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Explicit evaluations of the Rogers-Ramanujan continued fraction

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§ 1. Introduction

Let, for $|q| < 1$,

$$F(q) = \frac{q^{\frac{1}{5}}}{1} \, \frac{q}{+1} \, \frac{q^2}{+1} \, \frac{q^3}{+1} + \cdots$$

denote the famous Rogers-Ramanujan continued fraction, and let $S(q) = -F(-q)$. In his first letter to G.H. Hardy, Ramanujan asserted that

$$F(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}, \tag{1.1}$$

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}, \tag{1.2}$$

and

(1.3) $F(e^{-\pi\sqrt{n}})$ *can be exactly found if n be any positive rational quantity.*

Identities (1.1) and (1.2) were first proved by G.N. Watson [15]. Watson vaguely discussed (1.3) and merely claimed that $F(e^{-\pi\sqrt{n}})$ is an algebraic number.

The first attempt to find a "uniform" method to evaluate $F(q)$ was made by K.G. Ramanathan [9]. By studying the ideal class groups of imaginary quadratic fields with the property that each genus contains a single class, Ramanathan was able to compute $F(e^{-\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for several rational numbers n using Kronecker's limit formula. In particular, he showed that

$$F(e^{-\pi\sqrt{2/5}}) = \left(-(18+5\sqrt{5}) + \sqrt{90(5+2\sqrt{5})}\right)^{1/5}$$

and

$$S(e^{-\pi\sqrt{3/5}}) = \left(\frac{-(3+5\sqrt{5})+\sqrt{30(5+\sqrt{5})}}{4}\right)^{1/5}.$$

Unfortunately, his method did not apply to the following continued fraction stated by Ramanujan [12], vol. 1, p. 311:

$$F(e^{-6\pi}) = \sqrt{c^2+1} - c, \tag{1.4}$$

where

$$2c = \frac{60^{1/4}+2-\sqrt{3}+\sqrt{5}}{60^{1/4}-2+\sqrt{3}-\sqrt{5}}\sqrt{5}+1.$$

Identity (1.4) was first proved by B.C. Berndt and H.H. Chan [2], but their proof is somewhat complicated. In an attempt to simplify the proof of (1.4), we discovered two theorems for evaluating $F(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ in terms of Ramanujan-Weber class invariants defined at the end of this introduction. These results will be proved in Section 2.

In Section 3, we will present a new proof of (1.4) and establish its companion continued fraction

$$S(e^{-3\pi}) = \sqrt{c_1^2+1} - c_1, \tag{1.5}$$

where

$$2c_1 = \frac{60^{1/4}+2+\sqrt{3}-\sqrt{5}}{60^{1/4}-2-\sqrt{3}+\sqrt{5}}\sqrt{5}-1.$$

In Section 4, we will discuss page 210 of Ramanujan's Lost Notebook, which contains evidence that Ramanujan had attempted to evaluate $S(e^{-\pi/\sqrt{5n}})$ and $S(e^{-\pi\sqrt{n/5}})$ for odd numbers between 1 and 15. In Section 5, we state a result which will give evaluations associated with imaginary quadratic fields that are not discussed in Ramanathan's work [9].

In the final section, we use a result in Stark's paper [14] to show that $F(e^{-\pi\sqrt{n}})$ is a unit when n is rational. This is clearly a stronger statement than that of Watson.

We complete this introduction with definitions of certain functions which will be used in the sequel. For $|q| < 1$, let

$$(a;q)_\infty := \prod_{k=1}^{\infty} (1-aq^{k-1}),$$

$$f(-q) := (q;q)_\infty, \tag{1.6}$$

$$\varphi(q) := \sum_{k=-\infty}^{\infty} q^{k^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{1.7}$$

(1.8) $$\chi(-q) = (q;q^2)_\infty ,$$

(1.9) $$G_n := 2^{-1/4} e^{\pi\sqrt{n}/24} \chi(e^{-\pi\sqrt{n}}),$$

and

(1.10) $$g_n := 2^{-1/4} e^{\pi\sqrt{n}/24} \chi(-e^{-\pi\sqrt{n}}).$$

We shall refer to G_n and g_n as the *Ramanujan-Weber class invariants*. We also use modular equations in the sequel, and refer to [1], pp. 213, 214, for this terminology.

§ 2. Explicit formulas for $F(q)$ and $S(q)$

It was stated by Ramanujan [1], p. 85, and proved by Watson [15] that

(2.1) $$F^{-1}(q) - 1 - F(q) = q^{-1/5} \frac{f(-q^{1/5})}{f(-q^5)}.$$

Replacing q by $-q$, we have

(2.2) $$S^{-1}(q) + 1 - S(q) = q^{-1/5} \frac{f(q^{1/5})}{f(q^5)}.$$

Consequently, in order to compute $F(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$, it suffices to evaluate

(2.3) $$A := e^{2\pi\sqrt{n}/5} \frac{f(-e^{-2\pi\sqrt{n}/5})}{f(-e^{-10\pi\sqrt{n}})}$$

and

(2.4) $$A_1 := e^{\pi\sqrt{n}/5} \frac{f(e^{-\pi\sqrt{n}/5})}{f(e^{-5\pi\sqrt{n}})},$$

respectively.

