$$

Since (3·8) is a well-known property of $S(n, h)$ [**9**, p. 33, equation (37)], this completes the proof of Lemma 3·1.

Recall from (2·6) that we can express $R_k(2m)$ in terms of S_{2m}, where S_n is given by (2·7). Note that (2·6) and (2·7) hold for any positive integer k. We now examine S_n.

Separating the term with $r = k$ in (2·7) and using the multiplication formula for Bernoulli polynomials [**7**, p. 590, equation (24·4·18)]

$$B_n(kx) = k^{n-1}\sum_{\ell=1}^{k-1}B_n\left(x + \frac{\ell}{k}\right)$$

with $x = 0$, we deduce that

$$S_n = k^{1-n}B_n + \sum_{r=1}^{k-1}e^{2\pi i r^2/k}A_n(r), \tag{3·9}$$

where

$$A_n(r) = \sum_{\ell=1}^{k}B_n\left(\frac{\ell}{k}\right)\omega_r^{\ell}, \qquad \omega_r = e^{2\pi i r/k}. \tag{3·10}$$

Substituting $t = \ell/k$ in (1·5), multiplying by ω_r^ℓ, and summing over ℓ, $1 \leqslant \ell \leqslant k$, we deduce that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{\ell=1}^{k} B_n\left(\frac{\ell}{k}\right) \omega_r^\ell \right) x^n = \frac{x}{e^x - 1} \sum_{\ell=1}^{k} e^{x\ell/k} \omega_r^\ell. \tag{3·11}$$

Since

$$\sum_{\ell=1}^{k} \left(e^{x/k}\omega_r\right)^\ell = \frac{1-e^x}{1-e^{x/k}\omega_r} e^{x/k}\omega_r,$$

we find, from (3·10) and (3·11), that

$$\sum_{n=0}^{\infty} \frac{1}{n!} A_n(r) x^n = \frac{x e^{x/k}}{e^{x/k} - \omega_r^{-1}}.$$

Replacing x by kx (with $|x|$ sufficiently small), we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} k^{n-1} A_n(r) x^n = \frac{x e^x}{e^x - \alpha}, \tag{3·12}$$

where $\alpha = \omega_r^{-1} \neq 1$.

We are now ready to prove Theorem 1·2.

Proof of Theorem 1·2. Lemma 3·1 and (3·12) give a representation of $A_n(r)$ in terms of Stirling numbers of the second kind, namely,

$$k^{n-1} A_n(r) = n(1-u) \sum_{h=1}^{n-1} \sigma_{n-1,h} u^h (-1)^{h-1}, \tag{3·13}$$

where $u = 1/(1-\omega_r^{-1})$. Set $n = 2m$ in (3·13) and then substitute (3·13) in (3·9). Using also (2·6), we then find that

$$\begin{aligned} R_k(2m) &= \sum_{n=1}^{\infty} \frac{e^{2\pi i n^2/k}}{n^{2m}} = \frac{(-1)^{m+1}\pi(2\pi)^{2m-1}}{k(2m)!} S_{2m} \\ &= \frac{(-1)^{m+1}\pi(2\pi)^{2m-1}}{k(2m)!} \left\{ \frac{B_{2m}}{k^{2m-1}} + \frac{1}{k^{2m-1}} \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \right. \\ &\quad \left. \times\, 2m \sum_{h=1}^{2m-1} (-1)^h \sigma_{2m-1,h} \frac{\omega_r^h}{(\omega_r - 1)^{h+1}} \right\} \\ &= \frac{(-1)^{m+1}}{2(2m)!} \left(\frac{2\pi}{k}\right)^{2m} \left\{ B_{2m} + 2m \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \right. \\ &\quad \left. \times \sum_{h=1}^{2m-1} (-1)^h \sigma_{2m-1,h} \frac{\omega_r^h}{(\omega_r - 1)^{h+1}} \right\}. \end{aligned}$$

Now replace r by $k-r$ in the summation over r on the far right-hand side above. Employing (1·8), we then obtain the assertion (1·9).

In order to prove Theorem 1·3, we need to replace the terms

$$T_{s+1,1}, \qquad 1 \leqslant s \leqslant 2m-1,$$

in Theorem 1·2 by

$$T_{2j,j}, \qquad 1 \leqslant j \leqslant m.$$

To effect such a change, we need two lemmas.

