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A GENERALIZATION OF A BRAFMAN-BAILEY TYPE IDENTITY

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ABSTRACT. Recently, J. Wan and W. Zudilin discovered a generalization of an identity discovered by F. Brafman around 1951. In this article, we give a generalization of the identities of Brafman, Wan and Zudilin.

1. INTRODUCTION

The hypergeometric series ${}_pF_q$ is given by

$$ {}_pF_q\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, $$

with

$$ (a)_k = \begin{cases} 1 & \text{if } k = 0, \\ (a)(a+1)\cdots(a+k-1) & \text{if } k \geq 1. \end{cases} $$

Clausen-type identities are identities that involve expressing the product of two ${}_2F_1$ functions as a ${}_3F_2$. One of the simplest examples is given by

$$ \left({}_2F_1\left(\begin{matrix} a, \quad b \\ a+b+\frac{1}{2} \end{matrix}; x\right)\right)^2 = {}_3F_2\left(\begin{matrix} 2a, & 2b, & a+b \\ a+b+\frac{1}{2}, & & 2a+2b \end{matrix}; x\right). $$

In order to extend Clausen-type identities to an identity involving two independent variables, it is natural to consider the expression

$$ {}_2F_1\left(\begin{matrix} a, \quad b \\ c \end{matrix}; x\right) \cdot {}_2F_1\left(\begin{matrix} a', \quad b' \\ c' \end{matrix}; y\right) = \sum_{m,n=0}^{\infty} \frac{(a)_m (a')_n (b)_m (b')_n}{(c)_m (c')_n} \frac{x^m y^n}{m! n!}. \tag{1.1} $$

Unfortunately, this direct approach does not seem to yield interesting extension of Clausen-type identities. Instead of considering (1.1), P. Appell (see [1, Chapter IX]) replaced one, two or three of the products $(a)_m(a')_n, (b)_m(b')_n, (c)_m(c')_n$ by $(a)_{m+n}, (b)_{m+n}$ and $(c)_{m+n}$ and studied the corresponding functions. There are five possibilities but one of them is not interesting as it is simply a ${}_2F_1$, namely,

$$ \sum_{m,n=0}^{\infty} \frac{(a)_{m+n} (b)_{m+n}}{(c)_{m+n}} \frac{x^m y^n}{m! n!} = {}_2F_1\left(\begin{matrix} a, \quad b \\ c \end{matrix}; x+y\right). $$

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The remaining four are new functions of 2 variables and they are now known as Appell's hypergeometric functions. Among these functions, the most interesting is perhaps F_4 defined by

$$F_4(a,b,;c,c';x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n} \frac{x^m y^n}{m!n!}.$$

The function F_4 satisfies W.N. Bailey's identity [1, (1), p. 81]

(1.2)

$$F_4(a,b;c;a+b-c+1;z(1-Z),Z(1-z)) = {}_2F_1\left(\begin{matrix}a, & b \\ & c\end{matrix};z\right)\cdot {}_2F_1\left(\begin{matrix}a, & b \\ a+b-c+1\end{matrix};Z\right).$$

Around 1951, almost 16 years after Bailey's discovery, F. Brafman [2, a variant of (12)] showed, using a specialization of Bailey's identity, that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(a)_n(1-a)_n}{(1)_n^2} P_n\left(\frac{u+v-2uv}{u-v}\right)(u-v)^n \\ = {}_2F_1\left(\begin{matrix}a, & 1-a \\ & 1\end{matrix};u\right)\cdot {}_2F_1\left(\begin{matrix}a, & 1-a \\ & 1\end{matrix};v\right),\end{aligned} \tag{1.3}$$

where

$$P_n(x) = \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{x-1}{2}\right)^n \left(\frac{x+1}{2}\right)^{n-m}.$$

Observe that when $u = v$, Brafman's identity is reduced to the Clausen-type identity

$$\left({}_2F_1\left(\begin{matrix}a, & 1-a \\ & 1\end{matrix};u\right)\right)^2 = {}_3F_2\left(\begin{matrix}\frac{1}{2}, & a, & 1-a \\ & 1, & 1\end{matrix};4u(1-u)\right), \tag{1.4}$$

where we have used the identity

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}. \tag{1.5}$$

Brafman's identity was almost forgotten until recently. In early January 2011, Z.-W. Sun [6] announced his discovery of several new series for $1/\pi$ and subsequently produced over 170 new series for $1/\pi$ grouped according to the coefficients of the series. One of these groups has series associated with $T_n(b,c)$ where

$$T_n(b,c) = \sum_{k=0}^{n} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k.$$

