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LEGENDRE POLYNOMIALS AND RAMANUJAN-TYPE SERIES FOR $1/\pi$

BY

Heng Huat Chan

Department of Mathematics, National University of Singapore
2 Science Drive 2, Singapore 117543, Singapore
e-mail: matchh@nus.edu.sg

AND

James Wan and Wadim Zudilin*

School of Mathematical and Physical Sciences
The University of Newcastle, Callaghan, NSW 2308, Australia
e-mail: james.wan@newcastle.edu.au and wadim.zudilin@newcastle.edu.au

On Jon Borwein's 60th birthday

ABSTRACT

We resolve a family of recently observed identities involving $1/\pi$ using the theory of modular forms and hypergeometric series. In particular, we resort to a formula of Brafman which relates a generating function of the Legendre polynomials to a product of two Gaussian hypergeometric functions. Using our methods, we also derive some new Ramanujan-type series.

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1. Introduction

Recently, Z.-W. Sun [16] and G. Almkvist experimentally observed several new identities for $1/\pi$ of the form

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2}(A+Bn)T_n(b,c)\lambda^n = \frac{C}{\pi}, \tag{1}$$

where $s \in \{1/2, 1/3, 1/4\}$, $A, B, b, c \in \mathbb{Z}$, $T_n(b,c)$ denotes the coefficient of x^n in the expansion of $(x^2+bx+c)^n$, viz.

$$T_n(b,c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k, \tag{2}$$

while λ and C are either rational or (linear combinations of) quadratic irrationalities. All such equalities from [16] are compactly listed in Table 1 in Section 2.

The binomial sums (2) can be expressed via the classical Legendre polynomials

$$P_n(x) = {}_2F_1\left(\begin{matrix} -n,\ n+1 \\ 1 \end{matrix} \,\middle|\, \frac{1-x}{2}\right)$$

by means of the formula

$$T_n(b,c) = (b^2-4c)^{n/2} P_n\left(\frac{b}{(b^2-4c)^{1/2}}\right),$$

so that equalities (1) assume the form

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2}(A+Bn)P_n(x_0)z_0^n = \frac{C}{\pi}. \tag{3}$$

Here and throughout the paper we use a standard notation for hypergeometric series,

$${}_mF_{m-1}\left(\begin{matrix} a_1,\ a_2,\ \dots,\ a_m \\ b_2,\ \dots,\ b_m \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_m)_n}{(b_2)_n\cdots(b_m)_n} \frac{z^n}{n!},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol.

The sequence of Legendre polynomials can be alternatively defined by the ordinary generating function

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)z^n.$$

In the rest of the paper, we will make heavy use of another generating function for the Legendre polynomials due to F. Brafman. This and our general approach is described in Section 2. In Sections 3, 4, 5 and 6, we will examine the conjectures for $s = 1/2, 1/3, 1/4$ and $1/6$, respectively, and indicate six new identities (46)–(51) for $s = 1/4$ and $1/6$. Then in Sections 7 and 8 we show that "companion series" involving derivatives of Legendre polynomials can be obtained, and some of them, as well as a few series examined in the previous sections, are expressible in terms of known constants.

Our main result is the following, which we prove in Section 2:

THEOREM 1: *All the series for* $1/\pi$ *listed in Table* 1 *are true.*

2. Brafman's formula and modular equations

In [10], Brafman proved the following elegant hypergeometric formula for a generating function of the Legendre polynomials.

PROPOSITION 1 (Brafman's formula [10]):

$$(4) \quad \sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} P_n(x) z^n = {}_2F_1\left(\begin{matrix} s,\ 1-s \\ 1 \end{matrix} \,\middle|\, \frac{1-\rho-z}{2}\right) \cdot {}_2F_1\left(\begin{matrix} s,\ 1-s \\ 1 \end{matrix} \,\middle|\, \frac{1-\rho+z}{2}\right),$$

where $\rho = \rho(x,z) := (1-2xz+z^2)^{1/2}$.

This result, in an even more general form involving Jacobi polynomials, is a consequence of Bailey's identity for a special case of Appell's hypergeometric function of the fourth type [1, Section 9.6]. In [17] we present a generalization of Bailey's identity and follow the lines of Brafman's derivation to prove a new type of generating functions of the Legendre polynomials.

By introducing the compact notation for the involved hypergeometric function and its derivative,

$$(5) \qquad F(t) = F(s,t) := {}_2F_1\left(\begin{matrix} s,\ 1-s \\ 1 \end{matrix} \,\middle|\, t\right), \quad G(t) = G(s,t) := t\,\frac{\mathrm{d}}{\mathrm{d}t}F(t),$$

and differentiating both sides of (4) with respect to z, we immediately deduce

Proposition 2:

$$\sum_{n=0}^{\infty}\frac{(s)_n(1-s)_n}{n!^2}P_n(x)z^n = F(t_-)F(t_+), \tag{6}$$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(s)_n(1-s)_n}{n!^2}nP_n(x)z^n &= \frac{z(x-z-\rho)}{\rho(1-\rho-z)}\,G(t_-)F(t_+)\\ &\quad+\frac{z(x-z+\rho)}{\rho(1-\rho+z)}\,F(t_-)G(t_+),\end{aligned} \tag{7}$$

where $t_\pm = t_\pm(x,z) := (1-\rho\pm z)/2$.

For $s\in\{1/2,1/3,1/4,1/6\}$ the right-hand side of Brafman's formula represents the product of two arithmetic hypergeometric series: the modular functions

$$\begin{aligned} t_4(\tau) &= \left(1+\frac{1}{16}\left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8\right)^{-1}, \quad t_3(\tau) = \left(1+\frac{1}{27}\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12}\right)^{-1},\\ t_2(\tau) &= \left(1+\frac{1}{64}\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24}\right)^{-1}, \quad t_1(\tau) = \frac{1}{2}-\frac{1}{2}\sqrt{1-\frac{1728}{j(\tau)}} \end{aligned} \tag{8}$$

(with subscripts denoting the levels) translate the respective series $F(t)$ into a weight 1 modular form $F(t(\tau))$. Here $\eta(\tau)$ and $j(\tau)$ are classical Dedekind's eta function and the modular invariant, respectively. For the rest of the paper we will omit the subscript in $t_\ell(\tau)$ when the modular function used is clear from the context. The inversion formula is given [4, p. 91] by

$$\tau = iC_s\,\frac{F(1-t)}{F(t)}, \quad \text{where } C_s = \frac{1}{2\sin\pi s} = \begin{cases} \frac{1}{2} & \text{if } s=\frac{1}{2},\\ \frac{1}{\sqrt{3}} & \text{if } s=\frac{1}{3},\\ \frac{1}{\sqrt{2}} & \text{if } s=\frac{1}{4},\\ 1 & \text{if } s=\frac{1}{6}. \end{cases} \tag{9}$$

The elliptic nome is defined throughout the paper as $q=e^{2\pi i\tau}$. Note that for any of the four modular functions in (8) we have

$$\frac{1}{2\pi i}\,\frac{\mathrm{d}t}{\mathrm{d}\tau} = q\frac{\mathrm{d}t}{\mathrm{d}q} = t(1-t)\,F^2(t), \tag{10}$$

the result already known to Ramanujan [4, Chap. 33], [5], [13].

Table 1. Identities (3), and the corresponding choice of τ_0 and N such that $(1-\rho_0-z_0)/2 = t(\tau_0)$ and $(1-\rho_0+z_0)/2 = t(\tau_0/N)$ or $1-t(\tau_0/N)$ (the latter option is for entries marked by asterisk).