LEMMA 3·2. *Let j be a positive integer. Then*

$$T_{2j,j} = \sum_{\nu=0}^{j-1} \binom{j-1}{\nu} T_{j+\nu+1,1}. \tag{3·14}$$

Proof. From the definition (1·8),

$$\begin{aligned}
T_{2j,j} &= \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \frac{\omega_r^j}{(\omega_r - 1)^{2j}} \\
&= \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \frac{\omega_r (\omega_r)^{j-1}}{(\omega_r - 1)^{2j}} \\
&= \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \frac{\omega_r}{(\omega_r - 1)^{2j}} \left(\sum_{\nu=0}^{j-1} \binom{j-1}{\nu} (\omega_r - 1)^{\nu} \right) \\
&= \sum_{\nu=0}^{j-1} \binom{j-1}{\nu} T_{j+\nu+1,1},
\end{aligned}$$

where in the penultimate line we replaced ν by $j-1-\nu$.

LEMMA 3·3. *Let j be a positive integer. Then*

$$T_{2j+1,1} = -\frac{1}{2} \sum_{h=0}^{2j-2} \binom{2j-1}{h} T_{h+2,1}. \tag{3·15}$$

Proof. Replacing r by $k-r$ and then introducing the notation $v = \omega_r - 1$, we find that

$$\begin{aligned}
T_{2j+1,1} &= \frac{1}{2} \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \left(\frac{\omega_r}{(\omega_r - 1)^{2j+1}} + \frac{\omega_r^{-1}}{(\omega_r^{-1} - 1)^{2j+1}} \right) \\
&= -\frac{1}{2} \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \frac{\omega_r (\omega_r^{2j-1} - 1)}{(\omega_r - 1)^{2j+1}} \\
&= -\frac{1}{2} \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \frac{\omega_r ((v+1)^{2j-1} - 1)}{v^{2j+1}} \\
&= -\frac{1}{2} \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \frac{\omega_r}{v^{2j+1}} \left(\sum_{h=0}^{2j-2} \binom{2j-1}{h} v^{2j-1-h} \right) \\
&= -\frac{1}{2} \sum_{h=0}^{2j-2} \binom{2j-1}{h} T_{h+2,1}.
\end{aligned}$$

Proof of Theorem 1·3. Let

$$\mathcal{M}_s = \{T_{j,1} | 2 \leqslant j \leqslant s\}$$

and

$$\mathcal{N}_s = \{T_{2j,j} | 1 \leqslant j \leqslant s\}.$$

We claim that for fixed t, every term in $\mathcal{M}_{2t}$ is a linear combination of elements in $\mathcal{N}_t$.

We prove this by induction on $t \geqslant 1$. Keeping in mind that $\mathcal{M}_2 = \{T_{2,1}\}$, we can easily see that the case $t = 1$ is trivial. By induction, it suffices to show that both $T_{2t-1,1}$ and $T_{2t,1}$ can be expressed as linear combinations of elements in $\mathcal{N}_t$. Now, by (3·15), $T_{2t-1,1}$ is a linear combination of elements in $\mathcal{M}_{2(t-1)}$, and by induction, each term in $\mathcal{M}_{2(t-1)}$ is a linear combination of elements in $\mathcal{N}_{t-1}$. Next, by (3·14), $T_{2t,1}$ is a linear combination of elements in $\mathcal{M}_{2(t-1)} \cup \{T_{2t,t}\}$. By induction again, we conclude that $T_{2t,1}$ is a linear combination of elements in $\mathcal{N}_t$.

Let $\widetilde{S_{2m}}$ denote the sum on the right-hand side of (1·9), namely, with s replaced by $j+1$,

$$\widetilde{S_{2m}} = \sum_{j=0}^{2m-2} \sigma_{2m-1,\, j+1} T_{j+2,1}. \tag{3·16}$$

From the argument above, we can conclude that

$$\widetilde{S_{2m}} = \sum_{j=1}^{m} \alpha_{2m,\, j} T_{2j,j} = \sum_{j=0}^{m-1} \alpha_{2m,\, j+1} T_{2j+2,\, j+1}, \tag{3·17}$$

for certain rational numbers $\alpha_{2m,\, j+1}$.

We substitute (3·14) into (3·17) and obtain

$$\begin{aligned} \widetilde{S_{2m}} &= \sum_{j=0}^{m-1} \alpha_{2m,\, j+1} \sum_{h=0}^{j} \binom{j}{h} T_{j+h+2,1} \\ &= \sum_{\ell=0}^{2m-2} T_{\ell+2,1} \left(\sum_{j=0}^{m-1} \alpha_{2m,\, j+1} \binom{j}{\ell - j} \right). \end{aligned} \tag{3·18}$$

Note that in (3·18) the binomial coefficient $\binom{j}{\ell-j}$ vanishes when $\ell - j < 0$ and $j < \ell - j$. Comparing (3·18) and (3·16), we conclude that if $\alpha_{2m,\, j+1}$ can be chosen so as to satisfy the relations

$$\sum_{j=0}^{m-1} \alpha_{2m,\, j+1} \binom{j}{\ell - j} = \sigma_{2m-1,\ell+1}, \qquad 0 \leqslant \ell \leqslant 2m-2, \tag{3·19}$$

then (3·17) holds with these $\alpha_{2m,\, j+1}$.