One of the simplest series from this group is

$$\sum_{n=0}^{\infty} \frac{15n-4}{18^{3n}} \binom{2n}{n}\binom{3n}{n} T_n(198,1) = \frac{135\sqrt{3}}{2\pi}.$$

A few months later, W. Zudilin observed that

$$T_n(b,c) = (b^2-4c)^{n/2} P_n\left(\frac{b}{\sqrt{b^2-4c}}\right)$$

and proved series belonging to this group in a joint work with H.H. Chan and J. Wan [5]. The results needed in their proofs are precisely Bailey-Brafman identity (1.3) and Ramanujan-type series for $1/\pi$.

In an attempt to prove series from other groups discovered by Sun, Wan and Zudilin derived a generalization of Brafman's identity[1] and their result is

Theorem 1.1. *Let a, b, c be integers. Let $\{t_n\}_{n=-1}^{\infty}$ be the sequence defined by $t_{-1} = 0, t_0 = 1$ and for $n \geq 2$ by the relation*

$$(n+1)^2 t_{n+1} = (an^2 + an + b)t_n - cn^2 t_{n-1}. \tag{1.6}$$

Then there is a neighborhood of $X = Y = 0$ such that

$$\begin{aligned}(1.7)\qquad & \left(\sum_{n=0}^{\infty} t_n X^n\right)\left(\sum_{n=0}^{\infty} t_n Y^n\right) \\ &= \frac{1}{1-cXY} \sum_{n=0}^{\infty} t_n \sum_{m=0}^{n} \binom{n}{m}^2 (g(X,Y))^m (g(Y,X))^{n-m},\end{aligned}$$

where

$$g(X,Y) = \frac{X(1-aY+cY^2)}{(1-cXY)^2}. \tag{1.8}$$

For proofs of Theorem 1.1, see [7] or [3].

The main aim of this article is to present a generalization of Theorem 1.1. Before stating the result, we need to define a few quantities.

Let a, b, c, α and β be complex numbers such that α and β are not negative integers. Define $\{w_n\}_{n=-1}^{\infty}$ by the recurrence relation

$$(n+1+\alpha)(n+1+\beta)w_{n+1} = (an^2 + a(1+\alpha+\beta)n + b)w_n - cn^2 w_{n-1} \tag{1.9}$$

for $n \geq 1$, where $w_{-1} = 0$ and $w_0 = 1$. Let $\mathbf{u} = \{u_n\}_{n=-1}^{\infty}$ and $\mathbf{v} = \{v_n\}_{n=-1}^{\infty}$ be defined by

$$u_n := u_n(\alpha, \beta) = \frac{(1+\beta)_n}{n!} w_n \tag{1.10}$$

and

$$v_n := u_n(\beta, \alpha) = \frac{(1+\alpha)_n}{n!} w_n. \tag{1.11}$$

It is immediate to see that for $n \geq 2$, the terms in $\mathbf{u}$ and $\mathbf{v}$ satisfy

$$(n+1+\alpha)(n+1)u_{n+1} = (an^2 + a(1+\alpha+\beta)n + b)u_n - c(n+\beta)nu_{n-1} \tag{1.12}$$

and

$$(n+1+\beta)(n+1)v_{n+1} = (an^2 + a(1+\alpha+\beta)n + b)v_n - c(n+\alpha)nv_{n-1}, \tag{1.13}$$

respectively. We are now ready to state our result.

Theorem 1.2. *Let $g(X, Y)$ be the function defined by (1.8) and*

$$\begin{aligned}(1.14)\qquad p_n(m,\alpha,\beta) &= \frac{1}{n!}\binom{n}{m}\frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1+m)}\frac{\Gamma(\beta+1+n)}{\Gamma(\beta+1+n-m)} \\ &= \frac{(1+\alpha)_n(1+\beta)_n}{m!(n-m)!(1+\alpha)_m(1+\beta)_{n-m}}.\end{aligned}$$

[1] We remark here that the discovery of the Wan-Zudilin identity was motivated, coincidentally, by an earlier work of Chan, Y. Tanigawa, Y.F. Yang and Zudilin [4].

