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The Bailey–Brafman identity and its analogues

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ABSTRACT

In this article, we discuss a new proof of an elegant identity discovered by J. Wan and W. Zudilin. This identity is an analogue of F. Brafman's identity discovered around 1951 and it is a generalization of a Clausen type identity discovered recently by Chan, Tanigawa, Yang and Zudilin.

1. Introduction

Around January 2011, Z.-W. Sun announced that he had discovered a new type of series for $1/\pi$. Subsequently, he produced over 170 new series for $1/\pi$. Among Sun's long list of series, there are series that involve $T_n(b,c)$ where

$$T_n(b,c)=\sum_{k=0}^{n}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k.$$

Zudilin observed that

$$T_n(b,c)=(b^2-4c)^{n/2}P_n\left(\frac{b}{\sqrt{b^2-4c}}\right),$$

where the Legendre polynomial $P_n(x)$ is given by

$$\sum_{m=0}^{n}\binom{n}{m}^2\left(\frac{x-1}{2}\right)^n\left(\frac{x+1}{2}\right)^{n-m}.$$

With this important observation, Zudilin established a one to one correspondence between series for $1/\pi$ involving $T_n(b,c)$ and series for $1/\pi$ involving $P_n(x)$. For example, Sun's series

$$\sum_{n=0}^{\infty}\frac{15n-4}{18^{3n}}\binom{2n}{n}\binom{3n}{n}T_n(198,1)=\frac{135\sqrt{3}}{2\pi}$$

is equivalent to

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(1)_n^2}(15n-4)P_n\left(\frac{99}{70\sqrt{2}}\right)\left(\frac{35}{27\sqrt{2}}\right)^n=\frac{135\sqrt{3}}{2\pi}.$$

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Together with Chan and J. Wan, Zudilin [1] proved Sun's series involving $T_n(b, c)$ using Brafman's identity (see [2, (12)], with $\alpha = \beta = 0$)

$$\sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(1)_n^2} P_n\left(\frac{u+v-2uv}{u-v}\right)(u-v)^n = {}_2F_1(a, 1-a; 1; u) \cdot {}_2F_1(a, 1-a; 1; v), \tag{1.1}$$

where

$${}_2F_1\,(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^k}{k!},$$

with

$$(a)_k = \begin{cases} 1 & \text{if } k = 0, \\ (a)(a+1)\cdots(a+k-1) & \text{if } k \geq 1. \end{cases}$$

Besides series for $1/\pi$ associated with $T_n(b, c)$, Sun discovered series associated with $T_{2n}(b, c)$ and $T_{3n}(b, c)$. Examples of these series, written in terms of $P_{2n}(x)$ and $P_{3n}(x)$, are

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(1)_n^2} \left(5 - \sqrt{6} + 20n\right) P_{2n}\left(\frac{17}{15}\right)\left(\frac{217 - 88\sqrt{6}}{25}\right)^n = \frac{3(4+\sqrt{6})}{2\pi}$$

and

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(1)_n^2} (1+9n)\, P_{3n}\left(\frac{4}{\sqrt{10}}\right)\left(\frac{1}{3\sqrt{10}}\right)^{3n} = \frac{\sqrt{15+10\sqrt{3}}}{\pi\sqrt{2}}.$$

The proofs of these series were given by Wan and Zudilin [3] using the following elegant generalization of (1.1).

Theorem 1.1. *Let a, b, c be integers. Let $u_{-1} = 0$, $u_0 = 1$ and define u_n for $n \geq 2$ by the relation*

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1}. \tag{1.2}$$

Then there is a neighborhood of $X = Y = 0$ such that

$$G(X)G(Y) = H(X, Y), \tag{1.3}$$

where

$$G(Z) = \sum_{n=0}^{\infty} u_n Z^n$$

and

$$H(X, Y) = \frac{1}{W} \sum_{n=0}^{\infty} u_n \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{UL}{W^2}\right)^m \left(\frac{KV}{W^2}\right)^{n-m},$$

with

$$U = X, \qquad V = 1 - aX + cX^2, \qquad L = 1 - aY + cY^2, \qquad K = Y \quad \text{and} \quad W = 1 - cXY.$$

Unfortunately, many computational details of the proof of Theorem 1.1 were omitted in [3]. In Section 2, we will present a new proof of Theorem 1.1. The complete details are given with the hope that this method of proof can be used in future to establish identities similar to (1.3).

