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Complex series for $1/\pi$

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Abstract Many series for $1/\pi$ were discovered since the appearance of S. Ramanujan's famous paper "Modular equations and approximation to π" published in 1914. Almost all these series involve only real numbers. Recently, in an attempt to prove a series for $1/\pi$ discovered by Z.-W. Sun, the authors found that a series for $1/\pi$ involving complex numbers is needed. In this article, we illustrate a method that would allow us to prove series of this type.

Keywords Hypergeometric series · Singular moduli · Lambert series

Mathematics Subject Classification (2000) Primary 11F11 · Secondary 11F03 · 11Y60 · 33C05 · 33C20

1 Introduction

G. Bauer [2] is likely to be the first mathematician to have discovered a series for $1/\pi$ in the form

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$$\sum_{k=0}^{\infty} a_k(A+Bk)X^k = \frac{C}{\pi}, \tag{1.1}$$

where $\{a_k\}$ is a sequence of rational numbers, and A, B, C and X are *real* algebraic numbers. Bauer's series is

$$\frac{2}{\pi} = \sum_{k=0}^{\infty}(-1)^k \frac{(\frac{1}{2})_k^3}{k!^3}(4k+1),$$

where

$$(a)_0 = 1 \quad \text{and} \quad (a)_k = a(a+1)\cdots(a+k-1) \quad \text{for } k \geq 1.$$

It was, however, the paper of S. Ramanujan [10] that popularize the study of series of the type (1.1).

Many new series of the form (1.1) are found after Ramanujan's work (see [4, 6, 8], and [11]), with the most recent discovery being those found empirically by Z.-W. Sun [12] and proved by these authors in [7]. All such series share one common property that the coefficients are all real.

In [9], J. Guillera and W. Zudilin discovered the first series for $1/\pi$ with complex coefficients, namely,

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k^3}{k!^3}\left(\frac{49-13\sqrt{-7}}{64} + \frac{105-21\sqrt{-7}}{32}k\right)\left(\frac{47+45\sqrt{-7}}{128}\right)^k = \frac{\sqrt{7}}{\pi}. \tag{1.2}$$

This series was shown to be equivalent to another series involving only real numbers and the proof of the latter series follows from application of the Wilf–Zeilberger method.

Let

$${}_pF_{p-1}\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_2, \ldots, b_p \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_2)_k\cdots(b_p)_k}\frac{z^k}{k!}, \quad |z|<1,$$

and let

$$P_k(x) = {}_2F_1\left(\begin{matrix} -k, k+1 \\ 1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right). \tag{1.3}$$

Recently, Z.-W. Sun [12] discovered many new series for $1/\pi$ associated with $T_k(b,c)$, where

$$T_k(b,c) = \left(b^2-4c\right)^{k/2} P_k\left(\frac{b}{(b^2-4c)^{1/2}}\right).$$

Using (1.3), we [7] converted Sun's series to series involving $P_k(x)$, one of which is

$$\sum_{k=0}^{\infty} \frac{(\frac{1}{4})_k(\frac{3}{4})_k}{k!^2} P_k\left(\frac{-7i}{33\sqrt{15}}\right)(13+80k)\left(\frac{-11\sqrt{-15}}{147}\right)^k = \frac{7\sqrt{42}(3+2\sqrt{5})}{8\pi}. \tag{1.4}$$

To prove (1.4), we need two series analogous to (1.2), namely,

$$\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{4})_k(\frac{3}{4})_k}{k!^3}\left(52\mp 12\sqrt{-3}+\left(320\mp 55\sqrt{-3}\right)k\right)$$
$$\times\left(\frac{2(5\pm\sqrt{-3})}{7\sqrt{3}}\right)^{4k}=\frac{98\sqrt{3}}{\pi}. \tag{1.5}$$

It suffices to prove any one of the above series since one is the conjugate of the other.

The proof of (1.5) were sketched briefly in [7]. In this note, we will discuss a method to establish identities such as (1.5). Our proof is different from that given in [7] and is applicable to a more general collection of series similar to (1.5).

2 Functions and forms associated with $\Gamma_0(2)$ and a transformation formula

Our main aim is to prove (1.5) and these series arise from the study of Ramanujan's quartic theory of elliptic functions [3]. We recall some of the facts from [3].