Then there exists a power series $G(X,Y)$ such that

(1.15)
$$\left(\sum_{n=0} u_n X^n\right)\left(\sum_{n=0}^{\infty} v_n Y_n\right) = \frac{1}{1-cXY}\sum_{n=0}^{\infty} w_n \sum_{m=0}^{n} p_n(m,\alpha,\beta) g(X,Y)^m g(Y,X)^{n-m} - c(\alpha+\beta)XYG(X,Y)$$

holds in a neighborhood of $X = Y = 0$.

By numerical computations, we find that

$$G(X,Y) = w_1(X+Y) + w_2\left((2+\beta)X^2 + (2+\alpha)Y^2\right) + \frac{1}{2}\left(w_2\left(\frac{9}{2}+\alpha+\beta\right) - 2aw_1\right)XY + \cdots .$$

We are not able to determine the explicit form of $G(X,Y)$.

Note that if we substitute $\alpha = \beta = 0$ in Theorem 1.2, we derive Theorem 1.1 immediately.

In [2], Brafman showed that

$$\sum_{n=0}^{\infty} \frac{(s)_n(1+\alpha+\beta-s)_n}{(1+\alpha)_n(1+\beta)_n} P_n^{(\alpha,\beta)}\left(\frac{u+v-2uv}{u-v}\right)(u-v)^n = {}_2F_1\left(\begin{matrix} s, 1+\alpha+\beta-s \\ 1+\alpha \end{matrix}; u\right) {}_2F_1\left(\begin{matrix} s, 1+\alpha+\beta-s \\ 1+\beta \end{matrix}; v\right), \tag{1.16}$$

where

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} p_n(k,\alpha,\beta)\left(\frac{x-1}{2}\right)^k \left(\frac{x+1}{2}\right)^{n-k}$$

is the Jacobi polynomial ([2, (5)]). He then deduced (1.3) by setting $\alpha = \beta = 0$ and using the identities

$$p_n(k,0,0) = \binom{n}{k}^2 \quad \text{and} \quad P_n(x) = P_n^{(0,0)}(x).$$

It is immediate that (1.16) follows from Theorem 1.2 by setting $a = 1, c = 0$ and $b = s(1+\alpha+\beta-s)$.

In Section 2, we will establish a few elementary relations satisfied by $p_n(m,\alpha,\beta)$. In Section 3, we prove Theorem 1.2 for $\alpha+\beta = 0$ and, in Section 4, we prove Theorem 1.2 for $c \neq 0$, with independent variables α and β. The proofs of the results in these sections would complete the proof of Theorem 1.2.

2. Some preliminary observations

From (1.14), it is clear that $p_n(m,\alpha,\beta)$ satisfies the relations

$$p_n(m,\alpha,\beta) = p_n(n-m,\beta,\alpha), \tag{2.1}$$

$$p_n(m,\alpha,\beta)(n-m)(n-m+\beta) = (n+\alpha)(n+\beta)p_{n-1}(m,\alpha,\beta), \tag{2.2}$$

and

$$p_n(m,\alpha,\beta)m(m+\alpha) = (n+\alpha)(n+\beta)p_{n-1}(m-1,\alpha,\beta). \tag{2.3}$$

Furthermore, we may write

$$u_n = w_n p_n(n,\alpha,\beta) \quad \text{and} \quad v_n = w_n p_n(0,\alpha,\beta). \tag{2.4}$$

To simplify notation, set

$$U = X,\ V = 1 - aX + cX^2,\ L = 1 - aY + cY^2,\ K = Y \quad \text{and} \quad W = 1 - cXY,$$

and write the first term on the right side of (1.15) as

$$H(X,Y;\alpha,\beta) = \frac{1}{W}\sum_{n=0}^{\infty} w_n \sum_{m=0}^{n} p_n(m,\alpha,\beta)\left(\frac{UL}{W^2}\right)^m \left(\frac{VK}{W^2}\right)^{n-m}.$$

Set

$$F_{\mathbf{u}}(X) = \sum_{n=0}^{\infty} u_n X^n \quad \text{and} \quad F_{\mathbf{v}}(Y) = \sum_{n=0}^{\infty} v_n Y^n.$$

From (2.4), we have

$$H(X,0;\alpha,\beta) = F_{\mathbf{u}}(X) \quad \text{and} \quad H(0,Y;\alpha,\beta) = F_{\mathbf{v}}(Y). \tag{2.5}$$

Hence, we find that XY divides $F_{\mathbf{u}}(X)F_{\mathbf{v}}(Y) - H(X,Y;\alpha,\beta)$.