2. Proof of Theorem 1.1

Proof. Let ϑ_X be the operator

$$\vartheta_X = X\frac{\partial}{\partial X}$$

and following [3], define

$$\Psi_X = \frac{1}{X}\left(\vartheta_X^2 - X\left(a\vartheta_X^2 + a\vartheta_X + b\right) + cX^2\left(\vartheta_X + 1\right)^2\right).$$

Note that

$$\Psi_X\left(G(X)G(Y)\right) = G(Y)\Psi_X\left(G(X)\right) = 0$$

since u_n satisfies (1.2).

We will verify that

$$\Psi_X\left(H(X,Y)\right) = 0.$$

Applying Ψ_X to $H(X,Y)$, we find that

$$\Psi_X\left(H(X,Y)\right) = \sum_{n=0}^{\infty} u_n \sum_{m=0}^{n} \binom{n}{m}^2 \xi\left(X,Y\right) \frac{U^{m-1}V^{n-m-1}L^m K^{n-m}}{W^{2n+3}}$$

where

$$\begin{aligned}
\xi\left(X,Y\right) = {} & c^4X^6m^2Y^2 + (c^3Y + 4mc^3nY + 2c^3nY - 2c^3m^2Y - 2mc^3naY^2 \\
& - c^3naY^2 + 2c^3mY - bc^3Y^2)X^5 + (nc^2Y^2a^2 - 2m^2c^3Y^2 - 2mc^2Ya \\
& + 4nc^3Y^2 + 4c^2n + c^3Y^2 - 6c^2naY - 3c^2Ya + 4mnc^3Y^2 + 2bc^2Y \\
& + 4n^2c^2 - 4n^2c^2Ya + c^2 - 2c^2m + c^2m^2 - 2mc^3Y^2 + bc^2Y^2a - 4mc^2n \\
& + n^2c^2Y^2a^2)X^4 + (2mnac^2Y^2 + 2c^2Y - 4n^2c^2Y^2a - bc^2Y^2 + 2mcna \\
& - 2bcYa + 4nc^2Y + 2amc - 8mnc^2Y + 2cYa^2 + 2mc^2Y^2a + 8n^2c^2Y \\
& - bc - c^2Y^2a + 4ncYa^2 - 5cna - ac + 4m^2c^2Y + 4mc^2Y - 4n^2ac \\
& + 2n^2a^2cY - 5nac^2Y^2)X^3 + (c^2Y^2 + 4nc - 2m^2c - 3cYa + na^2 \\
& - 2mc + 2bcY - 6nacY - 2mc^2Y^2 - 2mcYa - 4n^2acY + 4n^2c^2Y^2 \\
& - 4mc^2Y^2n + m^2c^2Y^2 + n^2a^2 + ba + c + 4mnc + 4c^2Y^2n)X^2 \\
& + (2mcY - 2m^2cY + cY - 2mna + 4mcYn + 2cYn - na - b)X + m^2.
\end{aligned}$$

It turns out that one can express $\xi\left(X,Y\right)$ as

$$\begin{aligned}
\xi\left(X,Y\right) = {} & \left(a(n-m)^2UV + (a^2-4c)(n-m)^2U^2 + m^2V^2 + am^2UV\right)W^2 \\
& + c(2n+1)(2n+1-2m)U^2VL + c(2n+1)(2m+1)UKV^2 - (an^2+an+b)UVW^2.
\end{aligned}$$