For $|q|<1$, define

$$f(-q)=\prod_{j=1}^{\infty}\left(1-q^j\right).$$

When $q=e^{2\pi i\tau}$ with $\mathrm{Im}\ \tau>0$, we find that

$$q^{1/24}f(-q)=\eta(\tau),$$

where $\eta(\tau)$ is the Dedekind η-function. It is well known that $\eta(\tau)$ [1, Theorem 3.1] satisfies the transformation formula

$$\eta\left(-\frac{1}{\tau}\right)=\sqrt{-i\tau}\,\eta(\tau). \tag{2.1}$$

Let

$$Z(q)=\frac{f^8(-q)+32qf^8(-q^4)}{f^4(-q^2)}$$

and

$$X(q)=4x(q)\left(1-x(q)\right), \tag{2.2}$$

where

$$\frac{1}{x(q)}=1+\frac{f^{24}(-q)}{64qf^{24}(-q^2)}. \tag{2.3}$$

From [3], we know that

$$Z(q)={}_3F_2\left(\left.\begin{matrix}\frac{1}{4},\frac{3}{4},\frac{1}{2}\\1,1\end{matrix}\right|X(q)\right).$$

To extract the number π from these functions, we need the transformation formula for $A(q)$ and this follows immediately from (2.1). More precisely, we have

$$Z\left(e^{2\pi i(\frac{-1}{\sqrt{2}\tau})}\right) = -\tau^2 Z\left(e^{2\pi i\tau/\sqrt{2}}\right).$$

Differentiating the above with respect to τ, we deduce that

$$\frac{1}{\tau}\cdot\frac{q}{Z}\frac{\mathrm{d}Z}{\mathrm{d}q}\bigg|_{q=e^{-2\pi i/(\sqrt{2}\tau)}} = \frac{\sqrt{2}}{\pi i} + \tau\cdot\frac{q}{Z}\frac{\mathrm{d}Z}{\mathrm{d}q}\bigg|_{q=e^{2\pi i\tau/\sqrt{2}}}.$$

To simplify notation, let

$$G(\tau) = \frac{q}{Z}\frac{\mathrm{d}Z}{\mathrm{d}q}\bigg|_{q=e^{2\pi i\tau/\sqrt{2}}}.$$

Then the transformation can be rewritten as

$$\frac{1}{\tau}G\left(-\frac{1}{\tau}\right) = \frac{\sqrt{2}}{\pi i} + \tau G(\tau). \tag{2.4}$$

In the next section, we will express $G(\tau)$ and $G(-1/\tau)$ in terms of hypergeometric function and its derivative.

3 Some intermediate identities

Set

$$\tau_1 = \frac{\sqrt{-15}-1}{2\sqrt{2}}, \qquad \tau_2 = \frac{\sqrt{-5/3}-1}{2\sqrt{2}}, \quad \text{and} \quad \tau_3 = \frac{\sqrt{-15}+1}{2\sqrt{2}}.$$

From (2.4), we deduce that

$$G\left(-\frac{1}{\tau_1}\right) = \tau_1\frac{\sqrt{2}}{\pi i} - \left(\frac{7}{4} + \frac{\sqrt{-15}}{4}\right)G(\tau_1)$$

and

$$G(\tau_2) = \left(\frac{1}{\tau_2}\right)^2 G(\tau_1) + \frac{1}{\tau_2}\frac{\sqrt{2}}{\pi i},$$

where we have used $-1/\tau_2 = \tau_1 + \sqrt{2}$, which implies that

$$G\left(-\frac{1}{\tau_2}\right) = G(\tau_1).$$

Hence, we find that

$$G\left(-\frac{1}{\tau_1}\right) = \frac{\sqrt{-15}-1}{2\pi i} - \left(\frac{7}{4} + \frac{\sqrt{-15}}{4}\right)G(\tau_1) \tag{3.1}$$

and

$$G(\tau_2) = \left(-\frac{3}{4} + \frac{3}{4}\sqrt{-15}\right) G(\tau_1) + \left(-\frac{3}{2} - \frac{1}{2}\sqrt{-15}\right) \frac{1}{\pi i}. \tag{3.2}$$