Our next task is to invert the relations (3·19). Replacing j by $r = \ell - j$ in (3·19), we find that

$$\sum_{0 \leqslant r \leqslant \ell/2} \alpha_{2m,\ell+1-r} \binom{\ell - r}{r} = \sigma_{2m-1,\ell+1}. \tag{3·20}$$

If we set $b_n = \alpha_{2m,n+1}$ and $a_n = \sigma_{2m-1,n+1}$, then (3·20) becomes

$$\sum_{0 \leqslant r \leqslant \ell/2} \binom{\ell - r}{r} b_{\ell - r} = a_\ell.$$

Hence, by the inversion formula [**10**, p. 62, formula 5, table 2·3], we find that

$$b_\ell = \sum_{h\geqslant 0}(-1)^h \left\{\binom{\ell+h-1}{h} - \binom{\ell+h-1}{h-1}\right\} a_{\ell-h}. \tag{3·21}$$

Note that, by (1·7),

$$\binom{\ell+h-1}{h} - \binom{\ell+h-1}{h-1} = \frac{\ell-h}{\ell}\binom{\ell+h-1}{h} = c_{\ell-1,h}.$$

From (3·21), the formula above, and our auxiliary notation, we obtain the explicit formula

$$\alpha_{2m,\ell+1} = \sum_{h\geqslant 0}(-1)^h c_{\ell-1,h}\sigma_{2m-1,\ell-h+1}. \tag{3·22}$$

If we now return to Theorem 1·2 and then use (3·16), (3·17), (3·19) and (3·2), we see that (3·22) enables us to complete the proof of Theorem 1·3.

We conclude this section with one further observation. Let s tend to $-2m$, $m \geqslant 1$, in (2·3). The limit of the left-hand side is $R_k(2m+1)$. Since $\lim_{s\to -2m}\cos(\pi s/2) = (-1)^m$, it follows, from the obvious analogue of (2·6) and (2·7), that

$$S_{2m+1}(k) = \sum_{\ell=1}^{k} B_{2m+1}\left(\frac{\ell}{k}\right)\sum_{r=1}^{k} e^{2\pi i(r^2+r\ell)/k} = 0, \quad m\geqslant 1. \tag{3·23}$$

However, we can show (3·23) directly. First, recalling that $B_{2m+1}=0$, $m\geqslant 1$, and secondly replacing r by $k-r$, we find that

$$\begin{aligned} S_{2m+1} &= \sum_{r=1}^{k-1} e^{2\pi i r^2/k}\sum_{\ell=1}^{k-1} B_{2m+1}\left(\frac{\ell}{k}\right) e^{2\pi i r\ell/k} \\ &= \sum_{r=1}^{k-1} e^{2\pi i r^2/k}\sum_{\ell=1}^{k-1} B_{2m+1}\left(\frac{\ell}{k}\right) e^{-2\pi i r\ell/k}. \end{aligned}$$

Next replace ℓ by $k-\ell$ in the inner sum and use the property $B_n(1-x) = (-1)^n B_n(x)$, $n\geqslant 2$ [**7**, p. 589, equation (24·4·3)], to conclude that

$$S_{2m+1} = \sum_{r=1}^{k-1} e^{2\pi i r^2/k}\sum_{\ell=1}^{k-1} B_{2m+1}\left(1-\frac{\ell}{k}\right) e^{2\pi i r\ell/k} = -S_{2m+1}.$$

Hence, (3·23) follows.

4. *Further representations in terms of cotangent function and its derivatives*

Toward the end of Section 1, we mentioned that $R_k(2m)$ could be represented in terms of certain cosecant sums (1·13). In this section, we shall provide another representation of $R_k(2m)$ in terms of the cotangent function and its derivatives.

THEOREM 4·1. (i) *Let k be a positive integer. Then*

$$R_k(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!\,k^{2m}} B_{2m} - \frac{\pi^{2m}}{2(2m-1)!\,k^{2m}} \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \cot^{(2m-1)}\left(\frac{\pi r}{k}\right). \tag{4·1}$$

(ii) *Assume that $k = 2a$, where a is an even positive integer. Then*

$$R_{2a}(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!\,a^{2m}} \left(1 + e^{\pi i a/4}(2^{2m}-1)\right) B_{2m} - \frac{\pi^{2m}}{(2m-1)!\,a^{2m}} \sum_{r=1}^{\frac{1}{2}a-1} e^{\pi i r^2/a} \cot^{(2m-1)}\left(\frac{\pi r}{a}\right). \tag{4·2}$$

Proof. Returning to (2·1) and singling out the term $r = k$, we find that

$$R_k(2m) = \frac{1}{k^{2m}}\zeta(2m) + \frac{1}{k^{2m}} \sum_{r=1}^{k-1} e^{2\pi i r^2/k} \sum_{n=0}^{\infty} \frac{1}{(n+r/k)^{2m}}. \tag{4·3}$$

