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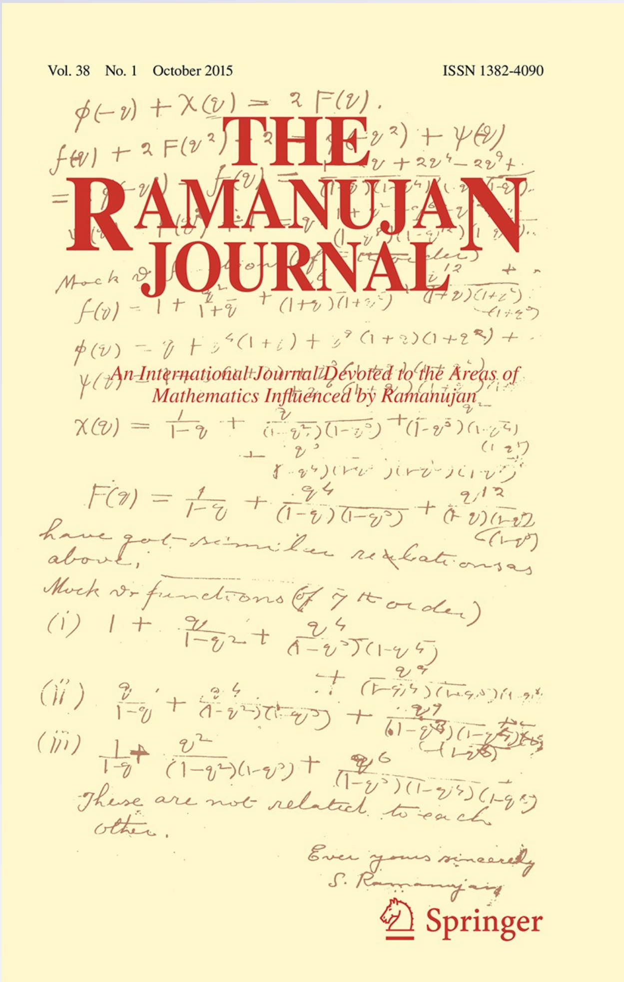

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Analogues of the Brent–Salamin algorithm for evaluating π

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Dedicated to Siew Lian, Si Min, Si Ya, and Si En

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Abstract In this article, we prove the Brent–Salamin algorithm for π using Jacobi's theta functions and use this approach to derive new analogues of the Brent–Salamin algorithm for elliptic functions to alternative bases.

Keywords Brent-Salamin algorithm · Elliptic integrals · Hypergeometric series · Theta series · Modular equations

Mathematical Subject Classification 11Y60 · 33E05

1 Introduction

In 1976, R. Brent [9] and E. Salamin [13] discovered independently an efficient algorithm, which is based on C.F. Gauss' 'Arithmetic–Geometric Mean' two-term recurrence, for evaluating π. The Brent–Salamin algorithm is stated as follows:

Iteration 1.1 *Let $a_0 = 1$ and $b_0 = 1/\sqrt{2}$. For $n \geq 0$, let*

$$a_{n+1} = \frac{a_n + b_n}{2} \tag{1.1}$$

$$b_{n+1} = \sqrt{a_n b_n}, \tag{1.2}$$

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and

$$\pi_n = \frac{2a_n^2}{1 - \sum_{j=0}^{n} 2^j \left(a_j^2 - b_j^2\right)}.$$

Then, π_n increases monotonically to π.

Both Brent and Salamin derived their algorithm using the elliptic integrals

$$K(\alpha) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \alpha \sin^2 \theta}}, \tag{1.3}$$

$$E(\alpha) = \int_0^{\pi/2} \sqrt{1 - \alpha \sin^2 \theta}\, d\theta, \tag{1.4}$$

$$I(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = \frac{1}{a} K\left(1 - \frac{b^2}{a^2}\right)$$

and

$$J(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = a E\left(1 - \frac{b^2}{a^2}\right).$$

The proof of the Brent–Salamin algorithm requires several identities involving these integrals. A sample of such identities is

$$E(\alpha) = (1 + \sqrt{1-\alpha}) E\left(\left(\frac{1 - \sqrt{1-\alpha}}{1 + \sqrt{1-\alpha}}\right)^2\right) - \sqrt{1-\alpha}\, K(\alpha), \tag{1.5}$$

$$2J(a_{n+1}, b_{n+1}) - J(a_n, b_n) = a_n b_n I(a_n, b_n) \tag{1.6}$$

and

$$E(\alpha)K(1-\alpha) + E(1-\alpha)K(\alpha) - K(\alpha)K(1-\alpha) = \frac{\pi}{2}. \tag{1.7}$$

Identity (1.5) is an example of duplication and dimidiation formulas for elliptic integrals, (1.6) plays a significant role in the derivation of Iteration 1.1 and (1.7) is known as the Legendre relation. For detailed proofs of these identities and Iteration 1.1, see [6, p. 12, Theorem 1.2, p. 13, Theorem 1.3, p. 24, Theorem 1.6 and p. 48]).

In this article, we will present a proof of the Brent–Salamin algorithm using theta functions instead of elliptic integrals. Our approach is motivated by known relations between elliptic integrals and theta functions. In the process of deriving the proof of the Brent–Salamin algorithm using theta functions, we will have to replace identities such as (1.5)–(1.7) by identities involving theta functions. We then use this alternative approach to derive algorithms for the computations of π associated with the analogues of Gauss' AGM iteration and compare them with those discovered by the Borweins [7].

2 Jacobi theta functions

We first recall some important identities satisfied by the Jacobi theta functions.

Theorem 2.1 *Let*

$$\theta_3(q) = \sum_{n\in\mathbf{Z}} q^{n^2},$$
$$\theta_2(q) = \sum_{n\in\mathbf{Z}} q^{(n+\frac{1}{2})^2}$$
$$\theta_4(q) = \sum_{n\in\mathbf{Z}} (-1)^n q^{n^2}$$

and

$${}_2F_1\,(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

with

$$(a)_k = \begin{cases} 1 & \text{if } k = 0, \\ (a)(a+1)\cdots(a+k-1) & \text{if } k \geq 1. \end{cases}$$

Then,

$$2\theta_3^2(q^2) = \theta_3^2(q) + \theta_4^2(q) \tag{2.1}$$
$$\theta_4^4(q^2) = \theta_3^2(q)\theta_4^2(q) \tag{2.2}$$
$$\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q) \tag{2.3}$$
$$\theta_3^2(q) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{\theta_2^4(q)}{\theta_3^4(q)}\right). \tag{2.4}$$

For proofs of (2.1)–(2.4), see [2, Chapter 16, Chapter 17], or [6, Chapter 2].

For simplicity, let

$$\alpha = \alpha(q) = \frac{\theta_2^4(q)}{\theta_3^4(q)} \tag{2.5}$$

and

$$F(\alpha) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right) = \theta_3^2(q). \tag{2.6}$$

We emphasize here that $F(\alpha)$ is a theta series as well as a hypergeometric function.