# in [16]	s	x_0	z_0	ρ_0	A	B	C	τ_0	N
(I1)*	1/2	$\frac{-i}{3\sqrt{7}}$	$\frac{-3i\sqrt{7}}{16}$	$\frac{15}{16}$	7	30	24	$\frac{i\sqrt{7}+1}{4}$	2
(I2)	1/2	$\frac{17}{12\sqrt{2}}$	$\frac{-3}{4\sqrt{2}}$	$\frac{5\sqrt{3}}{4\sqrt{2}}$	7	30	12	$\frac{2i\sqrt{3}-3}{2}$	3
(I3)	1/2	$\frac{97}{56\sqrt{3}}$	$\frac{7\sqrt{3}}{16}$	$\frac{\sqrt{15}}{16}$	-1	30	80	$\frac{i\sqrt{15}}{2}$	5
(I4)	1/2	$\frac{31}{8\sqrt{15}}$	$\frac{\sqrt{15}}{16}$	$\frac{7\sqrt{3}}{16}$	5	42	$16\sqrt{3}$	$\frac{i\sqrt{15}}{2}$	3
(II1)	1/3	$\frac{3\sqrt{3}}{5}$	$\frac{5}{6\sqrt{3}}$	$\frac{5}{6\sqrt{3}}$	2	15	$\frac{45\sqrt{3}}{4}$	$\frac{2i}{\sqrt{3}}$	2
(II2)	1/3	$\frac{5}{2\sqrt{6}}$	$\frac{27\sqrt{3}}{125\sqrt{2}}$	$\frac{91}{125}$	12	91	$\frac{75\sqrt{3}}{2}$	$i\sqrt{2}$	2
(II3)	1/3	$\frac{99}{70\sqrt{2}}$	$\frac{35}{27\sqrt{2}}$	$\frac{\sqrt{5}}{27}$	-4	15	$\frac{135\sqrt{3}}{2}$	$\frac{i\sqrt{10}}{\sqrt{3}}$	5
(II4)	1/3	$\frac{485}{198\sqrt{6}}$	$\frac{99\sqrt{3}}{125\sqrt{2}}$	$\frac{\sqrt{14}}{125}$	-41	42	$525\sqrt{3}$	$\frac{i\sqrt{14}}{\sqrt{3}}$	7
(II5)	1/3	$\frac{365}{364}$	$\frac{91}{125}$	$\frac{27\sqrt{3}}{125\sqrt{2}}$	1	18	$25\sqrt{3}$	$i\sqrt{2}$	3
(II6)	1/3	$\frac{51}{10\sqrt{26}}$	$\frac{5\sqrt{13}}{4913\sqrt{2}}$	$\frac{3465\sqrt{2}}{4913}$	559	6930	$\frac{1445\sqrt{6}}{2}$	$\frac{i\sqrt{26}}{\sqrt{3}}$	2
(II7)	1/3	$\frac{99}{70\sqrt{2}}$	$\frac{35}{35937\sqrt{2}}$	$\frac{8710\sqrt{17}}{35937}$	15724	222105	$\frac{114345\sqrt{3}}{4}$	$\frac{i\sqrt{34}}{\sqrt{3}}$	2
(II8)	1/3	$\frac{19601}{13860\sqrt{2}}$	$\frac{3465\sqrt{2}}{4913}$	$\frac{5\sqrt{13}}{4913\sqrt{2}}$	-3967	390	$56355\sqrt{3}$	$\frac{i\sqrt{26}}{\sqrt{3}}$	13
(II9)	1/3	$\frac{143649}{34840\sqrt{17}}$	$\frac{8710\sqrt{17}}{35937}$	$\frac{35}{35937\sqrt{2}}$	-7157	210	$114345\sqrt{3}$	$\frac{i\sqrt{34}}{\sqrt{3}}$	17
(II10)*	1/3	$\frac{-13i}{4\sqrt{35}}$	$\frac{i\sqrt{35}}{64}$	$\frac{27\sqrt{5}}{64}$	7	45	$\frac{8(3+\sqrt{5})}{\sqrt{3}}$	$\frac{i\sqrt{35}-1}{6}$	3
(II11)*	1/3	$\frac{-7i\sqrt{5}}{22}$	$\frac{-11i}{10\sqrt{5}}$	$\frac{27}{10\sqrt{5}}$	2	9	$\frac{15+\sqrt{5}}{2\sqrt{3}}$	$\frac{i\sqrt{5}+1}{3}$	2
(II12)*	1/3	$\frac{-10i\sqrt{2}}{23}$	$\frac{-23i}{125\sqrt{2}}$	$\frac{189}{125\sqrt{2}}$	11	63	$\frac{25(3+4\sqrt{2})}{4\sqrt{3}}$	$\frac{i\sqrt{8}+1}{3}$	3
(A1)	1/3	$\frac{9}{4\sqrt{5}}$	$\frac{\sqrt{5}}{27}$	$\frac{35}{27\sqrt{2}}$	5	42	$\frac{54\sqrt{3}}{5}$	$\frac{i\sqrt{10}}{\sqrt{3}}$	2
(A2)	1/3	$\frac{15}{4\sqrt{14}}$	$\frac{\sqrt{14}}{125}$	$\frac{99\sqrt{3}}{125\sqrt{2}}$	7	66	$\frac{50\sqrt{2}}{3}$	$\frac{i\sqrt{14}}{\sqrt{3}}$	2
(III1)	1/4	$\frac{52}{30\sqrt{3}}$	$\frac{160}{121\sqrt{3}}$	$\frac{85}{363}$	2	85	$33\sqrt{33}$	$i\sqrt{3}$	3
(III2)	1/4	$\frac{55}{12\sqrt{21}}$	$\frac{-\sqrt{21}}{6}$	$\frac{2\sqrt{7}}{3}$	5	28	$3\sqrt{6}$	$\frac{i\sqrt{21}+3}{2}$	3
(III3)	1/4	$\frac{49}{20\sqrt{6}}$	$\frac{10\sqrt{6}}{49}$	$\frac{10\sqrt{6}}{49}$	3	40	$\frac{70\sqrt{21}}{9}$	$\frac{3i}{\sqrt{2}}$	3
(III4)	1/4	$\frac{257}{255}$	$\frac{85}{363}$	$\frac{160}{121\sqrt{3}}$	9	80	$\frac{11\sqrt{66}}{2}$	$i\sqrt{3}$	2
(III5)*	1/4	$\frac{-7i}{33\sqrt{15}}$	$\frac{-11i\sqrt{15}}{147}$	$\frac{64\sqrt{5}}{147}$	13	80	$\frac{7\sqrt{42}(3+2\sqrt{5})}{8}$	$\frac{i\sqrt{15}+1}{4}$	2

When τ is a quadratic irrationality (with $\operatorname{Im}\tau > 0$), the value $t(\tau)$ is known to be an algebraic number; computation of such values is well discussed in the literature — see, for example, [4, Chap. 34]. A common feature of the Sun–Almkvist series (3) from [16] for $s \in \{1/2, 1/3, 1/4\}$ is that the algebraic numbers

$$(11)\quad \alpha = \frac{1-\rho_0-z_0}{2} \quad\text{and}\quad \beta = \frac{1-\rho_0+z_0}{2}, \quad \text{where } \rho_0 := (1-2x_0z_0+z_0^2)^{1/2},$$

are always values of the modular function $t(\tau)$ at two quadratic irrational points. In cases when x_0 and z_0 are real, we get $\alpha = t(\tau_0)$ and $\beta = t(\tau_0/N)$; while in cases when both x_0 and z_0 are purely imaginary (and there are five such cases in Table 1 marked by asterisk), we have $\alpha = t(\tau_0)$ and $\beta = 1 - t(\tau_0/N)$. The corresponding choice of quadratic irrational τ_0 and integer $N > 1$ is given in Table 1. We also note that $|\alpha| \le |\beta|$ for all entries, with the strict inequality when both x_0 and z_0 are real.

Remark 1: Observe the duality between several entries in Table 1, where the roles of z_0 and ρ_0 are swapped. These correspond to the same choice of τ_0 with different choices of N, which is often a prime factor of an integer inside the radical in τ_0.