Note that the sum with index r is identical to the sum with index $k - r$. Thus,

$$R_k(2m) = \frac{1}{k^{2m}}\zeta(2m) + \frac{1}{2k^{2m}} \sum_{r=1}^{k-1} e^{2\pi i r^2/k} U(r, k; 2m), \tag{4·4}$$

where we put

$$U(r, k; 2m) = \sum_{n=0}^{\infty} \left(\frac{1}{(n+r/k)^{2m}} + \frac{1}{(n+(k-r)/k)^{2m}} \right).$$

We observe that

$$U(r, k; 2m) = \sum_{n=0}^{\infty} \left(\frac{1}{(n+r/k)^{2m}} + \frac{1}{(-n-1+r/k)^{2m}} \right) = \sum_{n=-\infty}^{\infty} \frac{1}{(n+r/k)^{2m}}. \tag{4·5}$$

It therefore suffices to evaluate the bilateral sum in (4·5).

To evaluate $U(r, k; 2m)$, recall the partial fraction decomposition

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right).$$

Differentiating $\ell - 1$ times above, we find that

$$\pi^{\ell} \cot^{(\ell-1)}(\pi z) = (-1)^{\ell-1}(\ell-1)! \sum_{n=-\infty}^{\infty} \frac{1}{(z+n)^{\ell}} \tag{4·6}$$

for any positive integer $\ell \geqslant 2$. Putting $\ell = 2m$ and $z = r/k$ in (4·6), we deduce that

$$U(r, k; 2m) = -\frac{\pi^{2m}}{(2m-1)!} \cot^{(2m-1)}(\pi r/k). \tag{4·7}$$

Putting (4·7) in (4·4) and using Euler's formula for $\zeta(2m)$, we complete the proof of (4·1).

For (4·2), it is best to return to the definition (1·4). Since a is even, we can divide the sum over n into residue classes modulo a to deduce that

$$R_{2a}(2m) = \sum_{r=1}^{a} e^{\pi i r^2/a} \sum_{n=0}^{\infty} \frac{1}{(an+r)^{2m}}$$
$$= \frac{1}{a^{2m}}\zeta(2m) + \frac{1}{a^{2m}} \sum_{r=1}^{a-1} e^{\pi i r^2/a} \sum_{n=0}^{\infty} \frac{1}{(n+r/a)^{2m}}.$$

Singling out the term for $r = a/2$ in the sum over r in the right-hand side above, noting that the terms in the outer sum with indices r and $a-r$ are identical since a is even, and using the identity

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^{2m}} = (1-2^{-2m})\zeta(2m),$$

we find that

$$R_{2a}(2m) = \frac{1}{a^{2m}}\zeta(2m) + \frac{e^{\pi i a/4}}{a^{2m}}\left(2^{2m}-1\right)\zeta(2m)$$
$$+ \frac{1}{a^{2m}} \sum_{r=1}^{a/2-1} e^{\pi i r^2/a} \sum_{n=0}^{\infty} \left(\frac{1}{(n+r/a)^{2m}} + \frac{1}{(n+(a-r)/a)^{2m}} \right). \qquad (4·8)$$

Note that the innermost sum in the right-hand side of (4·8) is $U(r, a; 2m)$. Hence, using Euler's formula for $\zeta(2m)$, we have

$$R_{2a}(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}}{2(2m)!\,a^{2m}}\left(1 + e^{\pi i a/4}(2^{2m}-1)\right) B_{2m}$$
$$+ \frac{1}{a^{2m}} \sum_{r=1}^{a/2-1} e^{\pi i r^2/a} U(r, a; 2m).$$

An application of (4·7) to the right-hand side above then completes the proof of (4·2).

If we take real and imaginary parts in (4·2), we deduce the formulas

$$\sum_{n=1}^{\infty} \frac{\cos(\pi n^2/a)}{n^{2m}} = \frac{(-1)^{m-1}(2\pi)^{2m}}{2\,(2m)!\,a^{2m}} \left(1 + \cos\left(\frac{\pi a}{4}\right)(2^{2m}-1)\right) B_{2m}$$
$$- \frac{\pi^{2m}}{a^{2m}(2m-1)!} \sum_{r=1}^{\frac{1}{2}a-1} \cos\left(\frac{\pi r^2}{a}\right) \cot^{(2m-1)}\left(\frac{\pi r}{a}\right) \qquad (4·9)$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n^2/a)}{n^{2m}} = \frac{(-1)^{m-1}(2\pi)^{2m}}{2\,(2m)!a^{2m}} \sin\left(\frac{\pi a}{4}\right)(2^{2m}-1)B_{2m}$$
$$- \frac{\pi^{2m}}{a^{2m}(2m-1)!} \sum_{r=1}^{\frac{1}{2}a-1} \sin\left(\frac{\pi r^2}{a}\right) \cot^{(2m-1)}\left(\frac{\pi r}{a}\right). \qquad (4·10)$$