From [1, Chapter 11, p. 88], we know that

$$q\frac{d\alpha}{dq} = F^2(\alpha)\alpha(1-\alpha). \tag{2.7}$$

Using the product representations θ_3 and θ_4 (see for example [6, p. 64, Corollary 3.1]), one can show that (2.7) is equivalent to Jacobi's identity for the generating function of the sums of four squares, namely,

$$\theta_3^4(q) = 1 + 8\sum_{\substack{k=1 \\ 4\nmid k}}^{\infty} \frac{kq^k}{1-q^k}.$$

From (2.7), we conclude that if ℓ is a positive integer greater than 1, then

$$\frac{d\alpha(q^\ell)}{d\alpha(q)} = \ell\frac{\alpha(q^\ell)(1-\alpha(q^\ell))}{\alpha(q)(1-\alpha(q))}\frac{F^2(\alpha(q^\ell))}{F^2(\alpha(q))}. \tag{2.8}$$

It is known that ([2, Chapter 16, Entry 27(i),(ii)] or [6, Chapter 2])

$$\theta_3^2(e^{-\pi/t}) = t\theta_3^2(e^{-\pi t}), \theta_2^2(e^{-\pi/t}) = t\theta_4^2(e^{-\pi t}). \tag{2.9}$$

Hence,

$$\alpha(e^{-\pi/t}) = \frac{\theta_4^4(e^{-\pi/t})}{\theta_3^4(e^{-\pi/t})} = \frac{\theta_2^4(e^{-\pi t})}{\theta_3^4(e^{-\pi t})} = 1-\alpha(e^{-\pi t}), \tag{2.10}$$

where we have used (2.3). Since

$$\theta_3^2(e^{-\pi/t}) = F(1-\alpha),$$

we deduce that

$$t = \frac{F(1-\alpha)}{F(\alpha)}. \tag{2.11}$$

Prompted by (2.10) and (2.11), we also deduce that when $\alpha(q)$ is defined by (2.5),

$$q = e^{-\pi t} = \exp\left(-\pi\frac{F(1-\alpha)}{F(\alpha)}\right). \tag{2.12}$$

Differentiating (2.11) with respect to α and using (2.7), we find that

$$\alpha(1-\alpha)\left(F(\alpha)F'(1-\alpha) + F(1-\alpha)F'(\alpha)\right) = \frac{1}{\pi} \tag{2.13}$$

where

$$F'(x) = \frac{dF(x)}{dx}.$$

Identity (2.13) will play the role of Legendre's relation (1.7). Note that when $\alpha = 1/2$, we have from (2.13) that

$$F\left(\frac{1}{2}\right) F'\left(\frac{1}{2}\right) = \frac{2}{\pi}. \tag{2.14}$$

Now, from (2.1), (2.3) and (2.4), we find that if $\beta = \alpha(q^2)$, then

$$\frac{F(\alpha)}{F(\beta)} = \frac{2}{1+\sqrt{1-\alpha}}.$$

This implies that

$$(1+\sqrt{1-\alpha})F(\alpha) = 2F(\beta). \tag{2.15}$$

Differentiating (2.15) with respect to α, we deduce that

$$-\frac{1}{2\sqrt{1-\alpha}}F(\alpha) + (1+\sqrt{1-\alpha})F'(\alpha) = 4F'(\beta)\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\frac{F^2(\beta)}{F^2(\alpha)}, \tag{2.16}$$

where we have used (2.8) with $\ell = 2$. Multiplying (2.16) by $2\alpha(1-\alpha)$ and using (2.15), we deduce that

$$\begin{aligned} &-\frac{\alpha(1-\alpha)}{\sqrt{1-\alpha}}F(\alpha) + 2\alpha(1-\alpha)F'(\alpha)(1+\sqrt{1-\alpha}) \\ &= 2\beta(1-\beta)F'(\beta)(1+\sqrt{1-\alpha})^2. \end{aligned} \tag{2.17}$$

Next, set

$$G(\alpha) = 2\alpha(1-\alpha)F'(\alpha) + (1-\alpha)F(\alpha). \tag{2.18}$$

Our choice of $G(\alpha)$ is motivated by our attempt to replace the elliptic integral $E(\alpha)$ defined in (1.4) by expressions involving only $F(\alpha)$. We know from [6, p. 10, Exercise 3] that

$$E(\alpha) = 2(1-\alpha)\alpha K'(\alpha) + (1-\alpha)K(\alpha). \tag{2.19}$$

Motivated by (2.19), we let

$$G_{r,s}(\alpha) = r\alpha(1-\alpha)F'(\alpha) + s(1-\alpha)F(\alpha),$$

with $r, s \in \mathbf{Q}$ and choose r and s such that (2.17) simplifies (see (2.21)). This happens when $r = 2$ and $s = 1$ and leads to our choice of $G(\alpha)$. It is clear that in this case, $G(\alpha)$ coincides with $E(\alpha)$ when we express $K(\alpha)$ as the hypergeometric representation of $F(\alpha)$. We will see in the next few sections that our analogues of $G(\alpha)$ do not always coincide with the analogues of $E(\alpha)$, namely, E_s, $s = 1/6, 1/4$ (see [6, (5.5.5)]).

Now, from (2.1), (2.2) and (2.3), we deduce that

$$\beta = \left(\frac{1 - \sqrt{1-\alpha}}{1 + \sqrt{1-\alpha}} \right)^2. \tag{2.20}$$

Substituting (2.20) and using (2.18) and (2.15) in (2.17), we conclude that

$$G(\alpha) = (1 + \sqrt{1-\alpha})G(\beta) - \sqrt{1-\alpha}\,F(\alpha). \tag{2.21}$$

Identity (2.21) is equivalent to [6, Theorem 1.2 (d)].

We are now ready to establish the Brent–Salamin algorithm. Let

$$A_n = \theta_3^2(e^{-2^n\pi}),\ B_n = \theta_4^2(e^{-2^n\pi}) \quad \text{and} \quad C_n = \theta_2^2(e^{-2^n\pi}). \tag{2.22}$$

Note that by (2.1) and (2.2), we have

$$A_{n+1} = \frac{A_n + B_n}{2}, \quad \text{and} \quad B_{n+1} = \sqrt{A_n B_n}.$$

Let

$$\alpha_n = \left(\frac{C_n}{A_n} \right)^2.$$

By (2.3), we have

$$\sqrt{1-\alpha_n} = \frac{B_n}{A_n}. \tag{2.23}$$

Next, substitute $\alpha = \alpha_n$ in (2.21) and using (2.23), we deduce that

$$G(\alpha_n) = \left(1 + \frac{B_n}{A_n}\right) G(\alpha_{n+1}) - \frac{B_n}{A_n} F(\alpha_n)$$

or

$$A_n G(\alpha_n) = 2A_{n+1} G(\alpha_{n+1}) - B_n A_n \frac{F(\alpha_n)}{A_n}.$$