Proposition 3: *In notation* (11), *assume that both α and β are within the convergence domain of the hypergeometric function $F(t)$ (that is, $|\alpha|, |\beta| < 1$).*

(a) *Suppose that $\alpha = t(\tau_0)$ and $\beta = t(\tau_0/N)$ for a quadratic irrational τ_0 and an integer $N > 1$. Then there exist effectively computable algebraic numbers μ_0, λ_0 and λ_1 such that*

$$(12)\qquad F(\beta) = \mu_0 F(\alpha) \quad \textit{and} \quad G(\beta) = \lambda_0 F(\alpha) + \lambda_1 G(\alpha).$$

(b) *Suppose that $\alpha = t(\tau_0)$ and $\beta = 1 - t(\tau_0/N)$ for a quadratic irrational τ_0 and an integer $N > 1$. In addition, assume that $|1-\beta| < 1$. Then there exist effectively computable algebraic numbers μ_0, λ_0, λ_1 and λ_2 such that*

$$(13)\qquad F(\beta) = \mu_0 F(\alpha) \quad \textit{and} \quad G(\beta) = \lambda_0 F(\alpha) + \lambda_1 G(\alpha) + \frac{\lambda_2}{\pi F(\alpha)}.$$

Proof. (a) For N given, the two modular functions $t(\tau)$ and $t(\tau/N)$ are related by the modular equation of degree N; in particular, the function $t(\tau/N)$ is an algebraic function of $t(\tau)$. As both $F(t(\tau))$ and $F(t(\tau/N))$ are weight 1 modular forms, their quotient $F(t(\tau/N))/F(t(\tau))$ is a modular function, hence is an algebraic function of $t(\tau)$. The quotient specialized at $\tau = \tau_0$ is then an algebraic number, which we denote by μ_0.

Differentiating $F(t(\tau/N))/F(t(\tau))$ logarithmically and multiplying the result by $F^2(t(\tau))$, we arrive at a relation expressing $G(t(\tau/N))$ linearly via $F(t(\tau))$ and $G(t(\tau))$ with coefficients which are modular functions. Specializing at $\tau = \tau_0$ this yields the second equality in (12) with algebraic λ_0 and λ_1.

(b) Consider now $\beta = 1 - \beta'$ where $\beta' = t(\tau_0/N)$. By what is shown in part (a),

$$F(\beta') = \mu_0' F(\alpha) \quad \text{and} \quad G(\beta') = \lambda_0' F(\alpha) + \lambda_1' G(\alpha) \tag{14}$$

for certain algebraic μ_0', λ_0' and λ_1'. Relation (9) implies that

$$\frac{F(1-t)}{F(t)} = -\frac{i\tau}{C_s}, \tag{15}$$

which specialized to $\tau = \tau_0/N$, hence $t = \beta'$, results in

$$F(\beta) = -\frac{i\tau_0}{NC_s} F(\beta'). \tag{16}$$

Computing the logarithmic t-derivative of (15) and using (5), we find

$$\frac{tG(1-t)}{F(1-t)} + \frac{(1-t)G(t)}{F(t)} = -\frac{t(1-t)}{\tau}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1} = \frac{it(1-t)F(t)}{C_s F(1-t)}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1},$$

which, after multiplication by $F(1-t)/t$ and using (15), can be written as

$$G(1-t) = \frac{i\tau(1-t)}{C_s t} G(t) + \frac{i(1-t)}{C_s} F(t)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^{-1}. \tag{17}$$

Using now (10) and taking $\tau = \tau_0/N$ (so that $t = t(\tau_0/N) = \beta'$) in (17) we obtain

$$G(\beta) = \frac{i\tau_0\beta}{NC_s(1-\beta)} G(\beta') + \frac{1}{2\pi C_s(1-\beta)F(\beta')}. \tag{18}$$

Combining now (14), (16) and (18) we arrive at (13).

Finally note that all the above algebraicity is effectively computed by means of the involved modular equations. ∎

Now we appeal to a particular case of Clausen's formula (1828),

$${}_2F_1\left(\begin{matrix} s,\, 1-s \\ 1 \end{matrix} \,\middle|\, t\right)^2 = {}_3F_2\left(\begin{matrix} \frac12,\, s,\, 1-s \\ 1,\, 1 \end{matrix} \,\middle|\, 4t(1-t)\right), \tag{19}$$

which is valid for t within the left half of the lemniscate $4|t(1-t)|=1$. Differentiating (19) and expanding the ${}_3F_2$ hypergeometric function into series, we obtain

PROPOSITION 4: *For t satisfying $|t(1-t)| \le 1/4$ and $\operatorname{Re} t < 1/2$,*

$$F^2(t) = \sum_{n=0}^{\infty} \frac{(\frac12)_n (s)_n (1-s)_n}{n!^3} \big(4t(1-t)\big)^n,$$

$$F(t)G(t) = \frac{1-2t}{2(1-t)} \sum_{n=0}^{\infty} \frac{(\frac12)_n (s)_n (1-s)_n}{n!^3} \cdot n\big(4t(1-t)\big)^n.$$

Our final argument goes back to Ramanujan's discovery [15] of hypergeometric formulas for $1/\pi$. Its proof is outlined in [6], [7], [8] and [12].

PROPOSITION 5: *Let α be the value of the modular function $t(\tau)$ at a quadratic irrationality τ_0. Assume that $|\alpha(1-\alpha)| \le 1/4$ and $\operatorname{Re}\alpha < 1/2$. Then there exist effectively computable algebraic constants a, b and c such that*

$$\sum_{n=0}^{\infty} \frac{(\frac12)_n (s)_n (1-s)_n}{n!^3} (a+bn)\big(4\alpha(1-\alpha)\big)^n = \frac{c}{\pi}. \tag{20}$$

Remark 2: Observe that all the values $\alpha = (1-\rho_0-z_0)/2$ from Table 1 satisfy the hypothesis of Proposition 5, with the exception of (II11) which we treat separately in Section 4.

Proof of Theorem 1. For a given entry from Table 1, we choose $\alpha = (1-\rho_0-z_0)/2 = t(\tau_0)$ and $\beta = (1-\rho_0+z_0)/2$. Proposition 5 implies that we have a Ramanujan series (20). On invoking Proposition 4 for $t=\alpha$ we can write (20) in the form

$$aF^2(\alpha) + 2b\frac{1-\alpha}{1-2\alpha}F(\alpha)G(\alpha) = \frac{c}{\pi}. \tag{21}$$

On the other hand, by specializing the identities in Proposition 2 at $x=x_0$, $z=z_0$ and using then the algebraic relations obtained in Proposition 3, we obtain

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} P_n(x_0) z_0^n = \mu_0 F^2(\alpha),$$

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} nP_n(x_0) z_0^n = \lambda_0' F^2(\alpha) + \lambda_1' F(\alpha)G(\alpha) + \frac{\lambda_2'}{\pi},$$

with some algebraic (effectively computable) coefficients μ_0, λ_0', λ_1' and λ_2', where we simply choose $\lambda_2' = 0$ if $\beta = t(\tau_0/N)$.

Finally, taking

$$B' = \frac{2b(1-\alpha)}{\lambda_1'(1-2\alpha)} \quad \text{and} \quad A' = \frac{a - B'\lambda_0'}{\mu_0}$$

we derive from (21) that

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2}(A' + B'n)P_n(x_0)z_0^n = \frac{c - B'\lambda_2'}{\pi},$$

which assumes the required form (3) after stretching $A = CA'/(c - B'\lambda_2')$, $B = CB'/(c - B'\lambda_2')$. ∎

As verification of each entry in the table requires an explicit knowledge of all algebraic numbers involved and is therefore tedious, we give details for only some of the entries. In Section 3 we discuss in detail identity (I2) by using a parametrization of the corresponding modular equation. Section 4 describes the techniques without using an explicit parametrization on an example of identity (II1), and uses a hypergeometric transformation to treat (II11), an entry that does not satisfy the conditions of Proposition 5. Section 5 explains the derivation of identity (III5), which corresponds to imaginary x_0 and z_0, as well as outlines new identities for $s = 1/4$. In Section 6 we present two identities corresponding to $s = 1/6$, which are not from the list in [16].