3. The case $\alpha + \beta = 0$

Throughout this section, we assume that α is not an integer and

$$\alpha + \beta = 0.$$

For convenience, we recall the definitions of the series w_n, u_n and v_n:

$$(n+1+\alpha)(n+1-\alpha)w_{n+1} = (an^2 + an + b)w_n - cn^2 w_{n-1} \tag{3.1}$$

with $w_{-1} = 0, w_0 = 1$ and

$$u_n = u_n = \frac{(1-\alpha)_n}{n!}w_n \quad \text{and} \quad v_n = \frac{(1+\alpha)_n}{n!}w_n. \tag{3.2}$$

In this section, we prove the identity

$$F_{\mathbf{u}}(X)F_{\mathbf{v}}(Y) = H(X,Y;\alpha,-\alpha). \tag{3.3}$$

Let

$$\Psi_X^{(\alpha)} = \frac{1}{X}\left(\vartheta_X(\vartheta_X + \alpha) - X(a\vartheta_X^2 + a\vartheta_X + b) + cX^2\left(\vartheta_X + 1\right)\left(\vartheta_X + 1 - \alpha\right)\right),$$

where

$$\vartheta_X = X\frac{\partial}{\partial X}.$$

By (1.12) it follows immediately that

$$\Psi_X^{(\alpha)} F_{\mathbf{u}}(X) = 0. \tag{3.4}$$

On the other hand, $\Psi_X^{(\alpha)}(H(X,Y;\alpha,-\alpha))$ is an expression with 109 sums which can be written as

$$\Psi_X^{(\alpha)}(H(X,Y;\alpha,-\alpha)) = \sum_{n=0}^{\infty} w_n \sum_{m=0}^{n} p_n(m,\alpha,-\alpha)\xi(X,Y)\frac{U^{m-1}V^{n-m-1}L^m K^{n-m}}{W^{2n+3}},$$

where

$$\begin{aligned}\xi(X,Y) &= m(m+\alpha)V^2W^2 + am(m+\alpha)UVW^2 + a(n-m)(n-m-\alpha)UVW^2\\ &\quad + c(2n+1)(2m+\alpha+1)UKV^2 + c(2n+1)(2n+1-2m-\alpha)U^2VL\\ &\quad + (a^2-4c)(n-m)(n-m-\alpha)U^2W^2 - (an^2+an+b)UVW^2.\end{aligned}$$

Let

$$\begin{aligned}\zeta_1(X,Y) &= m(m+\alpha)V^2W^2 + am(m+\alpha)UVW^2 + a(n-m)(n-m-\alpha)UVW^2\\ &\quad + a^2(n-m)(n-m-\alpha)U^2W^2 - 4c(n-m)(n-m-\alpha)U^2W^2,\\ \zeta_2(X,Y) &= c(2n+1)(2m+\alpha+1)UKV^2 + c(2n+1)(2n+1-2m-\alpha)U^2VL\end{aligned}$$

and

$$\zeta_3(X,Y) = -(an^2+an+b)UVW^2.$$

Now, using the identities (2.2), (2.3) and

$$aVK + a^2UK + VL + aUL = W^2 + c(U+K)^2,$$

we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} w_n \sum_{m=0}^{n} p_n(m,\alpha,-\alpha)\zeta_1(X,Y)\frac{U^{m-1}V^{n-m-1}L^mK^{n-m}}{W^{2n+3}}\\ &\quad = \sum_{n=0}^{\infty}(n+\alpha+1)(n-\alpha+1)w_{n+1}\sum_{m=0}^{n} p_n(m,\alpha,-\alpha)\left(\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+1}}\right)\\ &\quad + c\sum_{n=0}^{\infty}(n+\alpha+1)(n-\alpha+1)w_{n+1}\\ &\quad \times \sum_{m=0}^{n} p_n(m,\alpha,-\alpha)(U-K)^2\left(\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}\right).\end{aligned}$$