Now, (2.6) and (2.22) show that for all $n \geq 0$,

$$\frac{F(\alpha_n)}{A_n} = 1.$$

Therefore,

$$2A_{n+1} G(\alpha_{n+1}) - A_n G(\alpha_n) = A_n B_n.$$

Next, we express $A_n B_n$ in terms of A_{n+1}^2 and A_n^2 and our final result is

$$2A_{n+1}^2 - A_n^2 = \frac{B_n^2 - A_n^2}{2} + A_n B_n.$$

Hence,

$$2(A_{n+1}G(\alpha_{n+1}) - A_{n+1}^2) - (A_n G(\alpha_n) - A_n^2) = 2^{-1}(A_n^2 - B_n^2). \tag{2.24}$$

Multiplying (2.24) by 2^n, we find that

$$2^{n+1}(A_{n+1}G(\alpha_{n+1}) - A_{n+1}^2) - 2^n(A_n G(\alpha_n) - A_n^2) = 2^{n-1}(A_n^2 - B_n^2). \tag{2.25}$$

We next sum both sides of (2.25) from 0 to N to deduce that

$$2^{N+1}\left(A_{N+1}G\left(\alpha_{N+1}\right) - A_{N+1}^2\right) - \left(A_0 G\left(\alpha_0\right) - A_0^2\right) = \sum_{n=0}^{N} 2^{n-1}\left(A_n^2 - B_n^2\right). \tag{2.26}$$

Now, from (2.18) and the fact that

$$\alpha_n = e^{-2^n\pi} + \cdots,$$

we find that

$$\lim_{N\to\infty} 2^{N+1}\left(A_{N+1}G(\alpha_{N+1}) - A_{N+1}^2\right) = 0.$$

Therefore, letting N tends to infinity in (2.26), we arrive at

$$-A_0 G(\alpha_0) + A_0^2 = \sum_{n=0}^{\infty} 2^{n-1}(A_n^2 - B_n^2) \tag{2.27}$$

or

$$A_0 G(\alpha_0) = A_0^2\left(1 - \sum_{n=0}^{\infty} 2^{n-1}\left(\left(\frac{A_n}{A_0}\right)^2 - \left(\frac{B_n}{A_0}\right)^2\right)\right). \tag{2.28}$$

Next, note that $\alpha_0 = 1/2$ if and only if

$$q = \exp\left(-\pi\frac{F(1-\alpha_0)}{F(\alpha_0)}\right) = e^{-\pi}.$$

Hence, by (2.18), we conclude, using

$$A_0 = F\left(\frac{1}{2}\right), \tag{2.29}$$

that

$$F\left(\frac{1}{2}\right)\left(\frac{1}{2}F'\left(\frac{1}{2}\right)+\frac{1}{2}F\left(\frac{1}{2}\right)\right)=A_0^2\left(1-\sum_{n=0}^{\infty}2^{n-1}\left(\left(\frac{A_n}{A_0}\right)^2-\left(\frac{B_n}{A_0}\right)^2\right)\right).$$

Using (2.14) and (2.29), we deduce that

$$\frac{2}{\pi}=F^2\left(\frac{1}{2}\right)\left(1-\sum_{n=0}^{\infty}2^n\left(\left(\frac{A_n}{A_0}\right)^2-\left(\frac{B_n^2}{A_0^2}\right)\right)\right).$$

Let

$$a_n=A_n/A_0, b_n=B_n/A_0.$$

Then, from (2.6) and (2.22), we find that

$$F(\alpha_n)A_0=F(\alpha_0)A_n.$$

From the theta function representation of $F(\alpha)$ (see (2.6)), we conclude that

$$\lim_{n\to\infty}F(\alpha_n)=1.$$

Hence,

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{A_n}{A_0}=\frac{1}{F\left(\frac{1}{2}\right)}.$$

This implies that

$$\frac{1}{\pi}=\frac{1}{2}(1-\sum_{j=0}^{\infty}2^j(a_j^2-b_j^2))F^2\left(\frac{1}{2}\right).$$

Identifying

$$F^2\left(\frac{1}{2}\right)=\lim_{n\to\infty}\frac{1}{a_n^2},$$

we have the following iteration:

Iteration 2.2 *(Brent–Salamin) Let* $a_0=1, b_0=\frac{1}{\sqrt{2}}$. *Let*

$$a_{n+1}=\frac{a_n+b_n}{2}, b_{n+1}=\sqrt{a_nb_n}$$

and

$$\pi_N = \frac{2a_N^2}{1 - \sum_{j=0}^{N} 2^j (a_j^2 - b_j^2)}.$$

Then,

$$\pi_N \to \pi.$$

The numbers π_N for $N = 1, 2, 3$ and 4 in Iteration 2.2 coincide with π for up to 1, 3, 9 and 20 decimal places, respectively.

In a similar way, we can derive the following iteration to π:

Iteration 2.3 *Let $a_0 = 1$ and $b_0 = \frac{1}{\sqrt[4]{2}}$. Let*

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt[4]{\frac{a_n^3 b_n + a_n b_n^3}{2}}$$

and

$$\pi_N = \frac{4a_N^4}{2 - \sum_{j=0}^{N} 4^j (a_j^2 - b_j^2)(b_j^2 + 3a_j^2)}.$$

Then, $\pi_N \to \pi$.

The numbers π_N for $N = 1, 2, 3$ and 4 in Iteration 2.3 coincide with π for up to 4, 20, 85 and 347 decimal places, respectively.

The derivation of the above is by applying the AGM iteration twice. More precisely, using (2.20) twice, we find that

$$\alpha(q^4) = \left(\frac{1 - \sqrt{1 - \alpha(q^2)}}{1 + \sqrt{1 - \alpha(q^2)}} \right)^2 = \left(\frac{1 - \sqrt[4]{1 - \alpha(q)}}{1 + \sqrt[4]{1 - \alpha(q)}} \right)^4.$$

This implies that

$$\hat{\beta} = \left(\frac{1 - \sqrt[4]{1 - \alpha}}{1 + \sqrt[4]{1 - \alpha}} \right)^4, \tag{2.30}$$

where

$$\hat{\beta} = \alpha(q^4) \tag{2.31}$$

and $\alpha = \alpha(q)$. Using (2.8) with $\ell = 4$ and (2.30), we deduce that

$$F(\alpha)(1 + \sqrt[4]{1 - \alpha})^2 = 4F(\hat{\beta}).$$

Differentiating both sides of the above with respect to α, we find that

$$-\frac{1}{2}\frac{(1+\sqrt[4]{1-\alpha})}{(1-\alpha)^{3/4}}F(\alpha)+(1+\sqrt[4]{1-\alpha})^2F'(\alpha)=F'(\hat{\beta})\frac{\hat{\beta}(1-\hat{\beta})}{\alpha(1-\alpha)}(1+\sqrt[4]{1-\alpha})^4. \tag{2.32}$$