3. Identities for $s = 1/2$

We illustrate our techniques outlined in Section 2 with (I2),

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2}(7 + 30n)P_n\left(\frac{17}{12\sqrt{2}}\right)\left(-\frac{3}{4\sqrt{2}}\right)^n = \frac{12}{\pi}.$$

Here we have $N = 3$, so that the values $\alpha = t(\tau_0) = (1 - \rho_0 - z_0)/2$ and $\beta = t(\tau_0/3) = (1 - \rho_0 + z_0)/2$ are related by the modular polynomial [3, Chap. 19]

$$(\alpha^2 + \beta^2 + 6\alpha\beta)^2 - 16\alpha\beta\big(4(1+\alpha\beta) - 3(\alpha+\beta)\big)^2 = 0$$

and admit the rational parametrization

$$\begin{aligned} \alpha &= \frac{4\sqrt{2}-5\sqrt{3}+3}{8\sqrt{2}} = \frac{p^3(2+p)}{1+2p}, \\ \beta &= \frac{4\sqrt{2}-5\sqrt{3}-3}{8\sqrt{2}} = \frac{p(2+p)^3}{(1+2p)^3}, \end{aligned} \qquad p = \frac{3-\sqrt{2}-\sqrt{3}}{2\sqrt{2}}.$$

In notation (5), recall now the identity [3, p. 238, Entry 6 (i)]

$$F\left(\frac{p(2+p)^3}{(1+2p)^3}\right) = (1+2p)F\left(\frac{p^3(2+p)}{1+2p}\right) \quad \text{for } p \in \left(-\frac{1}{2}, 1\right); \tag{22}$$

differentiating it we obtain

$$\begin{aligned} G\left(\frac{p(2+p)^3}{(1+2p)^3}\right) =& \frac{p(1+2p)(2+p)}{(1-p)^2}\, F\left(\frac{p^3(2+p)}{1+2p}\right) \\ &+ \frac{3(1+p)^2(1+2p)}{(1-p)^2}\, G\left(\frac{p^3(2+p)}{1+2p}\right). \end{aligned} \tag{23}$$

Substituting $p = (3-\sqrt{2}-\sqrt{3})/(2\sqrt{2})$ into (22) and (23) we obtain

$$\begin{aligned} F(\beta) =& \frac{-\sqrt{6}+3\sqrt{2}}{2}\, F(\alpha), \\ G(\beta) =& -\frac{85\sqrt{6}+120\sqrt{3}-147\sqrt{2}-208}{2}\, F(\alpha) \\ &+ \frac{(-19\sqrt{3}+33)(17\sqrt{2}+24)}{2}\, G(\alpha). \end{aligned}$$

Specializing (6), (7) by taking $x = 17/(12\sqrt{2})$, $z = -3/(4\sqrt{2})$ we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2}(A+Bn)P_n\left(\frac{17}{12\sqrt{2}}\right)\left(\frac{-3}{4\sqrt{2}}\right)^n =& \sqrt{6}\left(\frac{\sqrt{3}-1}{2}A - \frac{B}{30}\right)F^2(\alpha) \\ &+ \frac{15\sqrt{2}+8\sqrt{3}-3\sqrt{6}}{10} BF(\alpha)G(\alpha). \end{aligned}$$

In turn, the choice $A = 7$ and $B = 30$, Clausen's formula (19) (Proposition 4) and

$$4t(1-t)\big|_{t=(4\sqrt{2}-5\sqrt{3}+3)/(8\sqrt{2})} = -\frac{(\sqrt{3}-1)^6}{2^7}$$

imply

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^2}{n!^2}(7+30n)P_n\left(\frac{17}{12\sqrt{2}}\right)\left(\frac{-3}{4\sqrt{2}}\right)^n$$
$$=\frac{\sqrt{6}(7\sqrt{3}-9)}{2}\,{}_3F_2\left(\begin{matrix}\frac12,\,\frac12,\,\frac12\\ 1,\,1\end{matrix}\,\middle|\,\frac{-(\sqrt{3}-1)^6}{2^7}\right)$$
$$+\frac{9\sqrt{2}(101\sqrt{3}-175)}{128}\,{}_3F_2\left(\begin{matrix}\frac32,\,\frac32,\,\frac32\\ 2,\,2\end{matrix}\,\middle|\,\frac{-(\sqrt{3}-1)^6}{2^7}\right),$$

which is precisely $3/\sqrt{2}$ times the Ramanujan-type formula [2, eq. (8.3)]

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^3}{n!^3}\big(7-3\sqrt{3}+6(5-\sqrt{3})n\big)\frac{(-1)^n(\sqrt{3}-1)^{6n}}{2^{7n}}=\frac{4\sqrt{2}}{\pi}.$$

The derivation of (I4) is very similar, as the degree N is also 3 in this case (although we have to swap the rational p-parametric expressions of α and β). The choice of the parameter in the above rational parametrization is $p = -(2+\sqrt{3}+\sqrt{15})/4$, and the transformation (22) assumes the form

$$F\left(\frac{p(2+p)^3}{(1+2p)^3}\right)=-\frac{1+2p}{3}\,F\left(\frac{p^3(2+p)}{1+2p}\right)\quad \text{for } p\in(-\infty,-1).$$

This in fact follows from (22) by a change of variables, then by applying to both sides a transformation of the complete elliptic integral K (as $K(t)=\pi F(t^2)/2$),

$$K(x)=\frac{1}{\sqrt{1-x^2}}\,K\left(\sqrt{\frac{x^2}{x^2-1}}\right), \tag{24}$$

itself a result of Euler's hypergeometric transformation [1, Section 1.2, eq. (2)].

Finally, (I4) reduces to Ramanujan's identity [15, eq. (30)]

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^3}{n!^3}\big(5\sqrt{5}-1+6(7\sqrt{5}+5)n\big)\frac{(\sqrt{5}-1)^{8n}}{2^{14n}}=\frac{32}{\pi}.$$

For (I1) and (I3) we have to use the modular equations of degree 2 and 5, respectively [3, Chap. 19]; the corresponding "complex" Ramanujan-type series for $1/\pi$ required in the derivation of (I1) can be found in [14, Section 4].

4. Identities for $s = 1/3$

In this section we first prove (II1),

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2}(2+15n)P_n\left(\frac{3\sqrt{3}}{5}\right)\left(\frac{5}{6\sqrt{3}}\right)^n = \frac{45\sqrt{3}}{4\pi},$$

which is representative of identities in the large group for $s = 1/3$ in Table 1. Here

$$\alpha = \frac{1-\rho_0-z_0}{2} = \frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right)^3 \quad \text{and} \quad \beta = \frac{1-\rho_0+z_0}{2} = \frac{1}{2} \tag{25}$$

satisfy the modular equation of degree 2 in signature 3. Although a rational parametrization similar to the one we exploited in Section 3 exists, we will compute the algebraic relations of Proposition 3 by using the modular equation itself

$$(\alpha\beta)^{1/3} + ((1-\alpha)(1-\beta))^{1/3} = 1 \tag{26}$$

as well as the equation for the corresponding multiplier [4, p. 120, Theorem 7.1]

$$m = \frac{F(\beta)}{F(\alpha)} = \frac{(1-\alpha)^{2/3}}{(1-\beta)^{1/3}} - \frac{\alpha^{2/3}}{\beta^{1/3}}, \tag{27}$$

where $\alpha = \alpha(\tau) = t(\tau)$ has degree 2 over $\beta = \beta(\tau) = t(\tau/2)$.