Now, let

$$M_N(q) = \frac{Z(q)}{Z(q^N)}. \tag{3.3}$$

Then we find that

$$\frac{q}{M_N(q)} \frac{dM_N(q)}{dq} = \widetilde{Z}(q) - N\widetilde{Z}(q^N),$$

where

$$\widetilde{Z}(q) = \frac{q}{Z(q)} \frac{dZ(q)}{dq}.$$

Letting $q = e^{2\pi i\tau/\sqrt{2}}$, we deduce that

$$G(\tau) - NG(N\tau) = \widetilde{M}_N(\tau),$$

where

$$\widetilde{M}_N(\tau) = \frac{q}{M_N(q)} \frac{dM_N(q)}{dq}. \tag{3.4}$$

When $N = 2$, we have

$$G\left(-\frac{1}{\tau_1}\right) - 2G(\tau_3) = \widetilde{M}_2\left(-\frac{1}{\tau_1}\right) \tag{3.5}$$

and when $N = 3$,

$$G(\tau_2) - 3G(\tau_3) = \widetilde{M}_3(\tau_2). \tag{3.6}$$

Using (3.1) and (3.2), we would have two identities relating $G(\tau_1)$ and $G(\tau_3)$ (see (4.5) and (4.6) below).

4 Modular equations of degree 2 and 3 in the theory of signature 4 and explicit evaluations

A modular equation of degree N in the theory of signature 4 is a relation between $x(q)$ and $x(q^N)$, where $x(q)$ is given by (2.3). The modular equation of degree N is not unique. In order to establish the two series for $1/\pi$ in (1.5), we will need the following modular equations:

Theorem 4.1 *Let $\alpha = x(q)$ and $\gamma = x(q^2)$. Then*

$$64\gamma - 80\gamma\alpha + 18\gamma\alpha^2 - 81\gamma^2\alpha^2 + 144\gamma^2\alpha - 64\gamma^2 - \alpha^2 = 0. \tag{4.1}$$

Theorem 4.2 *Let $\alpha = x(q)$ and $\beta = x(q^3)$. Then*

$$\begin{aligned}
&\alpha^4+\beta^4+141056\beta^3\alpha^3+19206\beta^2\alpha^2-4096\alpha\beta+36864\beta^4\alpha^4\\
&\quad-3972\big(\beta^3\alpha+\alpha^3\beta\big)+36480\big(\alpha^4\beta^2+\beta^4\alpha^2\big)-73728\big(\beta^4\alpha^3+\alpha^4\beta^3\big)\\
&\quad+384\big(\alpha^4\beta+\beta^4\alpha\big)+7680\big(\alpha^2\beta+\beta^2\alpha\big)-63360\big(\alpha^3\beta^2+\beta^3\alpha^2\big)=0. \qquad (4.2)
\end{aligned}$$

Let $F(\tau) = x(q)$, with $q = e^{2\pi i\tau/\sqrt{2}}$. Since

$$3\tau_2 = \tau_3 - \sqrt{2},$$

we find that $F(\tau_2)$ and $F(\tau_3)$ satisfies (4.2). In a similar way, we conclude that $F(-1/\tau_1)$ and $F(\tau_3)$ satisfies (4.1). Now, using (2.1), we find that

$$F\left(-\frac{1}{\tau_1}\right) = 1 - F(\tau_1).$$

Hence, we deduce that

$$F(\tau_2) = 1 - F\left(-\frac{1}{\tau_2}\right) = 1 - F(\tau_1), \qquad (4.3)$$

where we have used

$$-\frac{1}{\tau_2} = \tau_1 + \sqrt{2}.$$

Hence, the two relations we obtained reduced to two equations involving $F(\tau_1)$ and $F(\tau_3)$. Solving these equations, we conclude that

$$F(\tau_1) = \frac{1}{2} - \frac{32}{147}\sqrt{5} - \frac{11}{294}\sqrt{-15}.$$

By taking conjugation, we find that

$$F(\tau_3) = \frac{1}{2} - \frac{32}{147}\sqrt{5} + \frac{11}{294}\sqrt{-15}.$$

By (4.3), we deduce that

$$F(\tau_2) = \frac{1}{2} + \frac{32}{147}\sqrt{5} + \frac{11}{294}\sqrt{-15}.$$

We next describe how we obtain an expression for M_N defined by (3.3). It is known [3] that if $x = x(q)$, then

$$q\frac{\mathrm{d}x}{\mathrm{d}q} = Zx(1-x) = \frac{Z}{4}X,$$

where X is defined by (2.2). If N is a positive integer greater than 1, then

$$q^N\frac{\mathrm{d}x(q^N)}{\mathrm{d}(q^N)} = \frac{Z(q^N)}{4}X\big(q^N\big).$$