Repeating what we did earlier and using the same $G(\alpha)$ defined in (2.18), we conclude that

$$G(\hat{\beta})(1+\sqrt[4]{1-\alpha})^2=G(\alpha)+\sqrt[4]{1-\alpha}(\sqrt{1-\alpha}+\sqrt[4]{1-\alpha}+1). \tag{2.33}$$

Let $\hat{A}_n=\theta_3(e^{-4^n\pi})$ and $\hat{B}_n=\theta_4(e^{-4^n\pi})$. Then, we write

$$\alpha(e^{-4^n\pi})=\alpha_n=1-\left(\frac{\hat{B}_n}{\hat{A}_n}\right)^4$$

and (2.33) is

$$4G(\hat{\beta}_n)\hat{A}_{n+1}^2=G(\alpha_n)\hat{A}_n^2+\hat{B}_n\hat{A}_n(\hat{B}_n^2+\hat{B}_n\hat{A}_n+\hat{A}_n^2).$$

Next, by simplifying $\hat{B}_n\hat{A}_n(\hat{B}_n^2+\hat{B}_n\hat{A}_n+\hat{A}_n^2)$ using $\hat{A}_{n+1}^4$ and $\hat{A}_n^4$, we observe that

$$\hat{B}_n\hat{A}_n(\hat{B}_n^2+\hat{B}_n\hat{A}_n+\hat{A}_n^2)=4\hat{A}_{n+1}^4-\hat{A}_n^4+\frac{1}{4}\left(\hat{A}_n^2-\hat{B}_n^2\right)\left(\hat{B}_n^2+3\hat{A}_n^2\right).$$

Hence, by repeating the argument as in the previous case, we conclude that

$$\hat{A}_0^2G(\alpha_0)=\hat{A}_0^4-\sum_{j=0}^{\infty}4^{j-1}\left(\hat{A}_j^2-\hat{B}_j^2\right)\left(\hat{B}_j^2+3\hat{A}_j^2\right). \tag{2.34}$$

As in the previous case, we can use (2.14) and (2.18) to conclude from (2.34) that

$$\frac{1}{\pi}=\hat{A}_0^4\left(\frac{1}{2}-\sum_{j=0}^{\infty}4^{j-1}\left(\hat{A}_j^2-\hat{B}_j^2\right)\left(\hat{B}_j^2+3\hat{A}_j^2\right)\right).$$

By letting

$$a_n=\frac{\hat{A}_n}{\hat{A}_0}\quad\text{and}\quad b_n=\frac{\hat{B}_n}{\hat{A}_0}$$

and truncating the series above, we conclude the proof of Iteration 2.3.

One should compare Iteration 2.3 with the Borweins' quartic iteration:

Iteration 2.4 *(The Borweins) Let $a_0 = 1$ and $b_0 = \sqrt[4]{12\sqrt{2} - 16}$. Let*

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt[4]{\frac{a_n^3 b_n + a_n b_n^3}{2}}$$

and

$$\pi_N = \frac{3a_N^4}{1 - \sum_{j=0}^{N} 4^{j+1}(a_j^4 - a_{j+1}^4)}.$$

Then, $\pi_N \to \pi$.

Note the difference between the initial values b_0 in Iterations 2.3 and 2.4. The Borweins used the initial value corresponding to $\alpha(q)$ at $q = e^{-2\pi}$ as the identity

$$1 - 24 \sum_{k=1}^{\infty} \frac{ke^{-2k\pi}}{1 - e^{-2k\pi}} = \frac{3}{\pi}$$

plays a crucial role in their proof of Iteration 2.4. The method of proof given in [7] is independent of elliptic integrals. It is straightforward once identities between theta functions and Eisenstein series

$$P(q) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^{2k}}{1 - q^{2k}}$$

are established. The only disadvantage of their method is that the derivations of their iterations depend on the explicit values of $P(q)$. For further development of this method, see [5].

Our method does not involve $P(q)$. It is, therefore, possible to derive a class of iterations with different b_0 since it is independent of our knowledge of explicit values of $P(q)$. To use the Borweins' method to produce an iteration with initial value for $\check{b}_0 = 1/\sqrt[4]{2}$, one would need to know the value of $P(e^{-\pi/2})$. We now produce an iteration with initial value $b_0 = (12\sqrt{2} - 16)^{1/4}$ (which is Borweins' initial values) using our new approach.

We have the following result that shows the difference between the Borweins' method and the present one:

Iteration 2.5 *Let $a_0 = 1$ and $b_0 = \sqrt[4]{12\sqrt{2} - 16}$. Let*

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt[4]{\frac{a_n^3 b_n + a_n b_n^3}{2}}$$

and

$$\pi_N = \frac{2a_N^4}{12 - 8\sqrt{2} - \sum_{j=0}^{N} 4^j (a_j^2 - b_j^2)(b_j^2 + 3a_j^2)}.$$

Then, $\pi_N \to \pi$.

The numbers π_N for $N = 1, 2, 3$ and 4 in both Iterations 2.4 and 2.5 coincide with π for up to 9, 42, 173 and 697 decimal places, respectively.

Proof If we study the proof of Iteration 2.3, we first observe that the difference between the condition of Iteration 2.3 and Iteration 2.5 is the initial value of $\hat{b}_0$ and the present value corresponds to setting $\hat{\hat{A}}_0 = \theta_3(e^{-2\pi})$ instead of $\theta_3(e^{-\pi})$. In other words, the value $\hat{\alpha}_0$ is replaced by

$$\hat{\hat{\alpha}}_0 = \alpha(e^{-2\pi}) = 17 - 12\sqrt{2}.$$

Now, (2.34) is still valid but we need to compute $\hat{\hat{A}}_0^2 G(\hat{\hat{\alpha}}_0)$. In our previous case, we compute $\hat{A}_0^2 G(1/2)$ using (2.14). We need another idea to compute $\hat{\hat{A}}_0^2 G(\hat{\hat{\alpha}}_0)$. It suffices to compute $F(\hat{\hat{\alpha}}_0)F'(\hat{\hat{\alpha}}_0)$. We recall (2.13) with $\alpha = \hat{\hat{\alpha}}_0$, namely,

$$\hat{\hat{\alpha}}_0(1 - \hat{\hat{\alpha}}_0)\left(F(\hat{\hat{\alpha}}_0)F'(1 - \hat{\hat{\alpha}}_0) + F(1 - \hat{\hat{\alpha}}_0)F'(\hat{\hat{\alpha}}_0)\right) = \frac{1}{\pi}. \tag{2.35}$$

Next, we multiply (2.32) by $F(\alpha)$ and set $\gamma = \alpha(e^{-\pi/2})$ to conclude that

$$\begin{aligned} -\frac{1}{2}\frac{(1 + \sqrt[4]{1-\gamma})}{(1-\gamma)^{3/4}} F^2(\gamma) + (1 + \sqrt[4]{1-\gamma})^2 F'(\gamma)F(\gamma) \\ = F'(\hat{\beta})F(\gamma)\frac{\hat{\beta}(1-\hat{\beta})}{\gamma(1-\gamma)}(1 + \sqrt[4]{1-\gamma})^4. \end{aligned} \tag{2.36}$$