On specializing (27) by taking $\tau = \tau_0$, we get

$$F\left(\frac{1}{2}\right) = \frac{2}{\sqrt{3}}\, F(\alpha)\big|_{\alpha=(1-1/\sqrt{3})^3/4}. \tag{28}$$

Computing the logarithmic t-derivative of (27) at $t = \alpha$, and using the notation of (5) result in

$$\begin{aligned}\frac{G(\beta)}{\beta F(\beta)}\,\frac{\mathrm{d}\beta}{\mathrm{d}\alpha} - \frac{G(\alpha)}{\alpha F(\alpha)} &= \frac{1}{m}\,\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{(1-\alpha)^{2/3}}{(1-\beta)^{1/3}} - \frac{\alpha^{2/3}}{\beta^{1/3}}\right)\\ &= \frac{F(\alpha)}{F(\beta)}\,\frac{1}{3}\left(\frac{\mathrm{d}\beta}{\mathrm{d}\alpha}\left(\frac{(1-\alpha)^{2/3}}{(1-\beta)^{4/3}} + \frac{\alpha^{2/3}}{\beta^{4/3}}\right)\right.\\ &\qquad \left. - \frac{2}{(1-\alpha)^{1/3}(1-\beta)^{1/3}} - \frac{2}{\alpha^{1/3}\beta^{1/3}}\right).\end{aligned} \tag{29}$$

The derivative $\mathrm{d}\beta/\mathrm{d}\alpha$ can be obtained by differentiating (26),

$$\frac{\mathrm{d}\beta}{\mathrm{d}\alpha}\left(\frac{\alpha^{1/3}}{\beta^{2/3}} - \frac{(1-\alpha)^{1/3}}{(1-\beta)^{2/3}}\right) + \frac{\beta^{1/3}}{\alpha^{2/3}} - \frac{(1-\beta)^{1/3}}{(1-\alpha)^{2/3}} = 0,$$

so that

$$\frac{\mathrm{d}\beta}{\mathrm{d}\alpha}\bigg|_{\tau=\tau_0} = 9.$$

Thus, with the choice $\tau = \tau_0$ in (29), we obtain

$$G\left(\frac{1}{2}\right) = \left(\frac{2}{9}\,F(\alpha) + \frac{3\sqrt{3}+5}{3}\,G(\alpha)\right)\bigg|_{\alpha=(1-1/\sqrt{3})^3/4}. \tag{30}$$

From now on we fix α and β as defined in (25). With the help of Proposition 2 and (28), (30) we find that

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2} P_n(x_0)z_0^n = F(\alpha)F(\beta) = \frac{2}{\sqrt{3}}\,F^2(\alpha),$$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2}\, nP_n(x_0)z_0^n &= \frac{3\sqrt{3}+5}{5}\,G(\alpha)F(\beta) + \frac{3\sqrt{3}}{5}\,F(\alpha)G(\beta)\\ &= \frac{2}{5\sqrt{3}}\,F^2(\alpha) + \frac{3\sqrt{3}+5}{\sqrt{3}}\,F(\alpha)G(\alpha).\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2}(2+15n)P_n(x_0)z_0^n &= \frac{10}{\sqrt{3}}\,F^2(\alpha) + \frac{15(3\sqrt{3}+5)}{\sqrt{3}}\,F(\alpha)G(\alpha)\\ &= \frac{5}{\sqrt{3}}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{n!^3}(2+15n)\big(4\alpha(1-\alpha)\big)^n,\end{aligned}$$

while the latter is a multiple of Ramanujan's series [15, eq. (31)]

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{3})_n(\frac{2}{3})_n}{n!^3}(2+15n)\left(\frac{2}{27}\right)^n = \frac{27}{4\pi},$$

and identity (II1) follows.

Remark 3: In Section 8 we show that in the discussed example we have closed form evaluations of $F(1/2)$ and $G(1/2)$, hence of

$$F(\alpha) = \frac{\sqrt{3}}{2}\,F\left(\frac{1}{2}\right), \quad G(\alpha) = \frac{5\sqrt{3}-9}{6}\,F\left(\frac{1}{2}\right) + \frac{9\sqrt{3}-15}{2}\,G\left(\frac{1}{2}\right) \tag{31}$$

(the relations follow from (28) and (30)). In particular, this gives a different way of deducing (II1), avoiding use of a Ramanujan-type series (Propositions 4 and 5).

We now turn our attention to (II11), shown below, for which $4|\alpha(1-\alpha)|>1$ and thus does not satisfy the conditions of Proposition 5. Our method employed is illustrative in dealing with more general situations when this occurs. It is also worth noting that this approach bypasses the computational difficulties encountered with purely imaginary x_0 and z_0 (see Section 5), as is the case here.

We are required to prove

$$\sum_{n=0}^{\infty}\frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2}(2+9n)P_n\left(\frac{-7i\sqrt{5}}{22}\right)\left(\frac{-11i}{10\sqrt{5}}\right)^n=\frac{15\sqrt{3}+\sqrt{15}}{6\pi},$$

with $\alpha=(10\sqrt{5}-27+11i)/(20\sqrt{5})$. We now take

$$p_0=\frac{(1+\sqrt{5})\sqrt{\sqrt{5}-2}-\sqrt{22-10\sqrt{5}}}{4}-\frac{1}{2},$$

and apply the transformation [4, p. 112, Theorem 5.6]

$$ {}_2F_1\left(\begin{matrix}\frac{1}{3},\ \frac{2}{3}\\ 1\end{matrix}\,\middle|\,\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right)=\frac{1+p+p^2}{\sqrt{1+2p}}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 1\end{matrix}\,\middle|\,\frac{p^3(2+p)}{1+2p}\right), \tag{32}$$

which is valid for real $p\in[0,1)$. By analytic continuation, the transformation remains valid in a domain surrounding the origin in which the absolute values of the arguments of both hypergeometric functions are less than 1; in particular, this domain contains p_0 and its conjugate $\overline{p_0}$.

In notation

$$\widetilde{F}(t):={}_2F_1\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2}\\ 1\end{matrix}\,\middle|\,t\right),\quad \widetilde{G}(t):=t\,\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{F}(t),$$

transformation (32) at $p=p_0$ gives us

$$F(\alpha)=\frac{(2\sqrt{5}-1+(32-14\sqrt{5})i)^{1/4}}{\sqrt{2}}\,\widetilde{F}(\alpha_0), \tag{33}$$

where $\alpha_0=1/2-\sqrt{\sqrt{5}-2}$ is real. Moreover, as β is the conjugate of α, it easily follows that at $p=\overline{p_0}$,

$$F(\beta)=\frac{(2\sqrt{5}-1-(32-14\sqrt{5})i)^{1/4}}{\sqrt{2}}\,\widetilde{F}(\alpha_0). \tag{34}$$

Therefore, $F(\alpha)$ and $F(\beta)$ are both algebraic multiples of $\widetilde{F}(\alpha_0)$, and we have transposed the problem to a simpler one in signature 2 with real argument. It remains to express $G(\alpha)$ and $G(\beta)$ in terms of $\widetilde{F}(\alpha_0)$ and $\widetilde{G}(\alpha_0)$.