This yields

$$M_N = \frac{1}{N}\frac{\mathrm{d}x(q)}{\mathrm{d}x(q^N)}\frac{X(q^N)}{X(q)}. \tag{4.4}$$

But if we are given a modular equation of degree N, then the right-hand side can be expressed explicitly in terms of $X(q)$ and $X(q^N)$ and hence we have an expression of M_N in terms of $X(q)$ and $X(q^N)$. From (4.4), we can then derive an explicit expression of $\mathrm{d}M_N/\mathrm{d}X(q)$ in terms of $X(q)$ and $X(q^N)$ and this in turn yields the expression for $\widetilde{M}_N$ defined by (3.4). We will carry out these computations and determine the right-hand side of (3.5).

Differentiating (4.1) with respect to α, we conclude that

$$\frac{\mathrm{d}\gamma}{\mathrm{d}\alpha} = \frac{80\gamma - 36\gamma\alpha + 162\gamma^2\alpha - 144\gamma^2 + 2\alpha}{64 - 80\alpha + 18\alpha^2 - 162\gamma\alpha^2 + 288\gamma\alpha - 128\gamma}.$$

Hence,

$$M_2 = \frac{1}{2}\frac{64 - 80\alpha + 18\alpha^2 - 162\gamma\alpha^2 + 288\gamma\alpha - 128\gamma}{80\gamma - 36\gamma\alpha + 162\gamma^2\alpha - 144\gamma^2 + 2\alpha}\frac{\gamma(1-\gamma)}{\alpha(1-\alpha)}.$$

Differentiating M_2 with respect to α,[1] and letting $\alpha = F(-1/\tau_1)$ and $\gamma = F(\tau_3)$, we conclude that

$$G\left(-\frac{1}{\tau_1}\right) - 2G(\tau_3) = \left(\frac{11}{49} + \frac{\sqrt{5}}{7} + \frac{\sqrt{-15}}{21} - \frac{\sqrt{-3}}{147}\right)Z(\tau_1). \tag{4.5}$$

In a similar way, we use (4.2) and the relation between $Z(\tau_1) = Z(-1/\tau_2)$ and $Z(\tau_2)$ to deduce that

$$G(\tau_2) - 3G(\tau_3) = \left(\frac{4}{49}\sqrt{5} - \frac{20}{49}\sqrt{-3} + \frac{2}{49}\sqrt{-15} + \frac{30}{49}\right)Z(\tau_1). \tag{4.6}$$

Next, using (3.1) and (3.2) in (4.5) and (4.6) to remove the term $G(\tau_3)$, we find that

$$-\left(\frac{15}{4} + \frac{9}{4}\sqrt{-15}\right)G(\tau_1) - \left(\frac{27}{49} - \frac{13}{49}\sqrt{5} - \frac{3}{49}\sqrt{-15} - \frac{39}{49}\sqrt{-3}\right)Z(\tau_1)$$
$$= -\frac{9i + \sqrt{15}}{2\pi}.$$

Finally, observing that

$$Z(\tau_1) = \sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k(\frac{1}{4})_k(\frac{3}{4})_k}{k!^3}\big(4F(\tau_1)\big(1 - F(\tau_1)\big)\big)^k$$

[1] This would be too complicated to present here.

and

$$G(\tau_1) = \big(1 - 2F(\tau_1)\big) \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k (\frac{1}{4})_k (\frac{3}{4})_k}{k!^3} \, k\big(4F(\tau_1)\big(1 - F(\tau_1)\big)\big)^k,$$

we obtain the desired series for $1/\pi$.

Remarks

1. The degrees of modular equations to be used to prove complex series for $1/\pi$ are not as obvious as in the real series. In the real series for $1/\pi$, if $\tau = \sqrt{-pq/2}$ where p and q are primes, then it is clear that we need modular equations of degree p and q. In the complex case, we observe that the *norms* of

$$\frac{1}{\sqrt{2}} \frac{\tau_3}{(-1/\tau_1)} \quad \text{and} \quad \frac{1}{\sqrt{2}} \frac{\tau_3}{\tau_2}$$

are 2 and 3, respectively. These norms determine the degrees of modular equations we used. We stress that the method presented here can also be applied to series for $1/\pi$ with complex coefficients that belong to the theory of elliptic function in other alternative bases.