Now, by (2.31),

$$\hat{\beta} = \alpha(e^{-2\pi}) = \hat{\hat{\alpha}}_0.$$

Now, the transformation formulas (2.9) for $\theta_3(q)$ and $\theta_2(q)$ imply that

$$\gamma = 1 - \hat{\hat{\alpha}}_0.$$

From (2.35) and (2.36), we conclude that

$$8F(\hat{\hat{\alpha}}_0)F'(\hat{\hat{\alpha}}_0) = -\frac{1}{(280 - 198\sqrt{2})\pi} - (10 + 7\sqrt{2})F^2(\hat{\hat{\alpha}}_0).$$

This allows us to evaluate $\hat{\hat{A}}_0^2 G(\hat{\hat{\alpha}}_0)$, and we complete the proof of Iteration 2.5. □

We note that the value $F(\hat{\hat{\alpha}}_0)F'(\hat{\hat{\alpha}}_0)$ obtained in the proof of Iteration 2.5 will give rise to another quadratic iteration for π. Returning to (2.28) with α_0 replaced by $\hat{\hat{\alpha}}_0$, we obtain the following new iteration similar to Brent–Salamin algorithm but with different initial value.

Iteration 2.6 *Let* $a_0 = 1$, $b_0 = \sqrt{12\sqrt{2} - 16}$. *Let*

$$a_n = \frac{a_{n-1} + b_{n-1}}{2}, \, b_n = \sqrt{a_{n-1}b_{n-1}}$$

and

$$\pi_N = \frac{a_N^2}{6 - 4\sqrt{2} - \sum_{j=0}^{N} 2^j (a_j^2 - b_j^2)}.$$

Then,

$$\pi_N \to \pi.$$

From the proofs of Iterations 2.2 and 2.5, we see that identities such as (2.27) (see also (2.34)) play very important roles in this method. For (2.27), the identity is true if we replace

$$A_n = \theta_3^2(e^{-2^n\pi}) \quad \text{and} \quad B_n = \theta_4^2(e^{-2^n\pi})$$

by

$$A_{n,s} = \theta_3^2(e^{-2^n\sqrt{s}\pi}) \quad \text{and} \quad B_{n,s} = \theta_4^2(e^{-2^n\sqrt{s}\pi}).$$

To establish an iteration corresponding to $A_{0,s}$ and $B_{0,s}$ as initial values, one would need to compute $F(\lambda_s)F'(\lambda_s)$ where

$$\lambda_s = \left(\frac{\theta_2(e^{-\pi\sqrt{s}})}{\theta_3(e^{-\pi\sqrt{s}})}\right)^4.$$

The proof of Iteration 2.5 illustrates how this is done. The keys lie in identities such as (2.35) and (2.36). The identity (2.35) will be the same with $\hat{\hat{\alpha}}_0$ replaced by λ_s, namely,

$$\lambda_s(1-\lambda_s)\left(F(\lambda_s)F'(1-\lambda_s) + F(1-\lambda_s)F'(\lambda_s)\right) = \frac{1}{\pi}. \tag{2.37}$$

We will also need an identity to replace (2.36).

We say that β has degree s over α if

$$\frac{F(1-\beta)}{F(\beta)} = s\frac{F(1-\alpha)}{F(\alpha)}. \tag{2.38}$$

This is equivalent to saying that if $\alpha = \alpha(q)$ then $\beta = \alpha(q^s)$ by the identification (2.12). In this section, we have already worked with β when $s = 2$ and 4.

The multiplier of degree s is defined by

$$m_s = \frac{F(\alpha)}{F(\beta)}. \tag{2.39}$$

Note that m_s can be expressed in terms of α and β. Now, by theory of modular functions, β and α are connected via some polynomial of two variables and the resulting equation is called a modular equation of degree s. An example of a modular equation of degree 4 is (2.30). In general, β can also be expressed in terms of α via modular equation of degree s and hence, m_s may also be viewed as a function of α. Next, note that when β has degree s over α, one has when $\alpha = 1-\beta$, we find from (2.38) that $\beta = \lambda_s$ since

$$\frac{F(1-\lambda_s)}{F(\lambda_s)} = \sqrt{s}.$$

Now, differentiating (2.39) with respect to α and simplifying with (2.8), we have

$$F'(\alpha) = \frac{dm_s}{d\alpha}F(\beta) + \frac{s}{m_s}\frac{\beta(1-\beta)}{\alpha(1-\alpha)}F'(\beta). \tag{2.40}$$

Multiplying by $F(\beta)$ and setting $\alpha = 1-\beta$, we conclude from (2.40) the following generalization of (2.36):

$$F'(1-\lambda_s)F(\lambda_s) = \left.\frac{dm_s}{d\alpha}\right|_{\alpha=1-\lambda_s} F^2(\lambda_s) + \frac{s}{m_s|_{\alpha=1-\lambda_s}}F'(\lambda_s)F(\lambda_s). \tag{2.41}$$

Identities (2.37) and (2.41) then imply that

$$F(\lambda_s)F'(\lambda_s) = \frac{1}{2\sqrt{s}}\frac{1}{\pi\lambda_s(1-\lambda_s)} - \frac{1}{2\sqrt{s}}\left.\frac{dm_s}{d\alpha}\right|_{\alpha=1-\lambda_s} F^2(\lambda_s). \tag{2.42}$$

Using (2.42), we find that

$$-A_{0,s}G(\lambda_s) = -\frac{1}{\pi\sqrt{s}} + \left.\frac{dm_s}{d\alpha}\right|_{\alpha=1-\lambda_s}\frac{\lambda_s(1-\lambda_s)}{\sqrt{s}}A_{0,s}^2 - (1-\lambda_s)A_{0,s}^2.$$

Substituting this into (2.27) where A_n and B_n are replaced by $A_{n,s}$ and $B_{n,s}$, respectively, we find that

$$\frac{1}{\sqrt{s}\pi} = A_{0,s}^2\left(\frac{dm_s}{d\alpha}\bigg|_{\alpha=1-\lambda_s}\frac{\lambda_s(1-\lambda_s)}{\sqrt{s}}+\lambda_s\right)-\sum_{j=0}^{\infty}2^{j-1}\left(A_{n,s}^2-B_{n,s}^2\right).$$

This yields the following iteration :

Iteration 2.7 *Let*

$$a_{0,s}=1 \quad and \quad b_{0,s}=\frac{\theta_4^2(e^{-\pi\sqrt{s}})}{\theta_3^2(e^{-\pi\sqrt{s}})}.$$

Let

$$a_{n,s}=\frac{a_{n-1,s}+b_{n-1,s}}{2} \quad and \quad b_{n,s}=\sqrt{a_{n-1,s}b_{n-1,s}}.$$

If

$$\pi_{N,s}=\frac{a_{N,s}^2}{\sqrt{s}\left(\frac{dm_s}{d\alpha}\bigg|_{\alpha=1-\lambda_s}\frac{\lambda_s(1-\lambda_s)}{\sqrt{s}}+\lambda_s\right)-\sqrt{s}\sum_{j=0}^{N}2^{j-1}\left(a_{j,s}^2-b_{j,s}^2\right)}, \tag{2.43}$$

then

$$\pi_{N,s}\to\pi.$$

For results similar to Iteration 2.7, see [6, p. 169, Theorem 5.5].