To this end, we differentiate (32) with respect to p, and obtain

$$\begin{aligned}(35)\quad &\frac{(1-p)(2+p)(1+2p)^{5/2}}{(1+p)(1+p+p^2)}\,G\left(\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right)\\ &\quad = 3p^2(1+p)\,\widetilde{F}\left(\frac{p^3(2+p)}{1+2p}\right)+\frac{6(1+p)^2(1+p+p^2)}{2+p}\,\widetilde{G}\left(\frac{p^3(2+p)}{1+2p}\right).\end{aligned}$$

Substituting p_0 and its complex conjugate $\overline{p_0}$, respectively, into (35) simplifies both $G(\alpha)$ and $G(\beta)$ in terms of the desired functions. Armed with this knowledge as well as with (33) and (34), we can use Proposition 2 to obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(\frac13)_n(\frac23)_n}{n!^2}(2+9n)P_n\left(\frac{-7i\sqrt5}{22}\right)\left(\frac{-11i}{10\sqrt5}\right)^n\\ &=\frac{\sqrt5\,\sqrt{41\sqrt5-89}}{\sqrt6}\,\widetilde{F}^2(\alpha_0)+\left(\frac{5(\sqrt3+3\sqrt5)}{6}+\frac{10\sqrt{17\sqrt5-31}}{\sqrt6}\right)\widetilde{F}(\alpha_0)\widetilde{G}(\alpha_0).\end{aligned}$$

This now satisfies the conditions of Proposition 4 with $s=1/2$, and the truth of (II11) is reduced to that of a classical Ramanujan series

$$(36)\qquad \sum_{n=0}^{\infty}\frac{(\frac12)_n^3}{n!^3}(5-\sqrt5+20n)\left(\frac{\sqrt5-1}{2}\right)^{6n}=\frac{2\sqrt5}{\pi}\sqrt{2+\sqrt5},$$

as $4\alpha_0(1-\alpha_0)=((\sqrt5-1)/2)^6$ — we comment on this remarkable numerical coincidence in Section 8.

5. Identities for $s=1/4$

In this section we choose to prove identity (III5),

$$\sum_{n=0}^{\infty}\frac{(\frac14)_n(\frac34)_n}{n!^2}(13+80n)P_n\left(-\frac{7i}{33\sqrt{15}}\right)\left(-\frac{11i\sqrt{15}}{147}\right)^n=\frac{7\sqrt{42}(3+2\sqrt5)}{8\pi},$$

and our new "rational" identity (46). However, it is worth commenting on the proof of (III3), which is very similar to the one of (II1) presented in Section 4. For (III3) we get

$$\alpha=\frac{(\sqrt6-2)^4}{2^3\cdot7^2},\quad \beta=\frac12,$$

the degree 3 modular equation reads

$$(37)\qquad (\alpha\beta)^{1/2}+((1-\alpha)(1-\beta))^{1/2}+4(\alpha\beta(1-\alpha)(1-\beta))^{1/4}=1,$$

while the underlying series

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{n!^3}(3+40n)\frac{1}{7^{4n}} = \frac{1}{3\pi\sqrt{3}}$$

is due to Ramanujan [15, eq. (42)]. A more elementary derivation of (II1), which we discuss in Section 8, is also available for (III3).

In the case of (III5), we have $\tau_0 = (1+i\sqrt{15})/4$, $N=2$,

$$\alpha = t(\tau_0) = \frac{1}{2} - \frac{32\sqrt{5}}{3\cdot 7^2} + \frac{11i\sqrt{15}}{6\cdot 7^2}, \quad \beta = \frac{1}{2} - \frac{32\sqrt{5}}{3\cdot 7^2} - \frac{11i\sqrt{15}}{6\cdot 7^2},$$

and $\beta' = 1-\beta = t(\tau_0/2)$. Note that for subsequent calculations,

$$\beta^{1/2} = \frac{4\sqrt{5}-5}{14} - i\frac{\sqrt{15}+4\sqrt{3}}{42}.$$

The degree 2 modular equation for $s = 1/4$ is [11, eq. (4.6)]

$$\alpha^{1/2}\big(1+3(1-\beta')^{1/2}\big) = 1-(1-\beta')^{1/2}, \tag{38}$$

and the multiplier is given by [11, eq. (4.5)]

$$m = \frac{F(\beta')}{F(\alpha)} = 2\big(1+3(1-\beta')^{1/2}\big)^{-1/2}. \tag{39}$$

Using (16) and (38), we can find the ratio between $F(\beta')$ and $F(\beta)$, as well as between $F(\beta')$ and $F(\alpha)$:

$$F(\beta) = \frac{3+2\sqrt{5}-(\sqrt{5}-2)\sqrt{3}i}{2\sqrt{14}}\, F(\alpha), \tag{40}$$

$$F(\beta') = \frac{2\sqrt{3}+\sqrt{15}+(2\sqrt{5}-3)i}{2\sqrt{7}}\, F(\alpha). \tag{41}$$

Relation (18) of Proposition 3 assumes the form

(42)

$$G(\beta) = \frac{(7-3\sqrt{5})(5\sqrt{15}+61i)}{128\sqrt{2}}\, G(\beta') + \frac{3(69+7\sqrt{5})+33i\sqrt{3}(15-7\sqrt{5})}{256\sqrt{2}\,\pi\, F(\beta')}.$$

It remains to express $G(\beta')$ as a linear combination of $G(\alpha)$ and $F(\alpha)$. Proceeding in a similar fashion as Section 4 (for (II1)), we differentiate both sides of (38) with respect to t at α, and obtain

$$(1+3\beta^{1/2})^2\beta^{1/2} = \big(1-\beta^{1/2}+3\alpha(1+3\beta^{1/2})\big)\frac{\mathrm{d}\beta'}{\mathrm{d}\alpha},$$

from which we easily solve for $\mathrm{d}\beta'/\mathrm{d}\alpha$; this we substitute into the next equation, obtained by differentiating both sides of (39):

$$G(\beta') = \frac{\beta' G(\alpha) F(\beta')}{\alpha F(\alpha)} \left(\frac{\mathrm{d}\beta'}{\mathrm{d}\alpha}\right)^{-1} + \frac{3\beta' F(\alpha)}{2\beta^{1/2}(1+3\beta^{1/2})^{3/2}}. \tag{43}$$

Now (43), when tidied up via (41), expresses $G(\beta')$ in terms of $G(\alpha)$ and $F(\alpha)$ as promised. Substituting the result into (42) and using (41) again, after much computational work we arrive at an expression of $G(\beta)$ in terms of $G(\alpha)$ and $F(\alpha)$:

$$\begin{aligned} G(\beta) = & \frac{3\sqrt{7}\big(23\sqrt{15} - 39\sqrt{3} - (3\sqrt{5}+1)i\big)}{256\sqrt{2}\pi\, F(\alpha)} \\ & - \frac{15 + 18\sqrt{5} + (38\sqrt{3} - 23\sqrt{15})i}{112\sqrt{14}}\, F(\alpha) \\ & - \frac{513 + 323\sqrt{5} + (153\sqrt{3} - 361\sqrt{15})i}{448\sqrt{14}}\, G(\alpha). \end{aligned} \tag{44}$$

Combining (40) and (44) with Proposition 2 allows us to invoke Proposition 4 to arrive at a series equivalent to (III5); the corresponding Ramanujan-type series and its conjugate are given by

$$\sum_{n=0}^{\infty} \frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3} \big((52 \mp 12i\sqrt{3}) + (320 \mp 55i\sqrt{3})n\big) \left(\frac{2(5 \pm i\sqrt{3})}{7\sqrt{3}}\right)^{4n} = \frac{98\sqrt{3}}{\pi}, \tag{45}$$

as we have

$$4\alpha(1-\alpha) = \left(\frac{2(5+i\sqrt{3})}{7\sqrt{3}}\right)^4$$

in this case.

Remark 4: We remark that the Ramanujan-type series (45) are rational over the ring $\mathbb{Z}[e^{2\pi i/3}]$. A possible way to establish them rests upon application of degree 2 modular equations (38), (39) with the different choice

$$\alpha = t\left(\frac{i\sqrt{15} \pm 1}{2}\right) = -\left(\frac{16 - 7\sqrt{5}}{11\sqrt{3}}\right)^2, \quad \beta' = t\left(\frac{i\sqrt{15} \pm 1}{4}\right),$$

so that α is real, and on using the real Ramanujan-type series

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}\big(1500-604\sqrt5+(6825-2240\sqrt5)n\big)(-1)^n\bigg(\frac{2(13-5\sqrt5)}{11\sqrt3}\bigg)^{4n} = \frac{121\sqrt{15}}{\pi}$$

for the argument $4\alpha(1-\alpha)$; this is very similar to what was done for (II11) in Section 4. A different approach is applying the general construction from [7].