2. In most of the proofs of Ramanujan-type series for $1/\pi$, the most complicated expression arises from differentiating M_N. In the quartic theory, this complication can be avoided by writing the derivative of M_N in terms of the expression

$$f_N(q) = \frac{NL(q^N) - L(q)}{(N-1)\sqrt{Z(q)Z(q^N)}},$$

where

$$L(q) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k}.$$

This identity, after some simplifications, is [3, (4.36)]

$$\frac{NL(q^N) - L(q)}{Z(q)} = \big(1 - 3x\big(q^N\big)\big) \frac{N}{M_N} - \big(1 - 3x(q)\big) - 6 \frac{\widetilde{M}_N}{Z(q)}.$$

The expression f_N can then be expressed in terms of $x(q)$ and $x(q^N)$ and we can then derive the value of $\widetilde{M}_N$ at corresponding values of q without differentiating M_N. In this article, we would require the formulas

$$f_2^2(q) = 1 + 3\sqrt{x\big(q^2\big)}$$

and

$$f_3^2(q) = 1 + 3\sqrt{x(q)x\big(q^3\big)}.$$

The two values that we needed are then

$$f_2\big(e^{2\pi i(-1/\tau_1)}\big) = \frac{\sqrt{105}+2\sqrt{21}+2\sqrt{-35}-\sqrt{-7}}{14}$$

and

$$f_3\big(e^{2\pi i\tau_2}\big) = \frac{\sqrt{105}-3\sqrt{-7}}{14}.$$

These values would then lead to the values on the right-hand sides of (4.5) and (4.6).

3. A different approach to derive series for $1/\pi$ with complex coefficients and argument is based on algebraic transformations of hypergeometric and related series of *modular* origin; the required details of the method can be found in [5]. For example, starting from *real* Ramanujan's series

$$\sum_{k=0}^{\infty}\frac{(\frac12)_k^3}{k!^3}(5-\sqrt5+20k)\bigg(\frac{\sqrt5-1}{2}\bigg)^{6k} = \frac{2\sqrt5}{\pi}\sqrt{2+\sqrt5},$$

and applying subsequently the transformation

$$\begin{aligned}&{}_3F_2\bigg(\begin{matrix}\frac13,\ \frac12,\ \frac23\\ 1,\ 1\end{matrix}\ \bigg|\ \frac{27p^4(1-p^2)(2+p)^4(1+2p)}{(2+2p-p^2)^6}\bigg)\\ &\quad=\frac{(2+2p-p^2)^2}{4(1+2p)}\,{}_3F_2\bigg(\begin{matrix}\frac12,\ \frac12,\ \frac12\\ 1,\ 1\end{matrix}\ \bigg|\ \frac{4p^3(1-p^2)(1+p)^2(2+p)}{(1+2p)^2}\bigg)\end{aligned}$$

at

$$p = \frac{(1+\sqrt5)\sqrt{\sqrt5-2}-\sqrt{22-10\sqrt5}}{4}-\frac12$$

(cf. [7]) and then the generating function

$$\sum_{k=0}^{\infty}\delta_k u^k = \frac{1}{1-4u}\,{}_3F_2\bigg(\begin{matrix}\frac13,\ \frac12,\ \frac23\\ 1,\ 1\end{matrix}\ \bigg|\ \frac{108u^2}{(1-4u)^3}\bigg)$$

of the *Domb numbers* [5] at $u=(3-2i-\sqrt{5-10i})/32$, we obtain the following two *complex* series:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(\frac13)_k(\frac12)_k(\frac23)_k}{k!^3}\big(3\big(401-i-(109-69i)\sqrt{1+2i}\big)+5830k\big)\\ &\qquad\times\bigg(\frac{27(2530+1451i-65(30-i)\sqrt{1+2i})}{495-4888i}\bigg)^k\\ &\quad=\frac{3321-381i+81(33-17i)\sqrt{1+2i}}{4\pi}\end{aligned}$$

and

$$\sum_{k=0}^{\infty}\big(69+13i-(23-7i)\sqrt{1+2i}+170k\big)\delta_k\left(\frac{3-2i-(1-2i)\sqrt{1+2i}}{32}\right)^k$$
$$=\frac{66+42i+12(1-4i)\sqrt{1+2i}}{\pi}.$$

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