Next, observe that

$$\begin{aligned}&\sum_{n=0}^{\infty} w_n \sum_{m=0}^{n} p_n(m,\alpha,-\alpha)\zeta_3(X,Y)\frac{U^{m-1}V^{n-m-1}L^mK^{n-m}}{W^{2n+3}}\\ &\qquad = -\sum_{n=0}^{\infty}(an^2+an+b)w_n\sum_{m=0}^{n} p_n(m,\alpha,-\alpha)\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+1}}.\end{aligned}$$

Hence, we may write

$$\begin{aligned}\Psi_X^{(\alpha)}(H(X,Y)) &= \sum_{n=0}^{\infty}((n+1+\alpha)(n+1-\alpha)w_{n+1} - (an^2+an+b)w_n + cn^2w_{n-1})\\ &\qquad \times \sum_{m=0}^{n} p_n(m,\alpha,-\alpha)\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+1}} + \mathcal{E},\end{aligned}$$

where

$$\begin{aligned}\mathcal{E} = c\sum_{n=0}^{\infty}&(n+\alpha+1)(n-\alpha+1)w_{n+1}\\ &\times\sum_{m=0}^{n} p_n(m,\alpha,-\alpha)(U-K)^2\left(\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}\right)\\ &+\sum_{n=0}^{\infty} w_n\sum_{m=0}^{n} p_n(m,\alpha,-\alpha)\zeta_2(X,Y)\left(\frac{U^{m-1}V^{n-m-1}L^mK^{n-m}}{W^{2n+3}}\right)\\ &-c\sum_{n=0}^{\infty}(n+1)^2w_n\sum_{m=0}^{n+1} p_{n+1}(m,\alpha,-\alpha)VK\left(\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}\right).\end{aligned}$$

We now observe that

$$U-K=\frac{UL-VK}{W}$$

and write each term in $\mathcal{E}$ in the form

$$\sum_{n=0}^{\infty} w_n\sum_{m=0}^{n+1} B(m,n)VK\frac{U^mV^{n-m}L^mK^{n-m}}{W^{2n+3}}.$$

The final result is

$$\mathcal{E}=c\sum_{n=0}^{\infty}\sum_{m=0}^{n+1} R(n,m)VK\frac{U^mK^{n-m}L^mK^{n-m}}{W^{2n+3}}$$

where

$$\begin{aligned}R(n,m) = {}&(n-\alpha)(n+\alpha)p_{n-1}(m-2,\alpha,-\alpha)+(n-\alpha)(n+\alpha)p_{n-1}(m,\alpha,-\alpha)\\ &-2(n-\alpha)(n+\alpha)p_{n-1}(m-1,\alpha,-\alpha)\\ &+(2n+1)(2n+3-2m-\alpha)p_n(m-1,\alpha,-\alpha)\\ &+(2n+1)(2m+\alpha+1)p_n(m,\alpha,-\alpha)-(n+1)^2p_{n+1}(m,\alpha,-\alpha).\end{aligned}$$

We claim that

$$R(n,m)=0.$$

To show this explicitly, write

$$R(n,m)=R_1(n,m)+R_2(n,m)-(n+1)^2p_{n+1}(m,\alpha,-\alpha),$$

where

$$\begin{aligned}R_1(n,m)={}&(n-\alpha)(n+\alpha)p_{n-1}(m-2,\alpha,-\alpha)-(n-\alpha)(n+\alpha)p_{n-1}(m-1,\alpha,-\alpha)\\ &+(2n+1)(2n+3-2m-\alpha)p_n(m-1,\alpha,-\alpha)\end{aligned}$$

and

$$\begin{aligned}R_2(n,m) = {}&(n-\alpha)(n+\alpha)p_{n-1}(m,\alpha,-\alpha)-(n-\alpha)(n+\alpha)p_{n-1}(m-1,\alpha,-\alpha)\\ &+(2n+1)(2m+\alpha+1)p_n(m,\alpha,-\alpha).\end{aligned}$$