When $s=4$,

$$m_4=\frac{4}{(1+\sqrt[4]{1-\alpha})^2}$$

and we recover Iteration 2.6. When $s=3$, we use the modular equation [2, p. 230, Entry 5(iii)]

$$m_3=1+2\sqrt[8]{\frac{\beta}{\alpha}},$$

where β has degree 3 over α and obtain the iteration

Iteration 2.8 *Let*

$$a_{0,3}=1 \quad and \quad b_{0,3}=\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{4}}.$$

Let

$$a_{n,3} = \frac{a_{n-1,3} + b_{n-1,3}}{2} \quad \textit{and} \quad b_{n,3} = \sqrt{a_{n-1,3} b_{n-1,3}}.$$

If

$$\pi_{N,3} = \frac{a_{N,3}^2}{\frac{\sqrt{3}}{2} - \frac{1}{2} - \sqrt{3} \sum_{j=0}^{N} 2^{j-1} \left(a_{j,3}^2 - b_{j,3}^2\right)},$$

then

$$\pi_{N,3} \to \pi.$$

The numbers $\pi_{N,3}$ for $N = 1, 2, 3$ and 4 in Iteration 2.8 coincide with π for up to 3, 8, 17 and 36 decimal places, respectively.

In [7], the Borweins indicated that any iteration to π that we have shown so far leads naturally to an iteration to $1/\pi$. We end this section by obtaining iterations to $1/\pi$ from Iteration 2.7. First, note that from (2.43), we find that

$$\frac{1}{\pi_{N,s}} = \frac{1}{a_{N,s}^2} \sqrt{s} \left(\left. \frac{dm_s}{d\alpha} \right|_{\alpha=1-\lambda_s} \frac{\lambda_s(1-\lambda_s)}{\sqrt{s}} + \lambda_s \right) - \sqrt{s} \sum_{j=0}^{N} 2^{j-1} \left(a_{j,s}^2 - b_{j,s}^2\right) \tag{2.44}$$

and

$$\frac{1}{\pi_{N-1,s}} = \frac{1}{a_{N-1,s}^2} \sqrt{s} \left(\left. \frac{dm_s}{d\alpha} \right|_{\alpha=1-\lambda_s} \frac{\lambda_s(1-\lambda_s)}{\sqrt{s}} + \lambda_s \right) - \sqrt{s} \sum_{j=0}^{N-1} 2^{j-1} \left(a_{j,s}^2 - b_{j,s}^2\right). \tag{2.45}$$

Subtracting (2.45) from (2.44) and letting $k_{N,s} = 1/\pi_{N,s}$, we conclude after simplifying that

$$k_{N,s} = \left(\left. m_s^2 \right|_{\alpha=1-\lambda_{N,s}} \right) \cdot k_{N-1,s} - \sqrt{s} \cdot 2^{N-1} \lambda_{N,s},$$

where

$$\lambda_{0,s} = \lambda_s \quad \text{and} \quad \lambda_{N,s} = \alpha(e^{-2^N \pi \sqrt{s}}).$$

Hence, we have the following iteration :

Iteration 2.9 *(For $1/\pi$) Let*

$$m_s = \frac{\theta_3^2(e^{-\pi t})}{\theta_3^2(e^{-s\pi t})}.$$

Let $\lambda_{0,s} = \alpha(e^{-\pi\sqrt{s}})$,

$$k_{0,s} = \sqrt{s}\left(\frac{dm_s}{d\alpha}\bigg|_{\alpha=1-\lambda_s} \frac{\lambda_s(1-\lambda_s)}{\sqrt{s}} + \lambda_s\right) - \frac{\sqrt{s}}{2}\lambda_{0,s}$$

and

$$\lambda_{N,s} = \left(\frac{1-\sqrt{1-\lambda_{N-1,s}}}{1+\sqrt{1-\lambda_{N-1,s}}}\right)^2.$$

If

$$k_{N,s} = \left(m_s^2\Big|_{\alpha=1-\lambda_{N,s}}\right)\cdot k_{N-1,s} - \sqrt{s}\cdot 2^{N-1}\lambda_{N,s},$$

then

$$k_{N,s} \to \frac{1}{\pi}.$$

As indicated earlier, the method of obtaining iteration for $1/\pi$ from an iteration for π is due to the Borweins [7]. For more iterations of this type, see [5] and [10].

We illustrate Iteration 2.9 using an example. When $s = 2$, $\lambda_{0,2} = 3 - 2\sqrt{2}$ (substitute $\alpha = 1 - \beta$ into (2.20)) and hence $b_{0,2} = \sqrt{1 - \lambda_{0,2}}$. In terms of $\lambda_{N,2}$, we have

$$m_2 = 1 + \sqrt{\lambda_{N,2}}.$$

Now, setting $s_{N,2} = \sqrt{\lambda_{N,2}}$, we obtain the following iteration :

Iteration 2.10 *Let* $s_{0,2} = \sqrt{3-2\sqrt{2}}$, $k_{0,2} = 1 - \frac{\sqrt{2}}{2}$ *and*

$$s_{n,2} = \frac{1-\sqrt{1-s_{n-1,2}^2}}{1+\sqrt{1-s_{n-1,2}^2}}.$$

If

$$k_{N,2} = (1+s_{N,2})^2 k_{N-1,2} - \sqrt{2}\cdot 2^{N-1} s_{N,2}^2,$$

then

$$k_{N,s} \to \frac{1}{\pi}.$$

The numbers $k_{N,2}$ for $N = 1, 2, 3$ and 4 in Iteration 2.2 coincide with $1/\pi$ for up to 3, 7, 15 and 30 decimal places, respectively.

We end this section by showing that identity (2.42) gives rise to Ramanujan's series for $1/\pi$ associated with the hypergeometric series $F(u)$. More precisely, since [12, (25)]

$$F^2(u) = {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; 1; 4u(1-u)\right) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3}(4u(1-u))^k,$$

we deduce from (2.42) that

$$\frac{1}{\sqrt{s}\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3} (4\lambda_s(1-\lambda_s))^k \left((1-2\lambda_s)k + \frac{\lambda_s(1-\lambda_s)}{\sqrt{s}} \cdot \left.\frac{dm_s}{d\alpha}\right|_{\alpha=1-\lambda_s}\right).$$

In the special case when $s = 2$, we have

$$\frac{1}{\pi} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^3}{(1)_k^3}(40\sqrt{2}-56)^k \left((8-5\sqrt{2})k + 3 - 2\sqrt{2}\right).$$

3 The Borweins cubic theta functions

There are analogues of the iterations we discussed in the previous sections for elliptic functions to alternative bases. We will not give too many details as the proofs are similar. We give the relevant identities which allow us to construct analogues of Brent–Salamin algorithm.