Theorem 2.1. *Let*

(2.5) $$V := \sqrt{\frac{G_{25n}}{G_{n/25}}}$$

and

(2.6) $$U := \sqrt{\frac{g_{25n}}{g_{n/25}}}.$$

(i) *If A is defined by* (2.3), *then*

(2.7) $$\frac{A}{\sqrt{5}V} - \frac{\sqrt{5}V}{A} = (V - V^{-1})^2 \left(\frac{V - V^{-1}}{\sqrt{5}} + \frac{\sqrt{5}}{V - V^{-1}} \right)$$

and

$$(2.8)\qquad \frac{A}{\sqrt{5}\,U}+\frac{\sqrt{5}\,U}{A}=(U+U^{-1})^2\left(\frac{U+U^{-1}}{\sqrt{5}}-\frac{\sqrt{5}}{U+U^{-1}}\right).$$

(ii) *If A_1 is defined by* (2.4), *then*

$$(2.9)\qquad \frac{A_1 V}{\sqrt{5}}-\frac{\sqrt{5}}{A_1 V}=(V-V^{-1})^2\left(\frac{V-V^{-1}}{\sqrt{5}}+\frac{\sqrt{5}}{V-V^{-1}}\right).$$

Proof. Let

$$q^{1/5}=\exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}\right)$$

and

$$q^5=\exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)}\right),$$

so that β is of degree 25 over α. Then ([1], p. 291, Entry 15(i), (ii))

$$(2.10)\qquad \left(\frac{\beta}{\alpha}\right)^{1/8}+\left(\frac{1-\beta}{1-\alpha}\right)^{1/8}-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}-2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}=\sqrt{mm'}$$

and

$$(2.11)\qquad \left(\frac{\alpha}{\beta}\right)^{1/8}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}-2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}=\frac{5}{\sqrt{mm'}},$$

where $\sqrt{mm'}=\dfrac{\varphi(q^{1/5})}{\varphi(q^5)}$. From (2.10) and (2.11), we deduce that, respectively,

$$(2.12)\qquad \frac{(\beta(1-\alpha))^{1/8}+(\alpha(1-\beta))^{1/8}}{(\alpha(1-\alpha))^{1/8}}=\sqrt{mm'}+\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}+2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}$$

and

$$(2.13)\qquad \frac{(\beta(1-\alpha))^{1/8}+(\alpha(1-\beta))^{1/8}}{(\beta(1-\beta))^{1/8}}=\frac{5}{\sqrt{mm'}}+\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}+2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}.$$

Eliminating $(\beta(1-\alpha))^{1/8}+(\alpha(1-\beta))^{1/8}$ from (2.12) and (2.13), we have

$$(2.14)\qquad \sqrt{mm'}+\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}+2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}$$
$$=\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left(\frac{5}{\sqrt{mm'}}+\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}+2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}\right).$$

From [1], p. 124, Entry 12(v), we have

(2.15) $$q^{-1/120}\chi(q^{1/5}) = 2^{1/6}\{\alpha(1-\alpha)\}^{-1/24}$$

and

(2.16) $$q^{-5/24}\chi(q^5) = 2^{1/6}\{\beta(1-\beta)\}^{-1/24}.$$

Hence, we can rewrite (2.14) as

(2.17) $$q^{-2/5}\frac{\varphi(q^{1/5})}{\varphi(q^5)}\left(\frac{\chi(q^5)}{\chi(q^{1/5})}\right)^2 + q^{1/5}\frac{\chi(q^{1/5})}{\chi(q^5)} + 2$$
$$= q^{-1/5}\frac{\chi(q^5)}{\chi(q^{1/5})}\left(5q^{2/5}\frac{\varphi(q^5)}{\varphi(q^{1/5})}\left(\frac{\chi(q^{1/5})}{\chi(q^5)}\right)^2 + q^{-1/5}\frac{\chi(q^5)}{\chi(q^{1/5})} + 2\right).$$

From the product representations of $f(-q)$, $\varphi(q)$, and $\chi(q)$ given in (1.6), (1.7), and (1.8), we have

(2.18) $$q^{-2/5}\frac{f(-q^{2/5})}{f(-q^{10})} = q^{-2/5}\frac{\varphi(q^{1/5})}{\varphi(q^5)}\left(\frac{\chi(q^5)}{\chi(q^{1/5})}\right)^2.$$

Substituting (2.18) into (2.17), and setting $q = e^{-\pi\sqrt{n}}$, we deduce that

(2.19) $$A + V^{-2} + 2 = 5A^{-1}V^2 + V^4 + 2V^2,$$

by (1.9), (2.3) and (2.5). Rearranging (2.19), we obtain (2.7).

To show (2.8), we first replace q by $-q$ in (2.17) and (2.18). Next, set $q = e^{-\pi\sqrt{n}}$. By (1.10) and (2.3), we have

(2.20) $$A - U^{-2} + 2 = -5A^{-1}U^2 + U^4 - 2U^2.$$

Rearranging (2.20), we deduce (2.8).

In order to show (2.9), we first observe that from (2.3)–(2.5),

(2.21) $$A_1 = e^{\pi\sqrt{n}/5}\frac{f(e^{-\pi\sqrt{n}/5})}{f(e^{-5\pi\sqrt{n}})} = e^{2\pi\sqrt{n}/5}\frac{f(-e^{-2\pi\sqrt{n}/5})}{f(-e^{-10\pi\sqrt{n}})}V^{-2}$$
$$= AV^{-2}.$$

Substituting (2.21) into (2.7), we arrive at (2.9). This completes the proof of Theorem 2.1.