Using (2.2) and (2.3), we may rewrite $R_1(n,m)$ and $R_2(n,m)$ as

$$\begin{aligned} R_1(n,m) =&(m-1)(m-1+\alpha)p_n(m-1,\alpha,-\alpha) \\ &-(n-(m-1))(n-(m-1)-\alpha)p_n(m-1,\alpha,-\alpha) \\ &+(2n+1)(2n+3-2m-\alpha)p_n(m-1,\alpha,-\alpha) \\ =&(3(n+1)^2-2nm-n\alpha-2m-\alpha)p_n(m-1,\alpha,-\alpha) \\ =&(n+1)(3(n+1)-(2m+\alpha))p_n(m-1,\alpha,-\alpha) \end{aligned}$$

and

$$\begin{aligned} R_2(n,m) =&(n-m)(n-m-\alpha)p_n(m,\alpha,-\alpha)-m(m+\alpha)p_n(m,\alpha,-\alpha) \\ &+(2n+1)(2m+\alpha+1)p_n(m) \\ =&(n+1)(n+1+2m+\alpha)p_n(m,\alpha,-\alpha). \end{aligned}$$

Hence, we find that

$$R(n,m)=(n+1)R_3(n,m),$$

where

$$\begin{aligned} R_3(n,m) =&(3(n+1)-2m-\alpha)p_n(m-1,\alpha,-\alpha) \\ &+(n+1+2m+\alpha)p_n(m,\alpha,-\alpha)-(n+1)p_{n+1}(m,\alpha,-\alpha). \end{aligned}$$

Using (2.2) and (2.3) again, we deduce that

$$\begin{aligned} &R_3(n,m)=p_{n+1}(m) \\ &\times\left(\frac{(3(n+1)-2m-\alpha)m(m+\alpha)}{(n+1+\alpha)(n+1-\alpha)}\right. \\ &\qquad\left.+\frac{(n+1+2n+\alpha)(n+1-m)(n+1-m-\alpha)}{(n+1+\alpha)(n+1-\alpha)}-(n+1)\right). \end{aligned}$$

It can be verified directly that the expression in the parentheses is 0 and this shows that

$$R(n,m)=0.$$

Therefore we conclude that

$$\Psi_X^{(\alpha)}(H(X,Y;\alpha,-\alpha))=0. \tag{3.5}$$

If we define the differential operator $\Psi_Y^{(-\alpha)}$ by

$$\Psi_Y^{(-\alpha)}=\frac{1}{Y}\left(\vartheta_Y(\vartheta_Y-\alpha)-Y(a\vartheta_Y^2+a\vartheta_Y+b)+cY^2\left(\vartheta_Y+1\right)\left(\vartheta_Y+1+\alpha\right)\right),$$

then, by similar argument, we find that

$$\Psi_Y^{(-\alpha)}F_{\mathbf{v}}(Y)=0 \quad\text{and}\quad \Psi_Y^{(-\alpha)}(H(X,Y;\alpha,-\alpha))=0. \tag{3.6}$$

To complete the proof we follow the method of Wan and Zudilin [7] (see also Chan [3]). Let

$$\phi(X,Y)=F_{\mathbf{u}}(X)F_{\mathbf{v}}(Y)-H(X,Y;\alpha,-\alpha).$$

By (2.5), we have

$$\phi(X,0)=F_{\mathbf{u}}(X)-H(X,0;\alpha,-\alpha)=0$$

and

$$\phi(0,Y)=F_{\mathbf{v}}(Y)-H(0,Y;\alpha,-\alpha)=0.$$

Hence, if we write

$$\phi(X,Y)=\sum_{i,j=0}^{\infty} t_{i,j}X^iY^j,$$

we find that

$$t_{i,0}=t_{0,j}=0$$

for all integers $i,j\geq 0$. Since $(\Psi_X^{(\alpha)}+\Psi_Y^{(-\alpha)})(\phi(X,Y))$ is identically zero by (3.4), (3.5) and (3.6), every coefficient of $(\Psi_X^{(\alpha)}+\Psi_Y^{(-\alpha)})(\phi(X,Y))$ (in fact the coefficient of X^iY^j is given by (3.7) below) vanishes; namely, we have

$$\begin{aligned}(3.7)\qquad &(i+1)(i+1+\alpha)t_{i+1,j}-(ai^2+ai+b)t_{i,j}+ci(i-\alpha)t_{i-1,j}\\ &+(j+1)(j+1-\alpha)t_{i,j+1}-(aj^2+aj+b)t_{i,j}+cj(j+\alpha)t_{i,j-1}=0\end{aligned}$$