Hence, setting $m = 1$ in (4·9) and (4·10), we deduce that, respectively,

$$\sum_{n=1}^{\infty} \frac{\cos\left(\pi n^2/a\right)}{n^2} = \frac{\pi^2}{6a^2} + \frac{\pi^2 \cos(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2} \sum_{r=1}^{\frac{1}{2}a-1} \cos\left(\frac{\pi r^2}{a}\right) \csc^2\left(\frac{\pi r}{a}\right) \tag{4·11}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin\left(\pi n^2/a\right)}{n^2} = \frac{\pi^2 \sin(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2} \sum_{r=1}^{\frac{1}{2}a-1} \sin\left(\frac{\pi r^2}{a}\right) \csc^2\left(\frac{\pi r}{a}\right). \tag{4·12}$$

Recall that, at the end of the Introduction, we mentioned the formula (1·13) which provides a way to evaluate the series $R_k(2m)$ in terms of cosecant functions. For example, when $m = 1$ and $k = 2a$, where a is an even positive integer, formula (1·13) reads

$$\sum_{n=1}^{\infty} \frac{e^{\pi i n^2/a}}{n^2} = \frac{\pi^2}{24a^2} + \frac{\pi^2}{8a^2} \sum_{r=1}^{2a-1} e^{\pi i r^2/a} \csc^2\left(\frac{\pi r}{2a}\right). \tag{4·13}$$

Comparing (4·13) with (4·11) and (4·12), we see that the range of summation over r on the right-hand sides can be shortened in the case that a is even.

In Theorem 4·1, we find that $R_k(2m)$ is represented as a linear combination of the values of higher derivatives of the cotangent function. We are therefore motivated to find a more explicit representation for these derivatives. Comparing the expressions (1·13) and (4·1), we are led to the explicit representation of $\cot^{(2m-1)} x$ in terms of the cosecant function. Since it is interesting in itself, we shall describe such a formula below.

PROPOSITION 4·2. *Let* $\alpha_{2m,\,j}$ *be defined by* (1·11) *and* (1·12). *Then*

$$\cot^{(2m-1)} x = (-1)^{m+1} \sum_{j=1}^{m} (-1)^j 2^{2(m-j)} \alpha_{2m,\,j} \csc^{2j} x. \tag{4·14}$$

Proof. Since

$$\cot' x = -\csc^2 x$$

and

$$(\csc^n x)'' = -n^2 \csc^n x + n(n+1) \csc^{n+2} x, \tag{4·15}$$

it is easily seen that $\cot^{(2m-1)} x$ can be written as a linear combination of $\csc^{2j} x$, $1 \leqslant j \leqslant m$. To that end, define $p_m(j)$ by

$$\cot^{(2m-1)} x = \sum_{j=1}^{m} p_m(j) \csc^{2j} x. \tag{4·16}$$

For example, $p_1(1) = -1$, $p_2(1) = 4$, $p_2(2) = -6$, $p_3(1) = -16$, $p_3(2) = 120$, and $p_3(3) = -120$. In particular, $p_n(1) = (-1)^n 4^{n-1}$ and $p_n(n) = -(2n-1)!$. We shall show that

$$p_m(j) = (-1)^{m+j+1} 2^{2(m-j)} \alpha_{2m,\,j}. \tag{4·17}$$

Differentiating both sides of (4·16) twice and using (4·15), we find that

$$p_{m+1}(j+1) = 2j(2j+1) p_m(j) - 4(j+1)^2 p_m(j+1). \tag{4·18}$$

Define $p_m(0) = 0$ and $p_m(j) = 0$ for $j > m$, and observe that $p_m(j)$ is uniquely determined by the initial condition $p_1(1) = -1$ and (4·18). Therefore to establish (4·17), it is sufficient to prove the following lemma.