Proposition 2.2. *Let A and A_1 be defined by* (2.3) *and* (2.4), *respectively. Then*

(i) *if* $2c = A + 1$, *then* $F(e^{-2\pi\sqrt{n}}) = \sqrt{c^2+1} - c$,

(ii) *if* $2c = A_1 - 1$, *then* $S(e^{-\pi\sqrt{n}}) = \sqrt{c^2+1} - c$.

Proof. Solve the quadratic equations (2.1) and (2.2). This proves the proposition.

We next turn to the following relation discovered by Ramanujan [1], p. 267, and proved by Watson [15], namely,

$$F^{-5}(q) - 11 - F^{5}(q) = \frac{f^{6}(-q)}{q f^{6}(-q^{5})}. \tag{2.22}$$

Again, replacing q by $-q$, we have

$$S^{-5}(q) + 11 - S^{5}(q) = \frac{f^{6}(q)}{q f^{6}(q^{5})}. \tag{2.23}$$

Hence, in order to compute $F(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$, it suffices to evaluate

$$A' := e^{2\pi\sqrt{n}/6} \frac{f(-e^{-2\pi\sqrt{n}})}{f(-e^{-10\pi\sqrt{n}})} \tag{2.24}$$

and

$$A_1' := e^{\pi\sqrt{n}/6} \frac{f(e^{-\pi\sqrt{n}})}{f(e^{-5\pi\sqrt{n}})}, \tag{2.25}$$

respectively.

Theorem 2.3. *Let*

$$V' = \frac{G_{25n}}{G_n} \tag{2.26}$$

and

$$U' = \frac{g_{25n}}{g_n}. \tag{2.27}$$

(i) *If* A' *is defined by* (2.24), *then*

$$\frac{A'^2}{\sqrt{5}V'} - \frac{\sqrt{5}V'}{A'^2} = \frac{1}{\sqrt{5}}(V'^3 - V'^{-3}) \tag{2.28}$$

and

$$\frac{A'^2}{\sqrt{5}U'} + \frac{\sqrt{5}U'}{A'^2} = \frac{1}{\sqrt{5}}(U'^3 + U'^{-3}). \tag{2.29}$$

(ii) *If* A_1' *is defined by* (2.25), *then*

$$\frac{A_1'^2 V'}{\sqrt{5}} - \frac{\sqrt{5}}{A_1'^2 V'} = \frac{1}{\sqrt{5}}(V'^3 - V'^{-3}). \tag{2.30}$$

Proof. Let

$$q = \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}\right)$$

and

$$q^5 = \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \beta)} \right),$$

so that β is of degree 5 over α. Then ([1], pp. 281, 282, Entry 13 (xii))

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \tag{2.31}$$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}, \tag{2.32}$$

where $m = \dfrac{\varphi^2(q)}{\varphi^2(q^5)}$. From (2.31) and (2.32), we find that

$$(\alpha(1-\alpha))^{1/4}\left(m + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}\right) = (\beta(1-\beta))^{1/4}\left(\frac{5}{m} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}\right). \tag{2.33}$$

Using (2.15) (with $q^{1/5}$ replaced by q), (2.16), and the equality

$$\frac{f(-q^2)}{f(-q^{10})} = \frac{\varphi(q)}{\varphi(q^5)}\left(\frac{\chi(q^5)}{\chi(q)}\right)^2,$$

we can rewrite (2.33) in the form

$$\left(\frac{f(-q^2)}{f(-q^{10})}\right)^2 \left(\frac{\chi(q)}{\chi(q^5)}\right)^4 - 5q\left(\frac{f(-q^{10})}{f(-q^2)}\right)^2 \left(\frac{\chi(q)}{\chi(q^5)}\right)^2 = 1 - q\left(\frac{\chi(q)}{\chi(q^5)}\right)^6. \tag{2.34}$$

Next, set $q = e^{-\pi\sqrt{n}}$. By (2.34), (1.9), (2.24), and (2.26), we have

$$\frac{A'^2}{V'} - 5\frac{V'}{A'^2} = V'^3 - V'^{-3}.$$

Rearranging, we deduce (2.28).

To show (2.29), we simply replace q by $-q$ in (2.34) and set $q = e^{-\pi\sqrt{n}}$. By (1.10), (2.24), and (2.27), we have

$$\frac{A'^2}{U'} + 5\frac{U'}{A'^2} = U'^3 + U'^{-3},$$

which gives (2.29) after rearrangement.

Finally, we observe that by (2.24), (2.25), (2.26), and (1.9),

$$A_1' = \frac{A'}{V'}. \tag{2.35}$$

Substituting (2.35) into (2.28), we deduce (2.30). This completes the proof of Theorem 2.3.

Proposition 2.4. *Let A' and A_1' be defined by* (2.24) *and* (2.25) *respectively. Then*

(i) *if* $2c = A'^6 + 11$, *then* $F^5(e^{-2\pi\sqrt{n}}) = \sqrt{c^2+1} - c$,

(ii) *if* $2c = A_1'^6 - 11$, *then* $S^5(e^{-\pi\sqrt{n}}) = \sqrt{c^2+1} - c$.

Proof. Solve the quadratic equations (2.22) and (2.23).

§ 3. Evaluations of $F(e^{-6\pi})$ and $S(e^{-3\pi})$

We first restate and prove (1.4).