Theorem 3.1 *Let*

$$a(q) = \sum_{m,n\in\mathbf{Z}} q^{m^2+mn+n^2},$$
$$b(q) = \sum_{m,n\in\mathbf{Z}} e^{2\pi i(m-n)/3} q^{m^2+mn+n^2},$$
$$c(q) = \sum_{m,n\in\mathbf{Z}} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

Then,

$$a^3(q) = b^3(q) + c^3(q) \tag{3.1}$$
$$3a(q^3) = a(q) + 2b(q) \tag{3.2}$$
$$b^3(q^3) = \frac{b(q)\left(a^2(q) + a(q)b(q) + b^2(q)\right)}{3} \tag{3.3}$$
$$a(q) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \frac{c^3(q)}{a^3(q)}\right). \tag{3.4}$$

For proofs of (3.1)–(3.4), see [8]. For more information about elliptic functions to cubic base, see [3].

For simplicity, let

$$\tilde{\alpha} = \tilde{\alpha}(q) = \frac{c^3(q)}{a^3(q)} \tag{3.5}$$

and

$$\tilde{F}(\tilde{\alpha}) = {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; \tilde{\alpha}\right).$$

Next, we recall the following differential equation [11, (4.7)]

$$q\frac{d\tilde{\alpha}}{dq} = (\tilde{F}(\tilde{\alpha}))^2\tilde{\alpha}(1-\tilde{\alpha}).$$

This is of the same form as (2.7). As such, the following analogue of (2.8) also holds:

$$\frac{d\tilde{\alpha}(q^\ell)}{d\tilde{\alpha}(q)} = \ell\frac{\tilde{\alpha}(q^\ell)(1-\tilde{\alpha}(q^\ell))}{\tilde{\alpha}(q)(1-\tilde{\alpha}(q))}\frac{\tilde{F}^2(\tilde{\alpha}(q^\ell))}{\tilde{F}^2(\tilde{\alpha}(q))}.$$

It is known from the transformation formula of $\eta(t)$ [14] and η-product representations of $a(q)$, $b(q)$ and $c(q)$ [8] that

$$a(e^{-2\pi/(\sqrt{3}t)}) = ta(e^{-2\pi t/\sqrt{3}}), b(e^{-2\pi/(\sqrt{3}t)}) = tc(e^{-2\pi t/\sqrt{3}}).$$

This implies that

$$t = \frac{\tilde{F}(1-\tilde{\alpha})}{\tilde{F}(\tilde{\alpha})}. \tag{3.6}$$

As in the previous section, this implies that if (3.5) holds, then

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}}\frac{\tilde{F}(1-\tilde{\alpha})}{\tilde{F}(\tilde{\alpha})}\right).$$

Using exactly the same method as in the previous section, we obtain the following analogue of (2.13) :

$$\tilde{\alpha}(1-\tilde{\alpha})\left(\tilde{F}'(\tilde{\alpha})\tilde{F}(1-\tilde{\alpha}) + \tilde{F}'(1-\tilde{\alpha})\tilde{F}(\tilde{\alpha})\right) = \frac{\sqrt{3}}{2\pi} \tag{3.7}$$

where

$$\tilde{F}'(x) = \frac{d\tilde{F}(x)}{dx}.$$

Note that

$$\tilde{F}\,(1/2)\,\tilde{F}'\,(1/2) = \frac{\sqrt{3}}{\pi},$$

which is the analogue of (2.14).

We also need the modular equation of degree 3 in the cubic theory, which is

$$\frac{\tilde{F}(\tilde{\alpha})}{\tilde{F}(\tilde{\beta})} = \frac{3}{1+2\sqrt[3]{1-\tilde{\alpha}}}$$

where

$$\tilde{\beta} = \tilde{\alpha}(q^3).$$

This follows from (3.1), (3.2), (3.3) and (3.4).

Differentiating with respect to $\tilde{\alpha}$ yields

$$9\tilde{F}'(\tilde{\beta})\frac{\tilde{\beta}(1-\tilde{\beta})}{\tilde{\alpha}(1-\tilde{\alpha})}\left(\frac{\tilde{F}(\tilde{\beta})}{\tilde{F}(\tilde{\alpha})}\right)^2 = -\frac{2}{3\sqrt[3]{(1-\tilde{\alpha})^2}}\tilde{F}(\tilde{\alpha}) + \left(1+2\sqrt[3]{1-\tilde{\alpha}}\right)\tilde{F}'(\tilde{\alpha}). \tag{3.8}$$

The key now is to choose a suitable $\tilde{G}(\tilde{\alpha})$ which plays the role of $G(\alpha)$ in the previous section. This is where our choice differs slightly from the Borweins. According to [6, p. 178, (5.5.5)], the Borweins suggested

$$E_{1/6}(\tilde{\alpha}) = (1-\tilde{\alpha})\tilde{F}(\tilde{\alpha}) + \frac{3}{4}\tilde{\alpha}(1-\tilde{\alpha})\tilde{F}'(\tilde{\alpha})$$

but this choice appears to give complicated expressions. We propose to let

$$\tilde{G}(\tilde{\alpha}) = (1-\tilde{\alpha})\tilde{F}(\tilde{\alpha}) + 3\tilde{\alpha}(1-\tilde{\alpha})\tilde{F}'(\tilde{\alpha}).$$

This function is chosen to clear denominators that appear in (3.8), and the final result is

$$\tilde{G}(\tilde{\alpha}) = (1+2\sqrt[3]{1-\tilde{\alpha}})\tilde{G}(\tilde{\beta}) - \left(\sqrt[3]{1-\tilde{\alpha}} + \sqrt[3]{(1-\tilde{\alpha})^2}\right)\tilde{F}(\tilde{\alpha}). \tag{3.9}$$

Now, let

$$\tilde{A}_n = a(e^{-3^n\cdot 2\pi/\sqrt{3}}),\ \tilde{B}_n = b(e^{-3^n\cdot 2\pi/\sqrt{3}}),\ \tilde{C}_n = c(e^{-3^n\cdot 2\pi/\sqrt{3}}),$$

and

$$\tilde{\alpha}_n = \frac{(\tilde{C}_n)^3}{(\tilde{A}_n)^3}.$$

By (3.9), we find that

$$\tilde{A}_n\tilde{G}(\tilde{\alpha}_n) = 3\tilde{A}_{n+1}\tilde{G}(\tilde{\alpha}_{n+1}) - \left(\tilde{B}_n + \frac{(\tilde{B}_n)^2}{\tilde{A}_n}\right)\tilde{F}(\tilde{\alpha}_n).$$