At the end of this section we would like to present four new rational series that are analogous to (III2).

Our first series for $s=1/4$ corresponds to the choice

$$x_0=\frac{199}{60\sqrt{11}},\quad z_0=\frac{-5\sqrt{11}}{96},\quad \rho_0=\frac{65}{32\sqrt3},\quad \tau_0=\frac{i\sqrt{33}+3}{2},\quad\text{and}\quad N=3,$$

in the notation of Table 1. Then we have

$$\sum_{n=0}^{\infty}\frac{(\frac14)_n(\frac34)_n}{n!^2}(33+260n)P_n(x_0)z_0^n=\frac{32\sqrt6}{\pi},$$

or alternatively in the form involving T_n (as in (1)),

$$\sum_{n=0}^{\infty}\frac{33+260n}{(-384^2)^n}\binom{4n}{2n}\binom{2n}{n}T_n(398,1)=\frac{32\sqrt6}{\pi}. \tag{46}$$

The proof proceeds in the fashion of (II1) via the degree 3 modular equation and the multiplier in signature 4 (see [4, pp. 153–154]), and the Ramanujan-type series

$$\sum_{n=0}^{\infty}\frac{(\frac14)_n(\frac34)_n(\frac12)_n}{n!^3}\big(33\sqrt{33}-119+(260\sqrt{33}-220)n\big)\bigg(\frac{325\sqrt{33}-1867}{4608}\bigg)^n=\frac{128\sqrt3}{\pi}.$$

The three other new series are obtained by choosing

$$\tau_0\in\bigg\{\frac{i\sqrt{57}+3}{2},\frac{i\sqrt{93}+3}{2},\frac{i\sqrt{177}+3}{2}\bigg\}$$

and, again, $N=3$. They are:

$$\sum_{n=0}^{\infty}\frac{7\,331+83\,980n}{(-2\,688^2)^n}\binom{4n}{2n}\binom{2n}{n}T_n(2\,702,1)=\frac{80\sqrt{42^3}}{\pi}, \tag{47}$$

$$\sum_{n=0}^{\infty}\frac{71\,161+1\,071\,980n}{(-24\,288^2)^n}\binom{4n}{2n}\binom{2n}{n}T_n(24\,302,1)=\frac{135\sqrt{253^3}}{\pi\sqrt6}, \tag{48}$$

and

(49)
$$\sum_{n=0}^{\infty}\frac{30\,282\,753+632\,736\,260n}{(-1\,123\,584^2)^n}\binom{4n}{2n}\binom{2n}{n}T_n(1\,123\,598,1)=\frac{2944\sqrt{1463^3}}{\pi\sqrt{3}}.$$

The partial sum of (49) adds about four digits of accuracy per term.

In order to find these new series similar to (III2), we search for imaginary quadratic fields $\mathbb{Q}(\sqrt{-3\ell})$ with class number 4, where prime $\ell\equiv 3 \pmod 4$. It turns out that this is satisfied when $\ell=7$, 11, 19, 31 and 59 (this list seems exhaustive). The four new series correspond to the latter four discriminants, respectively.

Another curious observation is that, in the notation of
$$\sum_{n=0}^{\infty}\frac{A+Bn}{\Lambda^n}\binom{4n}{2n}\binom{2n}{n}T_n(b,1)=\frac{C}{\pi},$$
when $N=3$ we have $|b-|\Lambda|^{1/2}|=14$. This is observed in (III1)–(III3), as well as in (46)–(49), and in fact follows from the modular equation (37).

6. New identities for $s=1/6$

In this section, we illustrate two series corresponding to $s=1/6$, a case not considered in [16].

Our first example follows by taking $\tau_0=i\sqrt{6}$ and $N=2$. Then
$$\frac{1728}{j(\tau_0)}=\frac{1399-988\sqrt{2}}{4913}\quad\text{and}\quad\frac{1728}{j(\tau_0/2)}=\frac{1399+988\sqrt{2}}{4913},$$
and we have two Ramanujan-type series of Proposition 5,
$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac16)_n(\frac56)_n}{n!^3}\big(5+12(5\mp\sqrt{2})n\big)\left(\frac{1399\pm 988\sqrt{2}}{4913}\right)^n=\frac{3\pm 1}{2\pi}\sqrt{213\mp 24\sqrt{2}}.$$
Note that adding these two series gives a rational left-hand side. By using either of the two series, and with
$$x_0=\frac{17\sqrt{17}-46}{2\sqrt{1757-391\sqrt{17}}},\quad z_0=\frac{\sqrt{1757-391\sqrt{17}}}{17\sqrt{17}},$$
we obtain
$$\sum_{n=0}^{\infty}\frac{(\frac16)_n(\frac56)_n}{n!^2}\big(5(31+17\sqrt{17})+5928n\big)P_n(x_0)z_0^n=\frac{17\sqrt{6}}{\pi}\sqrt{1069\sqrt{17}-1683}.$$

In notation of (1), this becomes

$$\sum_{n=0}^{\infty}\binom{6n}{3n}\binom{3n}{n}\big(5(31+\sqrt{17^3})+5928n\big)T_n\big(2(\sqrt{17^3}-46),1\big)\frac{1}{(12\sqrt{17})^{3n}} = \frac{17\sqrt{6}}{\pi}\sqrt{1069\sqrt{17}-1683}. \tag{50}$$

In the second example we choose $\tau_0 = i\sqrt{7}+1$ and $N=2$, so that

$$\frac{1728}{j(\tau_0)} = \left(\frac{4}{85}\right)^3 \quad\text{and}\quad \frac{1728}{j(\tau_0/2)} = -\left(\frac{4}{5}\right)^3,$$

and the related Ramanujan-type series is

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac16)_n(\frac56)_n}{n!^3}(8+133n)\left(\frac{4}{85}\right)^{3n} = \frac{\sqrt{85^3}}{18\pi\sqrt{3}},$$

due to Ramanujan himself [15, eq. (34)]. The series and the corresponding choice

$$x_0 = \frac{323\sqrt{1785}}{13650} - \frac{\sqrt{105}}{40950}, \quad z_0 = \frac{171\sqrt{1785}}{14450} - \frac{3\sqrt{105}}{50}$$

generate the formula

$$\sum_{n=0}^{\infty}\frac{(\frac16)_n(\frac56)_n}{n!^2}(1687-15\sqrt{17^3}+6552n)P_n(x_0)z_0^n = \frac{85\sqrt{30}}{32\pi}\sqrt{19809\sqrt{17}-68425}.$$

In notation of (1), the identity can be stated in the form

$$\sum_{n=0}^{\infty}\binom{6n}{3n}\binom{3n}{n}(1687-15\sqrt{17^3}+6552n)T_n\left(\frac{10773-125\sqrt{17^3}}{32},-1\right)\frac{1}{(-15\sqrt{17})^{3n}} = \frac{85\sqrt{30}}{32\pi}\sqrt{19809\sqrt{17}-68425}. \tag{51}$$

The appearance of a negative c in (1) is not found on the list from [16].

7. Companion series

If we differentiate (4) with respect to x instead of z, a series involving the derivatives of Legendre polynomials is obtained:

Proposition 6: *In the notation of* (5),

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{n!^2} P_n'(x) z^n = \frac{z}{\rho}\left(\frac{G(t_-)F(t_+)}{1-\rho-z} + \frac{F(t_-)G(t_+)}{1-\rho+z}\right), \tag{52}$$

where $t_\pm = t_\pm(x,z) := (1-\rho \pm z)/2$.

We may then take a linear combination of the series (6) and (52), and apply Proposition 4 to match a series for $1/\pi$ (of the type in Proposition 5), thus obtaining what we call a "companion series".