Corollary 3.1. *Let* $a = 60^{1/4}$ *and* $b = 2 - \sqrt{3} + \sqrt{5}$. *If* $2c = \frac{a+b}{a-b}\sqrt{5} + 1$, *then*

$$F(e^{-6\pi}) = \sqrt{c^2+1} - c\,.$$

Proof. We first recall that ([11])

$$G_n = G_{1/n}\,. \tag{3.1}$$

By Theorem 2.1, it suffices to evaluate

$$V = \sqrt{\frac{G_{225}}{G_{9/25}}} = \sqrt{\frac{G_{225}}{G_{25/9}}}\,.$$

From [3], Theorem 1, we have

$$G_{9n} = G_n\left(p + \sqrt{p^2-1}\right)^{1/6} \tag{3.2}$$

$$\times\left\{\sqrt{\frac{p^2-2+\sqrt{(p^2-1)(p^2-4)}}{2}} + \sqrt{\frac{p^2-4+\sqrt{(p^2-1)(p^2-4)}}{2}}\right\}^{1/3},$$

where

$$p = G_n^4 + G_n^{-4}\,.$$

Since ([17], p. 722)

$$G_{25} = \frac{\sqrt{5}+1}{2},$$

we find that $p = 7$ and so deduce that

$$G_{225} = G_{25}(7+\sqrt{48})^{1/6}\left(\sqrt{\frac{47+12\sqrt{15}}{2}} + \sqrt{\frac{45+12\sqrt{15}}{2}}\right)^{1/3}, \tag{3.3}$$

by (3.2).

If we examine the proof of (3.2) carefully (see [3]), we can deduce a corresponding relation between $G_{n/9}$ and G_n, namely,

$$G_{n/9} = G_n(p+\sqrt{p^2-1})^{1/6} \times \left\{\sqrt{\frac{p^2-2+\sqrt{(p^2-1)(p^2-4)}}{2}} - \sqrt{\frac{p^2-4+\sqrt{(p^2-1)(p^2-4)}}{2}}\right\}^{1/3}. \tag{3.4}$$

Substituting $n = 25$ and $p = 7$ into (3.4) gives

$$G_{25/9} = G_{25}(7+\sqrt{48})^{1/6}\left(\sqrt{\frac{47+12\sqrt{15}}{2}} + \sqrt{\frac{45+12\sqrt{15}}{2}}\right)^{1/3}. \tag{3.5}$$

Dividing (3.3) by (3.5) yields

$$V^2 = \left(\frac{\sqrt{47+12\sqrt{15}}+\sqrt{45+12\sqrt{15}}}{\sqrt{47+12\sqrt{15}}-\sqrt{45+12\sqrt{15}}}\right)^{1/3} = \left(\sqrt{\frac{47+12\sqrt{15}}{2}} + \sqrt{\frac{45+12\sqrt{15}}{2}}\right)^{2/3}.$$

Hence,

$$V^3 = \sqrt{\frac{47+12\sqrt{15}}{2}} + \sqrt{\frac{45+12\sqrt{15}}{2}}$$

and

$$V^3 - V^{-3} = \sqrt{90+24\sqrt{15}} = 15^{3/4} + 3\cdot 15^{1/4}. \tag{3.6}$$

Since $V^3 - V^{-3} = (V-V^{-1})^3 + 3(V-V^{-1})$, we deduce from (3.6) that

$$V - V^{-1} = 15^{1/4}. \tag{3.7}$$

Therefore,

$$V = \frac{15^{1/4}}{2} + \sqrt{\frac{\sqrt{15}+4}{4}}. \tag{3.8}$$

Substituting (3.7) and (3.8) into Theorem 2.1 (i) gives

$$\frac{A}{\sqrt{5}V} - \frac{\sqrt{5}V}{A} = 15^{1/4}(\sqrt{3}+\sqrt{5}). \tag{3.9}$$

Solving for A, we find that

$$A = \frac{1}{2}\left(15^{1/4} + \sqrt{\sqrt{15}+4}\right)\left(\sqrt{\frac{8\sqrt{15}+34}{4}} + \sqrt{\frac{8\sqrt{15}+30}{4}}\right)\sqrt{5}.$$

To show Corollary 3.1, it suffices, by Proposition 2.2, to verify that

$$(3.10) \qquad \frac{1}{2}\left(15^{1/4} + \sqrt{\sqrt{15}+4}\right)\left(\sqrt{\frac{8\sqrt{15}+34}{4}} + \sqrt{\frac{8\sqrt{15}+30}{4}}\right) = \frac{60^{1/4}+2-\sqrt{3}+\sqrt{5}}{60^{1/4}-2+\sqrt{3}-\sqrt{5}}.$$

Before establishing (3.10), we first make a simple observation. When α is a unit, it can be written in the form $\sqrt{\gamma+1} \pm \sqrt{\gamma}$. Since $\sqrt{\alpha}$ is also a unit, it is of the form $\sqrt{\delta+1} \pm \sqrt{\delta}$. Therefore,

$$(3.11) \qquad \alpha = (\sqrt{\alpha})^2 = (\sqrt{\delta+1} \pm \sqrt{\delta})^2 = \frac{\sqrt{\delta+1} \pm \sqrt{\delta}}{\sqrt{\delta+1} \mp \sqrt{\delta}}.$$

This observation allows us to express a unit in the form $\dfrac{u+v}{u-v}$.