LEMMA 4·3. *Let* $\alpha_{2m,\,j}$ *be defined by* (1·11) *and* (1·12). *Then*

$$\alpha_{2m+2,\,j+1} = 2j(2j+1)\alpha_{2m,\,j} + (j+1)^2\alpha_{2m,\,j+1}. \tag{4·19}$$

Proof. Using (3·8) twice, we find that

$$S(n,h) = S(n-2,h-2) + (2h-1)S(n-2,h-1) + h^2 S(n-2,h) \tag{4·20}$$

(see [**10**, p. 227]). Substituting (4·20) in the definition (3·22) of $\alpha_{2m+2,\,j+1}$ and rearranging the terms, we find that

$$\begin{aligned}
&\alpha_{2m+2,\,j+1}\\
&\quad = c_{j-1,0}(j+1)!\,(j+1)^2 S(2m-1,\,j+1)\\
&\qquad + \left\{c_{j-1,0}(j+1)!(2j+1) - c_{j-1,1}\,j!\,j^2\right\}S(2m-1,\,j)\\
&\qquad + \sum_{s=0}^{j-3}(-1)^s\left\{c_{j-1,s}(j+1-s)! - c_{j-1,s+1}(j-s)!\,(2j-2s-1)\right.\\
&\qquad\qquad \left.+\, c_{j-1,s+2}(j-1-s)!\,(j-1-s)^2\right\}S(2m-1,\,j-1-s)\\
&\qquad + (-1)^{j-2}\left\{c_{j-1,\,j-2}3! - c_{j-1,\,j-1}2!\cdot 3\right\}S(2m-1,1)\\
&\qquad + (-1)^{j-1}c_{j-1,\,j-1}2!\,S(2m-1,0).
\end{aligned} \tag{4·21}$$

On the other hand, from the definition (3·22) of $\alpha_{2m,\,j}$, the right-hand side of (4·19) can be transformed into

$$\begin{aligned}
&2j(2j+1)\alpha_{2m,\,j} + (j+1)^2\alpha_{2m,\,j+1}\\
&\quad = (j+1)^2(j+1)!\,S(2m-1,\,j+1)\\
&\qquad + \sum_{s=0}^{j-2}(-1)^s\left\{2j(2j+1)c_{j-2,s} - (j+1)^2 c_{j-1,s+1}\right\}\\
&\qquad\qquad\qquad \times (j-s)!\,S(2m-1,\,j-s).
\end{aligned} \tag{4·22}$$

We shall show that the coefficients of $S(2m-1,s)$ in (4·21) and (4·22) are identical.

First, observe that on the right-hand side of (4·22), the terms $S(2m-1,0)$ and $S(2m-1,1)$ are absent. On the right-hand side of (4·21), $S(2m-1,0)$ appears, but we recall that $S(2m-1,0) = 0$. Second, using the definition (1·7), we find that the coefficient of $S(2m-1,1)$ on the right-hand side of (4·21) equals 0. Since $c_{j-1,0} = 1$, the coefficients of $S(2m-1,\,j+1)$ in (4·21) and (4·22) are also the same. Furthermore, both of the coefficients of $S(2m-1,\,j)$ reduce to

$$j!\,(-j^3 + 3j^2 + 3j + 1).$$

Therefore it remains to show that

$$\begin{aligned}
&(j+1-s)(j-s)c_{j-1,s} - (j-s)(2j-2s-1)c_{j-1,s+1} + (j-1-s)^2 c_{j-1,s+2}\\
&\quad = -2j(2j+1)c_{j-2,s+1} + (j+1)^2 c_{j-1,s+2}
\end{aligned} \tag{4·23}$$

for $0 \leqslant s \leqslant j-3$. Substituting the easily proved identity $c_{j-2,s+1} = c_{j-1,s+1} - c_{j-1,s}$ into the first term on the right-hand side of (4·23) and dividing both sides by $c_{j-1,s+1}$, we find that (4·23) reduces to

$$\begin{aligned}\big((j+1-s)(j-s) - 2j(2j+1)\big)&\frac{c_{j-1,s}}{c_{j-1,s+1}} \\ + \big((j-1-s)^2 - (j+1)^2\big)&\frac{c_{j-1,s+2}}{c_{j-1,s+1}} \\ = (j-s)(2j-2s-1) - 2j(2j+1).& \qquad (4\cdot24)\end{aligned}$$

Now, by (1·7),

$$\frac{c_{j,h}}{c_{j,h-1}} = \frac{(j-h+1)(j+h)}{(j-h+2)h}.$$

Thus, the left-hand side of (4·24) can be written in the form

$$\begin{aligned}\big((j+1-s)(j-s) - 2j(2j+1)\big)&\frac{(j-s)(s+1)}{(j-s-1)(j+s)} \\ + \big((j-1-s)^2 - (j+1)^2\big)&\frac{(j-s-2)(j+s+1)}{(j-s-1)(s+2)} \\ = -2j^2 - (3+4s)j + 2s^2 + s& \\ = (j-s)(2j-2s-1) - 2j(2j+1),&\end{aligned}$$

which establishes (4·24). Hence, the identity (4·23) has been proved for $0 \leqslant s \leqslant j-3$, and so the proof of (4·19) is complete.

With this lemma, we also complete the proof of Proposition 4·2.

5. *Evaluations using the theory of periodic zeta functions*

We now offer another approach to (1·1), (1·2), and their generalizations through the theory of periodic zeta functions developed in [**5**]. For the sake of completeness, we review the necessary terminology from [**5**].