for all $i,j\geq 0$. We first show that $t_{i,1}=0$ for all integers $i\geq 0$. If $i=0$, $t_{0,1}=0$ by our previous discussion. If $i\geq 1$, let $j=0$ in (3.7), then we get $(1-\alpha)t_{i,1}=0$. By assumption, α is not an integer, hence $t_{i,1}=0$ for all $i\geq 1$. Now assuming that $t_{i,j}=0$ for all integers $i\geq 0$ and $0\leq j\leq k$. By the similar argument with $j=k$, we readily find that $t_{i,k+1}=0$ for all i. This completes the proof of (3.3) and hence Theorem 1.2 when $\alpha+\beta=0$.

4. The case $c=0$

In this section, we consider the case $c=0$. Assume that α and β are arbitrary complex numbers which are not negative integers. Since this is easier than the identity proved in the previous section, we will only give an outline of the proof.

The recurrence relation for $\{w_n\}$ is given by

$$(n+1+\alpha)(n+1+\beta)w_{n+1}=(an^2+a(1+\alpha+\beta)n+b)w_n,\quad w_0=1$$

and $\{u_n\}$ and $\{v_n\}$ are defined by (1.10) and (1.11), respectively. Furthermore,

$$U=X,\ V=1-aX,\ L=1-aY,\ K=Y\ \text{ and }\ W=1$$

in this case.

We claim that

$$F_{\mathbf{u}}(X)F_{\mathbf{v}}(Y)=H(X,Y;\alpha,\beta), \tag{4.1}$$

where

$$H(X,Y;\alpha,\beta)=\sum_{n=0}^{\infty}w_n\sum_{m=0}^{n}p_n(m,\alpha,\beta)U^mL^mK^{n-m}V^{n-m}.$$

The corresponding differential operator is given by

$$\Theta_X^{(\alpha)}=\frac{1}{X}\left(\vartheta_X(\vartheta_X+\alpha)-X(a\vartheta_X^2+a(1+\alpha+\beta)\vartheta_X+b)\right).$$

Clearly, we have

$$\Theta_X^{(\alpha)}F_{\mathbf{u}}(X)=0.$$

On the other hand we get

$$\Theta_X^{(\alpha)}H(X,Y;\alpha,\beta)=\sum_{n=0}^{\infty}w_n\sum_{m=0}^{n}p_n(m,\alpha,\beta)\xi(X,Y)U^{m-1}V^{n-m-1}L^mK^{n-m},$$

where

$$\begin{aligned}\xi(X,Y) =& m(m+\alpha)V + a(n-m)(n-m+\beta)U \\ &- (an^2 + a(1+\alpha+\beta)n + b)UV.\end{aligned}$$

Using (2.2) and (2.3) we find that

$$\begin{aligned}&\Theta_X^{(\alpha)} H(X,Y;\alpha,\beta) \\ &= \sum_{n=0}^{\infty} w_{n+1}(n+1+\alpha)(n+1+\beta) \sum_{m=0}^{n} p_n(m,\alpha,\beta) U^m V^{n-m} K^{n-m} L^m \cdot L \\ &\quad + \sum_{n=0}^{\infty} w_{n+1}(n+1+\alpha)(n+1+\beta) \sum_{m=0}^{n} p_n(m,\alpha,\beta) U^m V^{n-m} K^{n-m} L^m \cdot aK \\ &\quad - \sum_{n=0}^{\infty} w_n(an^2 + a(1+\alpha+\beta)n + b) \sum_{m=0}^{n} p_n(m,\alpha,\beta) U^m V^{n-m} K^{n-m} L^m \\ &= 0,\end{aligned}$$

since $L + aK = 1$.

Similarly, we find that

$$\Theta_Y^{(\beta)} F_{\mathbf{v}}(Y) = 0 \quad \text{and} \quad \Theta_Y^{(\beta)} H(X,Y;\alpha,\beta) = 0,$$

where

$$\Theta_Y^{(\beta)} = \frac{1}{Y}\left(\vartheta_Y(\vartheta_Y+\beta) - Y(a\vartheta_Y^2 + a(1+\alpha+\beta)\vartheta_Y + b)\right).$$

By the same argument as in Section 2, we deduce (4.1). This completes the proof of Theorem 1.2 in the case $c = 0$.