Now, let

$$\alpha = \frac{15^{1/4} + \sqrt{\sqrt{15}+4}}{2}.$$

Then

$$\alpha + \alpha^{-1} = \sqrt{\sqrt{15}+4}.$$

This implies that

$$(3.12) \qquad (\sqrt{\alpha} + \sqrt{\alpha^{-1}})^2 = \sqrt{\sqrt{15}+4} + 2$$

and

$$(3.13) \qquad (\sqrt{\alpha} - \sqrt{\alpha^{-1}})^2 = \sqrt{\sqrt{15}+4} - 2.$$

Since

$$\sqrt{\frac{2\sqrt{15}+8}{2}} = \frac{\sqrt{3}+\sqrt{5}}{\sqrt{2}},$$

we may simplify (3.12) and (3.13) to obtain

$$\sqrt{\alpha} + \sqrt{\alpha^{-1}} = \sqrt{\frac{\sqrt{6}+\sqrt{10}+4}{2}}$$

and

$$\sqrt{\alpha}-\sqrt{\alpha^{-1}}=\sqrt{\frac{\sqrt{6}+\sqrt{10}-4}{2}}.$$

Hence

$$\sqrt{\alpha}=\sqrt{\frac{\sqrt{6}+\sqrt{10}+4}{8}}+\sqrt{\frac{\sqrt{6}+\sqrt{10}-4}{8}}.$$

From the discussion leading to (3.11), we conclude that

$$\alpha=\frac{\sqrt{\sqrt{6}+\sqrt{10}+4}+\sqrt{\sqrt{6}+\sqrt{10}-4}}{\sqrt{\sqrt{6}+\sqrt{10}+4}-\sqrt{\sqrt{6}+\sqrt{10}-4}} \tag{3.14}$$
$$=\frac{\sqrt{(\sqrt{6}+\sqrt{10})^2-16}+\sqrt{6}+\sqrt{10}-4}{\sqrt{(\sqrt{6}+\sqrt{10})^2-16}-\sqrt{6}-\sqrt{10}+4}$$
$$=\frac{60^{1/4}+\sqrt{3}+\sqrt{5}-2\sqrt{2}}{60^{1/4}-\sqrt{3}-\sqrt{5}+2\sqrt{2}}.$$

In a similar way, we find that

$$\sqrt{\frac{8\sqrt{15}+34}{4}}+\sqrt{\frac{8\sqrt{15}+30}{4}}=\frac{60^{1/4}+\sqrt{15}-\sqrt{10}+\sqrt{6}-1}{60^{1/4}-\sqrt{15}+\sqrt{10}-\sqrt{6}+1}. \tag{3.15}$$

Combining (3.14) and (3.15) and simplifying, we obtain (3.10), and the proof is complete.

Remark. The value V established in the proof of Corollary 3.1 can also be deduced from the values of G_{225} and $G_{25/9}$ found in [5], p. 149.

Corollary 3.2. *Let* $a=60^{1/4}$ *and* $b=2+\sqrt{3}-\sqrt{5}$. *If* $2c=\frac{a+b}{a-b}\sqrt{5}-1$, *then*

$$S(e^{-3\pi})=\sqrt{c^2+1}-c.$$

Proof. From Theorem 2.1 (ii) and the right hand side of (3.9), we find that

$$\frac{A_1V}{\sqrt{5}}-\frac{\sqrt{5}}{A_1V}=15^{1/4}(\sqrt{3}+\sqrt{5}).$$

Thus,

$$A_1=\frac{1}{2}\left(\sqrt{\sqrt{15}+4}-15^{1/4}\right)\left(\sqrt{\frac{8\sqrt{15}+34}{4}}+\sqrt{\frac{8\sqrt{15}+30}{4}}\right)\sqrt{5}$$
$$=\frac{60^{1/4}+2+\sqrt{3}-\sqrt{5}}{60^{1/4}-2-\sqrt{3}+\sqrt{5}}\sqrt{5},$$

by the same argument as in the previous proof. By Corollary 2.2 (ii), we complete the proof of Corollary 3.2.

Remarks. Note that (1.4) and (1.5) are truly companions. In fact, from Theorem 2.1, we observe that the values of $F(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ are similar since they both arise from the solutions of the same quadratic equation. Identities (1.1) and (1.2) are further examples of the observation above.

§ 4. Page 210 of Ramanujan's Lost Notebook

On page 210 of his Lost Notebook [13], Ramanujan defined $S(q)$ and constructed a table of values for $S(e^{-\pi\sqrt{n/5}})$ and $S(e^{-\pi/\sqrt{5n}})$ for odd integers n between 1 and 15. The table is incomplete and only three values are given, namely, $S(e^{-\pi/\sqrt{5}})$, $S(e^{-\pi/\sqrt{35}})$, and $S(e^{-\pi\sqrt{7/5}})$. Our aim in this section is to complete the table using Theorem 2.3 and some results proved in [4].

We first recall the relation [7]

$$\left\{\left(\frac{\sqrt{5}-1}{2}\right)^5+S^5(e^{-\pi\sqrt{n/5}})\right\}\left\{\left(\frac{\sqrt{5}-1}{2}\right)^5+S^5(e^{-\pi/\sqrt{5n}})\right\} = 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5. \tag{4.1}$$

In view of (4.1), it suffices to evaluate either $S(e^{-\pi\sqrt{n/5}})$ or $S(e^{-\pi/\sqrt{5n}})$. We choose to compute the former.