First, write

$$R_{2a}(r) = \sum_{n=1}^{\infty} \frac{\cos(\pi n^2/a)}{n^r} + i \sum_{n=1}^{\infty} \frac{\sin(\pi n^2/a)}{n^r} =: S_a(r) + iT_a(r), \qquad (5\cdot1)$$

where r and a are even positive integers. In order to effect these evaluations, we need to introduce periodic Bernoulli numbers.

DEFINITION 5·1. Let $\mathfrak{A} = \{a_n\}$, $-\infty < n < \infty$, denote a sequence of numbers with period k. Then the periodic Bernoulli numbers $B_n(\mathfrak{A})$, $n \geqslant 0$, can be defined by [**5**, p. 55, proposition 9·1], for $|z| < 2\pi/k$,

$$\frac{z\sum_{n=0}^{k-1} a_n e^{nz}}{e^{kz} - 1} = \sum_{n=0}^{\infty} \frac{B_n(\mathfrak{A})}{n!} z^n.$$

Furthermore [**5**, p. 56, equation (9·5)], for each positive integer n,

$$B_n(\mathfrak{A}) = k^{n-1} \sum_{j=0}^{k-1} a_{-j} B_n\left(\frac{j}{k}\right), \qquad (5\cdot2)$$

where $B_n(x), n \geqslant 0$, denotes the nth Bernoulli polynomial. We say that $\mathfrak{A}$ is **even** if $a_n = a_{-n}$ for every integer n. The complementary sequence $\mathfrak{B} = \{b_n\}$, $-\infty < n < \infty$, is defined by [**5**, p. 32]

$$b_n = \frac{1}{k}\sum_{j=0}^{k-1} a_j e^{-2\pi i jn/k}. \tag{5·3}$$

It is easily checked that if $\mathfrak{A}$ is even, then $\mathfrak{B}$ is even, and that (5·3) holds if and only if

$$a_n = \sum_{j=0}^{k-1} b_j e^{2\pi i jn/k}, \qquad -\infty < n < \infty. \tag{5·4}$$

Now set

$$\zeta(s; \mathfrak{A}) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad \operatorname{Re} s > 1.$$

If $\mathfrak{A}$ and r are even and if $r \geqslant 2$, then [**5**, p. 49, equation (6·25)]

$$\zeta(r; \mathfrak{B}) = \frac{(-1)^{r+1} B_r(\mathfrak{A})}{2\,r!}\left(\frac{2\pi i}{k}\right)^r.$$

From (5·3) and (5·4), we see that the sequences $\mathfrak{A}$ and $\mathfrak{B}$ are not symmetric. Thus, we note from above that, since $\mathfrak{A}$ is even,

$$\zeta(r; \mathfrak{A}) = \frac{(-1)^{r+1} B_r(\mathfrak{B})k}{2\,r!}\left(\frac{2\pi i}{k}\right)^r. \tag{5·5}$$

We are now ready to state general evaluations in closed form for $S_a(r)$ and $T_a(r)$.

THEOREM 5·1. *If $S_a(r)$ and $T_a(r)$ are defined by* (5·1) *and if r and a are even positive integers, then*

$$S_a(r) = \frac{(-1)^{1+r/2}2^{r-1}\pi^r}{r!\sqrt{a}} \sum_{m=0}^{a-1} B_r\left(\frac{m}{a}\right)\sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) \tag{5·6}$$

and

$$T_a(r) = \frac{(-1)^{1+r/2}2^{r-1}\pi^r}{r!\sqrt{a}} \sum_{m=0}^{a-1} B_r\left(\frac{m}{a}\right)\cos\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right). \tag{5·7}$$

In our work below, we need the value of the Gauss sum [**4**, p. 43, exercise 5; p. 15, corollary 1·2·3]

$$\sum_{n=0}^{c-1} e^{\pi i n^2/c} = e^{\pi i/4}\sqrt{c}, \tag{5·8}$$

where c is an even positive integer.

Before proceeding further, we show that (1·1) and (1·2) are special cases of (5·6) and (5·7), respectively. Let $r = 2$ in Theorem 5·1, and recall that $B_2(x) = x^2 - x + 1/6$. Then

$$\begin{aligned} S_a(2) &= \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left(\frac{m}{a}\right)^2 - \frac{m}{a} + \frac{1}{6} \right\} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) \\ &= \frac{\pi^2}{6\sqrt{a}} \sum_{m=0}^{a-1} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) + \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left(\frac{m}{a}\right)^2 - \frac{m}{a} \right\} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) \\ &= \frac{\pi^2}{6} + \frac{\pi^2}{\sqrt{a}} \sum_{m=0}^{a-1} \left\{ \left(\frac{m}{a}\right)^2 - \frac{m}{a} \right\} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right), \end{aligned}$$

upon the use of (5·8) twice.