5. Conclusion

We started our article with a Clausen-type identity, and so it is appropriate to conclude the article with two new Clausen-type identities.

Observe that Clausen-type identity (1.4) is obtained by specializing $u = v$ in (1.3). If we specialize $X = Y$ in Theorem 1.2, we deduce our first new Clausen-type identity; namely,

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} \frac{(1+\beta)_n}{n!} w_n X^n\right)\left(\sum_{n=0}^{\infty} \frac{(1+\alpha)_n}{n!} w_n X^n\right) \\ &= \frac{1}{1-cX^2} \sum_{n=0}^{\infty} w_n \frac{\Gamma(2n+\alpha+\beta+1)}{n!\,\Gamma(n+\alpha+\beta+1)} \left(\frac{X(1-aX+cX^2)}{(1-cX^2)^2}\right)^n - c(\alpha+\beta)X^2 G(X,X),\end{aligned}$$

which is a generalization of Proposition 1 of [7]. To prove this identity we need the identity

$$\sum_{m=0}^{n} p_n(m,\alpha,\beta) = \frac{\Gamma(2n+\alpha+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}, \tag{5.1}$$

which follows from computing the coefficients of x^n of both sides of

$$(1+x)^{n+\alpha}(1+x)^{n+\beta} = (1+x)^{2n+\alpha+\beta}.$$

Note that (5.1) is just an analogue of (1.5).

Our second identity is obtained by setting $\alpha = \beta = 0$, $a = 1$ and $b = s(1-s)$, $UL = -VK$ in (4.1). Using the identity

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}^2 = \begin{cases} (-1)^m \dfrac{(2m)!}{(m!)^2} & \text{if } n = 2m, \\ 0 & \text{otherwise,} \end{cases}$$

we deduce that

$$\sum_{n=0}^{\infty} \frac{(s)_{2n}(1-s)_{2n}}{(1)_{2n}^2} \frac{(1)_{2n}}{(1)_n^2}(-1)^n u^{2n}$$
$$= {}_2F_1\left(s, 1-s; 1; \frac{1}{2} - \frac{\sqrt{1+4u^2}}{2} + u\right) \cdot {}_2F_1\left(s, 1-s; 1; \frac{1}{2} - \frac{\sqrt{1+4u^2}}{2} - u\right).$$

One should compare this with (1.4) in the form

$$\sum_{n=0}^{\infty} \frac{(s)_n(1-s)_n}{(1)_n^2} \frac{(1)_{2n}}{(1)_n^2} u^n = {}_2F_1^2\left(s, 1-s; 1; \frac{1}{2} - \frac{\sqrt{1-u}}{2}\right).$$

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References

[1] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 32, Stechert-Hafner, Inc., New York, 1964. MR0185155 (32 #2625)
[2] Fred Brafman, *Generating functions of Jacobi and related polynomials*, Proc. Amer. Math. Soc. **2** (1951), 942–949. MR0045875 (13,649i)
[3] Heng Huat Chan, *The Bailey-Brafman identity and its analogues*, J. Math. Anal. Appl. **399** (2013), no. 1, 12–16, DOI 10.1016/j.jmaa.2012.09.015. MR2993832
[4] Heng Huat Chan, Yoshio Tanigawa, Yifan Yang, and Wadim Zudilin, *New analogues of Clausen's identities arising from the theory of modular forms*, Adv. Math. **228** (2011), no. 2, 1294–1314, DOI 10.1016/j.aim.2011.06.011. MR2822224 (2012h:11063)
[5] Heng Huat Chan, James Wan, and Wadim Zudilin, *Legendre polynomials and Ramanujan-type series for* $1/\pi$, Israel J. Math. **194** (2013), no. 1, 183–207, DOI 10.1007/s11856-012-0081-5. MR3047067
[6] Z.-W. Sun, *List of conjectural series for powers of* π *and other constants*, preprint, `arXiv:1102.5649v21 [math.CA]` (May 23, 2011).
[7] James Wan and Wadim Zudilin, *Generating functions of Legendre polynomials: a tribute to Fred Brafman [reprint of MR2885420]*, J. Approx. Theory **170** (2013), 198–213, DOI 10.1016/j.jat.2012.11.001. MR3044053

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