Corollary 4.1.

$$S^5(e^{-\pi/\sqrt{5}}) = \sqrt{\left(\frac{5\sqrt{5}-11}{2}\right)^2+1}-\frac{5\sqrt{5}-11}{2}.$$

Proof. Let $n=1/5$ in (2.26). Then

$$V' = \frac{G_5}{G_{1/5}} = 1 .$$

By Theorem 2.3 (ii), we have

$$A_1'^2 = \sqrt{5}. \tag{4.2}$$

We complete the proof upon substituting (4.2) into Proposition 2.4 (ii).

Remark. Corollary 4.1 was first proved by Ramanathan [8].

Corollary 4.2.

$$S^5(e^{-\pi\sqrt{3/5}}) = \frac{-5\sqrt{5}-3+\sqrt{30(5+\sqrt{5})}}{4}.$$

Proof. Let $n=3/5$ in Theorem 2.3 (ii). From Weber's table [17], p. 721, we have

$$G_{15} = 2^{-1/12}(1 + \sqrt{5})^{1/3}.$$

Using a modular equation of degree 5 ([1], p. 282, Entry 13 (xiv)), we deduce that

$$G_{3/5} = 2^{-1/12}(\sqrt{5} - 1)^{1/3}.$$

Hence,

$$V' = \left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)^{1/3} \tag{4.3}$$

and

$$V'^3 - V'^{-3} = 1. \tag{4.4}$$

Substituting (4.3) and (4.4) into (2.30) and solving for A_1', we deduce that

$$A_1'^6 = \frac{25 + 5\sqrt{5}}{2}.$$

We may now complete the proof using Proposition 2.4 (ii).

Remarks. Corollary 4.2 was first proved by Ramanathan [9]. For a proof of this corollary using modular equations, see [2].

Theorem 4.3.

$$S^5(e^{-\pi\sqrt{7/5}}) = -5\sqrt{5} - 7 + \sqrt{35(5 + 2\sqrt{5})}.$$

Theorem 4.3 does not follow from Theorem 2.3 since we do not have a simple expression for $G_{35}/G_{7/5}$. This result is stated here for completeness. For the proofs of Theorem 4.3, see [2] or [9].

Corollary 4.4. *Let* $a = 2\sqrt{15}$ *and* $b = 3\sqrt{5} - 1$. *If* $2c = \dfrac{a+b}{a-b} 5\sqrt{5} - 11$, *then*

$$S^5(e^{-\pi\sqrt{9/5}}) = \sqrt{c^2 + 1} - c.$$

Proof. Let $n = 9/5$ in Theorem 2.3 (ii). From [3], Theorem 1, we deduce that

$$V' = \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)^{1/3}. \tag{4.5}$$

Hence,

$$V'^3 - V'^{-3} = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} - \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} = 2\sqrt{15}. \tag{4.6}$$

Substituting (4.5) and (4.6) into Theorem 2.3 (ii), we find that

$$A_1'^6 = \frac{2\sqrt{15}+3\sqrt{5}-1}{2\sqrt{15}-3\sqrt{5}+1}\,5\sqrt{5}$$

after some simplification. Thus, by Proposition 2.4 (ii), we deduce Corollary 4.4.

Corollary 4.5. *If*

$$A_1'^2 = \frac{\sqrt{3\sqrt{5}+7}-\sqrt{3\sqrt{5}-1}}{\sqrt{9\sqrt{5}+27}-\sqrt{9\sqrt{5}+19}}\sqrt{5}$$

and $2c = A_1'^6 - 11$, *then*

$$S^5(e^{-\pi\sqrt{11/5}}) = \sqrt{c^2+1}-c\,.$$

Proof. It is known that ([16], [17])

$$G_{55} = 2^{1/4}(\sqrt{5}+2)^{1/6}\left(\sqrt{\frac{7+\sqrt{5}}{8}}+\sqrt{\frac{\sqrt{5}-1}{8}}\right).$$

Using a modular equation of degree 5 ([1], p. 282, Entry 13 (xiv)), we find that

$$G_{11/5} = 2^{1/4}(\sqrt{5}+2)^{1/6}\left(\sqrt{\frac{7+\sqrt{5}}{8}}-\sqrt{\frac{\sqrt{5}-1}{8}}\right).$$

Hence,

(4.7)
$$V' = \frac{\sqrt{7+\sqrt{5}}+\sqrt{\sqrt{5}-1}}{\sqrt{7+\sqrt{5}}-\sqrt{\sqrt{5}-1}} = \sqrt{\frac{3\sqrt{5}+7}{8}}+\sqrt{\frac{3\sqrt{5}-1}{8}}$$

and

(4.8)
$$V' - V'^{-1} = \sqrt{\frac{3\sqrt{5}-1}{2}}.$$

Now, by (4.8), we have

(4.9)
$$\begin{aligned}\frac{1}{\sqrt{5}}(V'^3 - V'^{-3}) &= \frac{1}{\sqrt{5}}(V'-V'^{-1})\left((V'-V'^{-1})^2+3\right)\\ &= \frac{1}{\sqrt{5}}\left(\sqrt{\frac{3\sqrt{5}-1}{2}}\right)\frac{3\sqrt{5}+5}{2}\\ &= \sqrt{\frac{19+9\sqrt{5}}{2}}.\end{aligned}$$

Substituting (4.7) and (4.9) into Theorem 2.3 (ii) and simplifying, we deduce Corollary 4.5.