The proof of (1·2) follows along the same lines, but note that in this case, by (5·8),

$$\sum_{m=0}^{a-1} \cos\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) = 0.$$

Proof of Theorem 5·1. Let

$$a_n = \cos\left(\frac{\pi n^2}{a}\right), \qquad -\infty < n < \infty,$$

which is an even periodic sequence with period a, since a is even. Then, from (5·3) and (5·8),

$$\begin{aligned} b_{-m} &= \frac{1}{a} \sum_{j=0}^{a-1} \cos\left(\frac{\pi j^2}{a}\right) e^{2\pi i j m/a} \\ &= \frac{1}{2a} e^{-\pi i m^2/a} \sum_{j=0}^{a-1} e^{\pi i (j+m)^2/a} + \frac{1}{2a} e^{\pi i m^2/a} \sum_{j=0}^{a-1} e^{-\pi i (j+m)^2/a} \\ &= \frac{1}{2a} e^{-\pi i m^2/a} \sum_{j=0}^{a-1} e^{\pi i j^2/a} + \frac{1}{2a} e^{\pi i m^2/a} \sum_{j=0}^{a-1} e^{-\pi i j^2/a} \\ &= \frac{1}{2a} e^{-\pi i m^2/a + \pi i/4} \sqrt{a} + \frac{1}{2a} e^{\pi i m^2/a - \pi i/4} \sqrt{a} \\ &= \frac{1}{\sqrt{a}} \cos\left(\frac{\pi m^2}{a} - \frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{a}} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right). \end{aligned}$$

Therefore, by (5·2), with $\mathfrak{B}$ in place of $\mathfrak{A}$,

$$B_n(\mathfrak{B}) = a^{n-3/2} \sum_{m=0}^{a-1} \sin\left(\frac{\pi m^2}{a} + \frac{\pi}{4}\right) B_n\left(\frac{m}{a}\right). \tag{5·9}$$

If we substitute (5·9) into (5·5) and simplify, we deduce (5·6).

The proof of (5·7) is analogous to that for (5·6). In this case we set

$$a_n = \sin\left(\frac{\pi n^2}{a}\right), \qquad -\infty < n < \infty,$$

which of course is even, and repeat the same kind of argument that we gave above.

In conclusion, our attempts to establish Ramanujan's original evaluations (1·1) and (1·2) and their generalizations in Theorems 1·1, 1·2, 1·3, 4·1 and 5·1 have given us various representations for these sums in terms of ballot numbers, Stirling numbers of the second kind, Bernoulli numbers and polynomials, and trigonometric functions. Equating different evaluations provide identities that would be surprising if we had not known of their origins. For example, let us return to the case $R_{2a}(2)$. Combining (1·1) and (1·2) with (4·11) and (4·12) or (4·9) and (4·10) with $m = 1$, respectively, we deduce the identities

$$\frac{\pi^2}{6a^2} + \frac{\pi^2\cos(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2}\sum_{j=1}^{\frac{1}{2}a-1}\cos\left(\frac{\pi j^2}{a}\right)\csc^2\left(\frac{\pi j}{a}\right) = \frac{\pi^2}{6} - \frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\sin\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right)$$

and

$$\frac{\pi^2\sin(\pi a/4)}{2a^2} + \frac{\pi^2}{a^2}\sum_{j=1}^{\frac{1}{2}a-1}\sin\left(\frac{\pi j^2}{a}\right)\csc^2\left(\frac{\pi j}{a}\right) = -\frac{\pi^2}{\sqrt{a}}\sum_{r=1}^{a}\frac{r}{a}\left(1-\frac{r}{a}\right)\cos\left(\frac{\pi}{4}+\frac{\pi r^2}{a}\right).$$

Note that on the left-hand sides above, the sums contain only trigonometric functions, while on the right-hand sides the sums contain both polynomials and trigonometric functions. Trigonometric identities involving polynomials in the summands appear to be rare. The sums on both sides of the identities may be regarded as new analogues of Gauss sums.

We record a few examples to illustrate our evaluations, namely,

$$S_2(2) = \frac{\pi^2}{24}, \qquad S_4(2) = -\frac{\pi^2}{48} + \frac{\pi^2\sqrt{2}}{16}, \qquad S_6(2) = -\frac{\pi^2}{72} + \frac{\pi^2\sqrt{3}}{18},$$

$$T_2(2) = \frac{\pi^2}{8}, \qquad T_4(2) = \frac{\pi^2\sqrt{2}}{16}, \qquad T_6(2) = \frac{\pi^2}{24} + \frac{\pi^2\sqrt{3}}{54}.$$

Lastly, we remark that (1·1) and (1·2) are discussed in [**2**], but considerably less thoroughly than in this paper. Related results are also examined in [**2**].

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