Let

$$\zeta_1\left(X,Y\right) = \left(a(n-m)^2UV + (a^2-4c)(n-m)^2U^2 + m^2V^2 + am^2UV\right)W^2,$$

$$\zeta_2\left(X,Y\right) = c(2n+1)(2n+1-2m)U^2VL + c(2n+1)(2m+1)UKV^2$$

and

$$\zeta_3\left(X,Y\right) = -(an^2+an+b)UVW^2.$$

Using the simple identities

$$(n-m)^2\binom{n}{m}^2 = n^2\binom{n-1}{m}^2 \quad \text{and} \quad m^2\binom{n}{m}^2 = n^2\binom{n-1}{m-1}^2,$$

we conclude that

$$\begin{aligned}
& \sum_{n=0}^{\infty} u_n \sum_{m=0}^{n} \binom{n}{m}^2 \zeta_1\left(X,Y\right) \frac{U^{m-1}V^{n-m-1}L^mK^{n-m}}{W^{2n+3}} \\
& \quad = \sum_{n=0}^{\infty}(n+1)^2u_{n+1}\sum_{m=0}^{n}\binom{n}{m}^2 \frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+1}} \\
& \qquad + c\sum_{n=0}^{\infty}(n+1)^2u_{n+1}\sum_{m=0}^{n}\binom{n}{m}^2 (U-K)^2\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}.
\end{aligned}$$

Since

$$U - K = \frac{UL - VK}{W},$$

we conclude that

$$\sum_{n=0}^{\infty} u_n \sum_{m=0}^{n} \binom{n}{m}^2 \zeta_1(X,Y) \frac{U^{m-1}V^{n-m-1}L^mK^{n-m}}{W^{2n+3}}$$
$$= \sum_{n=0}^{\infty}(n+1)^2u_{n+1} \sum_{m=0}^{n} \binom{n}{m}^2 \frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+1}}$$
$$+ c\sum_{n=0}^{\infty}(n+1)^2u_{n+1} \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{UL-VK}{W}\right)^2 \frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}. \quad (2.1)$$

Note that

$$\sum_{n=0}^{\infty} u_n \sum_{m=0}^{n} \binom{n}{m}^2 \zeta_3(X,Y) \frac{U^{m-1}V^{n-m-1}L^mK^{n-m}}{W^{2n+3}} = \sum_{n=0}^{\infty} -(an^2+an+b)u_n \sum_{m=0}^{n} \binom{n}{m}^2 \frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+1}}.$$

Hence, we can now write

$$\Psi_X(H(X,Y)) = \sum_{n=0}^{\infty} \left((n+1)^2u_{n+1} - (an^2+an+b)u_n + cn^2u_{n-1}\right) \sum_{m=0}^{n} \binom{n}{m}^2 \frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+1}} + \mathcal{E}, \quad (2.2)$$

where

$$\mathcal{E} = c\sum_{n=0}^{\infty}(n+1)^2u_{n+1} \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{UL-VK}{W}\right)^2 \frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}$$
$$+ \sum_{n=0}^{\infty} u_n \sum_{m=0}^{n} \binom{n}{m}^2 \zeta_2(X,Y) \frac{U^{m-1}V^{n-m-1}L^mK^{n-m}}{W^{2n+3}}$$
$$- c\sum_{n=0}^{\infty}(n+1)^2u_n \sum_{m=0}^{n+1} \binom{n+1}{m}^2 VK\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}. \quad (2.3)$$

We observe that the last term of $\mathcal{E}$ is inserted so that we can have the term

$$\sum_{n=0}^{\infty} cn^2u_{n-1} \sum_{m=0}^{n} \binom{n}{m}^2 \frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+1}}$$

in (2.2).

Note that since

$$\sum_{n=0}^{\infty} \left((n+1)^2u_{n+1} - (an^2+an+b)u_n + cn^2u_{n-1}\right) \sum_{m=0}^{n} \binom{n}{m}^2 \frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+1}} = 0,$$

to show that $\Psi_X(H(X,Y)) = 0$, it suffices to show that $\mathcal{E} = 0$.