Corollary 4.6. *If*

$$A_1'^2 = \frac{\sqrt{\sqrt{65}+7} - \sqrt{\sqrt{65}-1}}{\sqrt{\sqrt{65}+9} - \sqrt{\sqrt{65}+7}} \frac{\sqrt{5}}{2}$$

and $2c = A_1'^6 - 11$, *then*

$$S^5(e^{-\pi\sqrt{13/5}}) = \sqrt{c^2+1} - c.$$

Proof. From [4],

$$V' = \frac{G_{65}}{G_{13/5}} = \sqrt{\frac{\sqrt{65}+7}{8}} + \sqrt{\frac{\sqrt{65}-1}{8}}. \tag{4.10}$$

Using calculations similar to those in the proof of Corollary 4.5, we deduce Corollary 4.6.

In [4], where (4.10) was proved, the values

$$G_{145}/G_{29/5},\quad G_{205}/G_{41/5},\quad G_{265}/G_{53/5},\quad G_{445}/G_{89/5},\quad \text{and}\quad G_{505}/G_{101/5}$$

were also established. Using these values and Theorem 2.3 (ii), we can prove the following corollaries.

Corollary 4.7. *If*

$$A_1'^2 = \frac{\sqrt{17+\sqrt{145}} - \sqrt{9+\sqrt{145}}}{(\sqrt{\sqrt{29}+2\sqrt{5}+1} - \sqrt{\sqrt{29}+2\sqrt{5}-1})^2} \sqrt{\frac{5}{2}}$$

and $2c = A_1'^6 - 11$, *then*

$$S^5(e^{-\pi\sqrt{29/5}}) = \sqrt{c^2+1} - c.$$

Corollary 4.8. *If*

$$A_1'^2 = \frac{\sqrt{25+3\sqrt{41}} - \sqrt{17+3\sqrt{41}}}{(\sqrt{23+3\sqrt{41}} - \sqrt{19+3\sqrt{41}})^2} \sqrt{10}$$

and $2c = A_1'^6 - 11$, *then*

$$S^5(e^{-\pi\sqrt{41/5}}) = \sqrt{c^2+1} - c.$$

Corollary 4.9. *If*

$$A_1'^2 = \frac{\sqrt{16+\sqrt{265}} - \sqrt{12+\sqrt{265}}}{(\sqrt{5\sqrt{53}+17\sqrt{5}+2} - \sqrt{5\sqrt{53}+17\sqrt{5}-2})^2} 2\sqrt{5}$$

and $2c = A_1'^6 - 11$, *then*

$$S^5(e^{-\pi\sqrt{53/5}}) = \sqrt{c^2+1} - c.$$

Corollary 4.10. *If*

$$A_1'^2 = \frac{\sqrt{85+9\sqrt{89}} - \sqrt{77+9\sqrt{89}}}{(\sqrt{85+9\sqrt{89}} - \sqrt{83+9\sqrt{89}})^2} \sqrt{\frac{5}{2}}$$

and $2c = A_1'^6 - 11$, *then*

$$S^5(e^{-\pi\sqrt{89/5}}) = \sqrt{c^2+1} - c.$$

Corollary 4.11. *If*

$$A_1'^2 = \frac{\sqrt{113+5\sqrt{505}} - \sqrt{105+5\sqrt{505}}}{(\sqrt{13\sqrt{101}+58\sqrt{5}+1} - \sqrt{13\sqrt{101}+58\sqrt{5}-1})^2}\sqrt{\frac{5}{2}}$$

and $2c = A_1'^6 - 11$, *then*

$$S^5(e^{-\pi\sqrt{101/5}}) = \sqrt{c^2+1} - c.$$

To complete Ramanujan's table mentioned in the beginning of this section, it remains to evaluate $S(e^{-\pi\sqrt{3}})$.

Theorem 4.12.

$$S(e^{-\pi\sqrt{3}}) = \frac{-3-\sqrt{5}+\sqrt{6(5+\sqrt{5})}}{4}.$$

Theorem 4.12 does not follow from Theorem 2.3 (ii) for the same reason as given in the discussion of Theorem 4.3. This result was first stated by Ramanujan on page 46 of his Lost Notebook. It was first proved by Ramanathan [10] using Kronecker's limit formula. A more natural proof of Theorem 4.12 using modular equations can be found in [6].

§ 5. An application of Kronecker's limit formula

We have seen in Sections 2–4 that the determinations of $F(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ depend upon our ability to evaluate V' (or V). In this section, we will show that, in some cases, we can express V' in terms of fundamental units of certain real quadratic fields.

First, we need some notation. Let $K = \mathbb{Q}(\sqrt{-d})$, where d is a positive squarefree integer. Set

$$\Omega = \begin{cases} \sqrt{-d}, & \text{if } -d \equiv 2, 3 \pmod 4, \\ \dfrac{1+\sqrt{-d}}{2}, & \text{if } -d \equiv 1 \pmod 4. \end{cases}$$

The ring of integers in K is $\mathbb{Z}[\Omega]$. It is well known that any ideal of K can be expressed as a $\mathbb{Z}$-module of the form $I := [b + c\Omega, a]$, with $a > 0$, $c > 0$, $c|b$, $c|a$, and $a|N(b+c\Omega)$, where $N(\gamma)$ denotes the norm of an algebraic integer γ.