For instance, in the case of (II1), the resulting companion series is

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2}\left[P_n'\left(\frac{3\sqrt{3}}{5}\right) + \sqrt{3}\,P_n\left(\frac{3\sqrt{3}}{5}\right)\right]\left(\frac{5}{6\sqrt{3}}\right)^n = \frac{15}{2\pi}.$$

If we combine (II1), its companion, and the formula

$$P_n'(x) = \frac{n}{x^2-1}\big(xP_n(x) - P_{n-1}(x)\big),$$

we produce the new identity

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2}\, nP_{n-1}\left(\frac{3\sqrt{3}}{5}\right)\left(\frac{5}{6\sqrt{3}}\right)^n = \frac{3}{4\pi}.$$

Note that the second order recursion satisfied by the Legendre polynomials allows us to derive many identities of this kind.

As another example of a companion series, (I4) produces

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2}\left[\frac{7}{26\sqrt{5}}\,P_n'\left(\frac{31}{8\sqrt{15}}\right) + \frac{214}{13\sqrt{3}}\,P_n\left(\frac{31}{8\sqrt{15}}\right)\right]\left(\frac{\sqrt{15}}{16}\right)^n = \frac{32}{\pi}.$$

8. Closed forms

Here we give our elementary proof of (II1) as promised in Remark 3. Using the same notation as Section 4, applying Proposition 2 and relation (31), we obtain

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2}(2+15n)P_n\left(\frac{3\sqrt{3}}{5}\right)\left(\frac{5}{6\sqrt{3}}\right)^n = \frac{45}{2}\,F\left(\frac{1}{2}\right)G\left(\frac{1}{2}\right).$$

Note that both the hypergeometric series on the right-hand side can be summed by Gauss' second summation theorem [1, Section 2.4, eq. (2)]:

$$F\left(\frac{1}{2}\right) = {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix} \,\middle|\, \frac{1}{2}\right) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})},$$

$$G\left(\frac{1}{2}\right) = \frac{1}{9}\,{}_2F_1\left(\begin{matrix}\frac{4}{3}, \frac{5}{3} \\ 2\end{matrix} \,\middle|\, \frac{1}{2}\right) = \frac{2\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{3})\Gamma(\frac{1}{6})}.$$

Therefore,

$$F\left(\frac{1}{2}\right)G\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\pi},$$

and identity (II1) follows. As mentioned, a similar derivation is valid for (III3).

When $s = 1/2$, we can alternatively use the complete elliptic integrals $K(k)$ and $K'(k) := K(k') = K(\sqrt{1-k^2})$ to represent proofs of the identities in group I. This sometimes leads to unexpected closed form evaluations of the involved $F(\alpha)$ and $F(\beta)$, hence also of $G(\alpha)$ and $G(\beta)$ through the corresponding series for $1/\pi$ or by taking derivatives. Our evaluations depend on the Nth singular value of K, that is, a modulus k_N such that

$$\frac{K'(k_N)}{K(k_N)} = \sqrt{N}.$$

For a positive integer N, k_N is algebraic and can be effectively computed [15], and the values of K and its derivative at k_N (hence $F(k_N^2)$ and $G(k_N^2)$) are expressible in terms of gamma functions (see [8, Chap. 5], which also lists k_N for small N).

Consider, for example, the product $F(\alpha)F(\beta)$ for (I2); with the help of (22) we see that it is

$$\frac{2\sqrt{6}(\sqrt{3}+1)}{3\pi^2}\,K^2\left(\sqrt{\frac{4\sqrt{2}-5\sqrt{3}-3}{8\sqrt{2}}}\right).$$

We now apply the transformation (24) followed by [8, Chap. 1]

$$K(x) = \frac{1}{1+x}\,K\left(\frac{2\sqrt{x}}{1+x}\right),$$

and observe that the argument of the elliptic integral is transformed to k_3', where $k_3 = \sin(\pi/12)$ is the third singular value. As $K'(k_3)$ has a closed form, we obtain

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} P_n\left(\frac{17}{12\sqrt{2}}\right)\left(-\frac{3}{4\sqrt{2}}\right)^n = \frac{3\Gamma\left(\frac{1}{3}\right)^6}{2^{11/3}\pi^4}.$$

Curiously enough, the quantity on the right-hand side is exactly the value of the (-1)st moment of the distance from the origin in a uniform 3-step walk on the plane [9, Section 5, Example 5].

In (I3) and (I4), $\alpha = 16 - 7\sqrt{3} - \sqrt{15}$ is the square of the 15th singular value of K. In the proof of (II11), α_0 is the square of the fifth singular value. In all these cases, F and G all have computable closed forms at α and α_0; we can therefore complete their proofs without resorting to Propositions 4 and 5. In the case of (II11) we can use the fact to establish the series (36).

9. Conclusion

We have discussed the proofs of several Ramanujan-type series for $1/\pi$ that are associated with the Legendre polynomials. Our analysis in Sections 5 and 6 shows that the list in [16] does not exhaust all, even rational, examples of such series, and that the latter problem is related to investigation of imaginary quadratic fields with prescribed class groups. In particular, our work effectively gives a recipe to generate more series of the type by picking suitable τ in imaginary quadratic fields.

The techniques of the present paper also allow us to prove other identities in [16] of the forms

(53)
$$\sum_{n=0}^{\infty} \frac{(\frac12)_n^2}{n!^2}(A+Bn)P_{2n}(x_0)z_0^n = \frac{C}{\pi} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{(\frac13)_n(\frac23)_n}{n!^2}(A+Bn)P_{3n}(x_0)z_0^n = \frac{C}{\pi},$$

although computation becomes more involved and the next two identities need to be invoked. Brafman's generating function (4) in these cases is replaced by

$$\sum_{n=0}^{\infty} \frac{(\frac12)_n^2}{n!^2} P_{2n}\left(\frac{(u+v)(1-uv)}{(u-v)(1+uv)}\right)\left(\frac{u-v}{1+uv}\right)^{2n} = \frac{1+uv}{2}\, {}_2F_1\left(\begin{matrix}\frac12, \frac12\\ 1\end{matrix}\,\middle|\, 1-u^2\right) {}_2F_1\left(\begin{matrix}\frac12, \frac12\\ 1\end{matrix}\,\middle|\, 1-v^2\right) \tag{54}$$

and

$$\sum_{n=0}^{\infty} \frac{(\frac13)_n(\frac23)_n}{n!^2} P_{3n}\left(\frac{u+v-2u^2v^2}{(u-v)\sqrt{1+4uv(u+v)}}\right)\left(\frac{u-v}{\sqrt{1+4uv(u+v)}}\right)^{3n} = \frac{\sqrt{1+4uv(u+v)}}{3}\, {}_2F_1\left(\begin{matrix}\frac13, \frac23\\ 1\end{matrix}\,\middle|\, 1-u^3\right) {}_2F_1\left(\begin{matrix}\frac13, \frac23\\ 1\end{matrix}\,\middle|\, 1-v^3\right). \tag{55}$$

Identities (54) and (55), valid in a neighbourhood of $u = v = 1$, are new, and we prove them and several other previously unknown generating functions of Legendre polynomials in [17]; there we also discuss in more detail applications to formulas for $1/\pi$ including the forms (53). In particular, we prove in [17] new series such as:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2} (2 + 15n)\, P_{2n}\left(\frac{3\sqrt{3}}{5}\right)\left(\frac{2\sqrt{2}}{5}\right)^{2n} = \frac{15}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^2}{n!^2}\, n\, P_{2n}\left(\frac{45}{17\sqrt{7}}\right)\left(\frac{4\sqrt{14}}{17}\right)^{2n} = \frac{68}{21\pi},$$

and

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{2}{3})_n}{n!^2} (1 + 9n)\, P_{3n}\left(\frac{4}{\sqrt{10}}\right)\left(\frac{1}{\sqrt{10}}\right)^{3n} = \frac{\sqrt{15 + 10\sqrt{3}}}{\pi\sqrt{2}}.$$

Although parameters b, c and λ in notation (1) cannot all be made rational, the summands of the three series given are all rational numbers.

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