We now observe that the term

$$\sum_{n=0}^{\infty}(n+1)^2u_{n+1} \sum_{m=0}^{n} \binom{n}{m}^2 (UL)^2\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+5}},$$

which arises from the first term of $\mathcal{E}$ in (2.3), can be written (after replacing n by $n-1$ and m by $m-2$) as

$$\sum_{n=0}^{\infty} n^2u_n \sum_{m=0}^{n+1} \binom{n-1}{m-2}^2 VK\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}.$$

In fact, each term in $\mathcal{E}$ can be expressed in the form

$$\sum_{n=0}^{\infty} u_n \sum_{m=0}^{n+1} B(m,n)VK\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}$$

and the final expression for $\mathcal{E}$ is

$$\mathcal{E} = c\sum_{n=0}^{\infty} u_n \sum_{m=0}^{n+1} R(n,m)VK\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}$$

where

$$R(n,m)=\binom{n-1}{m-2}^2n^2+\binom{n-1}{m}^2n^2-2\binom{n-1}{m-1}^2n^2+\binom{n}{m-1}(2n+1)(2n-2m+3)$$
$$+\binom{n}{m}^2(2n+1)(2m+1)-\binom{n+1}{m}^2(n+1)^2.$$

But it is easy to verify that $R(n,m)=0$ and hence $\mathcal{E}=0$.

To complete the proof of Theorem 1.1, we follow the method of Wan and Zudilin [3]. Let

$$\phi(X,Y)=G(X)G(Y)-H(X,Y).$$

From our computations, we conclude that

$$(\Psi_X+\Psi_Y)(\phi(X,Y))=0.$$

Furthermore,

$$\phi(X,0)=G(X)-H(X,0)=G(X)-G(X)=0.$$

Hence, if we write

$$\phi(X,Y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}v_{i,j}X^iY^j,$$

then we conclude that

$$v_{i,0}=0$$

for all integers $i\geq 0$. Since the argument is symmetric in X and Y, we have

$$v_{0,j}=0$$

for all integers $j\geq 0$.

Applying $\Psi_X+\Psi_Y$ to the series expansion of $\phi(X,Y)$, we find that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Big((i+1)^2v_{i+1,j}-\left(ai^2+ai+b\right)v_{i,j}+ci^2v_{i-1,j}$$
$$+(j+1)^2v_{i,j+1}-\left(aj^2+aj+b\right)v_{i,j}+cj^2v_{i,j-1}\Big)X^iY^j=0. \tag{2.4}$$

We first show that $v_{i,1}=0$ for all integers $i\geq 0$. Let $j=0$. If $i=0$, then by our previous discussion, we conclude that $v_{0,1}=0$. If $i>0$, then from (2.4), we conclude that $v_{i,1}=0$. Hence we find that $v_{i,1}=0$ for all integers $i\geq 0$.

We now suppose that $v_{i,j}=0$ for all integers $i\geq 0$ and $0\leq j\leq k$. We want to show that $v_{i,k+1}=0$ for all integers $i\geq 0$. When $i=0$, we clearly have $v_{0,k+1}=0$. If $i>0$, then by (2.4), we find immediately that $v_{i,k+1}=0$. This completes the proof of Theorem 1.1. □

Remark 2.1. 1. The case $c=0$ yields a new proof of Brafman's identity (1.1).

2. If we set $UL=VK$, we find that $X=Y$ and this yields Clausen type identities

$$G^2(X)=\frac{1}{1-cX^2}\sum_{n=0}^{\infty}\binom{2n}{n}u_n\left(\frac{X(1-aX+cX^2)}{(1-cX^2)^2}\right)^n.$$

Several identities of this type for $c\neq 0$ were first discovered by Chan et al. [4].

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