Theorem 5.1. *Let* $n = m/5$, *where* $5m$ *is a squarefree positive integer with* $5m \equiv 1 \pmod 4$. *Let* $K = \mathbb{Q}(\sqrt{-5m})$ *be a complex quadratic field with the property that each genus contains exactly two classes and that* $[10, 5+\Omega]$ *is in the principal genus. Let* $-20m = d_1 d_2$, *where* $d_1 > 1$. *Then*

$$(V')^{h/2} = \prod_{\chi([2,1+\Omega]) = -1} \varepsilon_1^{w h_1 h_2 / w_2},$$

where h, h_1, and h_2 are the class numbers of K, $\mathbb{Q}(\sqrt{d_1})$, and $\mathbb{Q}(\sqrt{d_2})$, respectively, w and w_2 are the numbers of roots of unity in K and $\mathbb{Q}(\sqrt{d_2})$, respectively, ε_1 is the fundamental unit in $\mathbb{Q}(\sqrt{d_1})$, and the product is over all the characters χ (with $\chi([2, 1+\Omega] = -1)$), associated with the decomposition $d = d_1 d_2$, and therefore d_1, d_2, h_1, h_2, w_2, and ε_1 are dependent on χ.

Remarks. 1. The proof of Theorem 5.1, which depends on Kronecker's limit formula, is a special case ($p = 5$) of Theorem 3.1 in [4], and so we omit it. Theorem 5.1 may be viewed as an extension of Ramanathan's result [9], Theorem 4.

2. Since h_1 and ε_1 are not defined for $d_1 = 1$, we omit the decomposition $-20m = 1 \cdot d_2$. In fact, if $d_1 = 1$, then the corresponding character associated with such a decomposition is the trivial character. This character does not occur in the product of Theorem 5.1 since the product is taken over all characters for which $\chi([2, 1+\Omega]) = -1$.

Example. Let $n = 13/5$. Then $K = \mathbb{Q}(\sqrt{-65})$, which has class number 8, and the principal genus contains $[10, 5+\Omega]$, where $\Omega = \sqrt{-65}$. The genus characters which satisfy $\chi([2, 1+\Omega]) = -1$ are the characters corresponding to the decompositions $-260 = -52 \cdot 5$ and $-260 = -20 \cdot 13$. In each of the decompositions above, $wh_1h_2/w_2 = 2$. Since the fundamental units of $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{13})$ are $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{13}+3}{2}$, respectively, we conclude from Theorem 5.1 that

$$\frac{G_{65}}{G_{13/5}} = \left(\frac{\sqrt{5}+1}{2}\right)^{1/2} \left(\frac{\sqrt{13}+3}{2}\right)^{1/2}. \tag{5.1}$$

A simple calculation shows that (5.1) is equivalent to (4.10).

§ 6. $F(e^{-\pi\sqrt{n}})$ is a unit for rational n

Let $S_1(f)$ be the collection of 2 by 2 integral matrices of determinant f with relatively prime entries and let

$$\Delta(\tau) = q f^{24}(-q),$$

where $q = e^{2\pi i\tau}$ and $f(-q)$ is defined by (1.6). If $\tau = \frac{z_1}{z_2}$ with $\operatorname{Im}\tau > 0$, we set

$$\Delta\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_2^{-12}\Delta(\tau)$$

and

$$g_A(\tau) = f^{12} \frac{\Delta\left(A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right)}{\Delta\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}, \quad A \in S_1(f).$$

Lemma 6.1. *If τ is in a complex quadratic field,* $\operatorname{Im}\tau > 0$, *and* $A \in S_1(f)$, *then* $g_A(\tau)$ *is an algebraic integer which divides* f^{12}.

Proof. See [14], Lemma 15.

Theorem 6.2. *If τ is in a complex quadratic field, then $F(e^{2\pi i\tau})$ is a unit.*

Proof. Let $z_1 = 5\tau$, $z_2 = 1$, and $A = \begin{pmatrix} 1 & 0 \\ 0 & 25 \end{pmatrix}$. Clearly, $A \in S_1(25)$. By Lemma 6.1, we find that

$$g_A(5\tau) = 25^{12}\frac{\Delta\left(A\begin{pmatrix} 5\tau \\ 1 \end{pmatrix}\right)}{\Delta\begin{pmatrix} 5\tau \\ 1 \end{pmatrix}}$$

$$= 25^{12}\frac{\Delta\begin{pmatrix} 5\tau \\ 25 \end{pmatrix}}{\Delta\begin{pmatrix} 5\tau \\ 1 \end{pmatrix}}$$

$$= \frac{\Delta\left(\frac{\tau}{5}\right)}{\Delta(5\tau)}$$

is an algebraic integer dividing 25^{12}. Hence

$$e^{-2\pi i\tau/5}\frac{f(-e^{2\pi i\tau/5})}{f(-e^{10\pi i\tau})}$$

is an algebraic integer. In view of the identity (2.1), namely,

$$F^{-1}(e^{2\pi i\tau}) - 1 - F(e^{2\pi i\tau}) = e^{-2\pi i\tau/5}\frac{f(-e^{2\pi i\tau/5})}{f(-e^{10\pi i\tau})},$$

we conclude that $F(e^{2\pi i\tau})$ is a unit if τ is in a complex quadratic field.

Corollary 6.3. *For any rational number n,* $F(e^{-\pi\sqrt{n}})$ *and* $S(e^{-\pi\sqrt{n}})$ *are units.*

Corollary 6.3 follows from Theorem 6.2 by setting

$$\tau = \frac{i\sqrt{n}}{2} \quad \text{and} \quad \tau = \frac{1+i\sqrt{n}}{2},$$

respectively.

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