As before,

$$\tilde{F}(\tilde{\alpha}_n)/\tilde{A}_n = 1.$$

It remains to express

$$\tilde{A}_n\tilde{B}_n + (\tilde{B}_n)^2$$

in terms of $(\tilde{A}_{n+1})^2$ and $(\tilde{A}_n)^2$. It turns out that

$$\tilde{A}_n\tilde{B}_n + (\tilde{B}_n)^2 = 3(\tilde{A}_{n+1})^2 - (\tilde{A}_n)^2 - \frac{(\tilde{B}_n)^2 + \tilde{A}_n\tilde{B}_n - 2(\tilde{A}_n)^2}{3}.$$

As before, we conclude that

$$\tilde{A}_0G(\tilde{\alpha}_0) = (\tilde{A}_0)^2 - \sum_{j=0}^{\infty} 3^{j-1}\left(2\tilde{A}_j + \tilde{B}_j\right)\left(\tilde{A}_j - \tilde{B}_j\right).$$

Following the previous case, we have the following iteration:

Iteration 3.2 *Let* $a_0 = 1$ *and* $b_0 = 1/\sqrt[3]{2}$. *Let*

$$a_{n+1} = \frac{a_n + 2b_n}{3},\ b_{n+1} = \sqrt[3]{\frac{b_n((a_n)^2 + a_nb_n + (b_n)^2)}{3}},$$

and

$$\pi_N = \frac{3\sqrt{3}(a_N)^2}{2 - 4\sum_{j=0}^{N} 3^{j-1}\left(2a_j + b_j\right)\left(a_j - b_j\right)}.$$

then, $\pi_N \to \pi$.

Truncated in this way and letting $k_n = 1/\pi_N$ (see the proof of Iteration 2.9), we obtain a new iteration for $1/\pi$ given by the following :

Iteration 3.3 *(For $1/\pi$) Let*

$$k_0 = \frac{2\sqrt{3}}{27}\left(-1 + \sqrt[3]{2} + \sqrt[3]{4}\right), \; s_0 = \frac{1}{\sqrt[3]{2}},$$

$$s_n = \sqrt[3]{1 - \left(\frac{1 - s_{n-1}}{1 + 2s_{n-1}}\right)^3}$$

and

$$k_n = \left(\frac{3}{1 + 2s_{n-1}}\right)^2 k_{n-1} - \frac{4\sqrt{3}}{9} 3^{n-1}(2 - s_n - (s_n)^2).$$

Then, k_n tends to $1/\pi$.

The numbers k_n for $n = 1, 2, 3$ and 4 in Iteration 3.3 coincide with $1/\pi$ for up to 4, 13, 41 and 127 decimal places, respectively.

It is clear that general result similar to Iteration 2.7 exists for cubic theory of elliptic functions and we leave this as an exercise for the reader.

4 Quartic theory

In this section, we list down the relevant functions for the 'quartic theory of elliptic functions'.

Theorem 4.1 *Let*

$$A(q) = \theta_3^4(q) + \theta_2^4(q),$$
$$B(q) = \theta_3^4(q) - \theta_2^4(q),$$
$$C(q) = 2\theta_3^2(q)\theta_2^2(q).$$

Then,

$$A^2(q) = B^2(q) + C^2(q) \tag{4.1}$$

$$4A(q^2) = A(q) + 3B(q) \tag{4.2}$$

$$B^2(q^2) = \sqrt{\frac{B(q)(A(q) + B(q))}{2}} \tag{4.3}$$

$$\sqrt{A(q)} = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \frac{C^2(q)}{A^2(q)}\right). \tag{4.4}$$

The proofs of all the above identities can be found in [4].

For simplicity, let

$$\breve{\alpha} = \breve{\alpha}(q) = \frac{C^2(q)}{A^2(q)} \tag{4.5}$$

and

$$\breve{F}(\breve{\alpha}) = {}_2F_1\left(\frac{1}{4}, \frac{3}{4}; 1; \breve{\alpha}\right).$$

Next, we recall the following differential equation [4]

$$q\frac{d\breve{\alpha}}{dq} = (\breve{F}(\breve{\alpha}))^2\breve{\alpha}(1-\breve{\alpha}).$$

This is of the same form as (2.7). As such, the following analogue of (2.8) also holds:

$$\frac{d\breve{\alpha}(q^\ell)}{d\breve{\alpha}(q)} = \ell\frac{\breve{\alpha}(q^\ell)(1-\breve{\alpha}(q^\ell))}{\breve{\alpha}(q)(1-\breve{\alpha}(q))}\frac{\breve{F}^2(\breve{\alpha}(q^\ell))}{\breve{F}^2(\breve{\alpha}(q))}.$$

It can be shown using transformation formulas (2.9) that

$$A(e^{-\pi\sqrt{2}/t}) = t^2A(e^{-\pi\sqrt{2}t}),\ B(e^{-\pi\sqrt{2}/t}) = t^2C(e^{-\pi\sqrt{2}t}).$$

This implies using (4.4) that

$$t = \frac{\breve{F}(1-\breve{\alpha})}{\breve{F}(\breve{\alpha})}$$

since

$$\breve{\alpha}(e^{-\sqrt{2}\pi/t}) = 1-\breve{\alpha}(e^{-\sqrt{2}\pi t}).$$

Therefore, if (4.5) holds, then

$$q = \exp\left(-\sqrt{2}\pi\frac{\breve{F}(1-\breve{\alpha})}{\breve{F}(\breve{\alpha})}\right).$$

Using exactly the same method as in Sect. 2, we obtain the following analogue of (2.13) :

$$\breve{\alpha}(1-\breve{\alpha})\left(\breve{F}'(\breve{\alpha})\breve{F}(1-\breve{\alpha}) + \breve{F}'(1-\breve{\alpha})\breve{F}(\breve{\alpha})\right) = \frac{1}{\pi\sqrt{2}}$$

where

$$\breve{F}'(x) = \frac{d\breve{F}(x)}{dx}.$$

Note that

$$\breve{F}\,(1/2)\,\breve{F}'\,(1/2) = \frac{\sqrt{2}}{\pi},$$

which is the analogue of (2.14).

We also need the modular equation of degree 2 in the quartic theory, which is

$$\frac{\breve{F}(\breve{\alpha})}{\breve{F}(\breve{\beta})} = \frac{2}{\sqrt{1+3\sqrt{1-\breve{\alpha}}}}$$

where

$$\breve{\beta} = \breve{\alpha}(q^2).$$

Differentiating with respect to $\breve{\alpha}$ yields

$$\begin{aligned}&\breve{\beta}(1-\breve{\beta})\breve{F}'(\breve{\beta})\left(1+3\sqrt{1-\breve{\alpha}}\right)\\ &\quad = -\frac{3}{4}\frac{\breve{\alpha}(1-\breve{\alpha})}{\sqrt{1-\breve{\alpha}}\sqrt{1+3\sqrt{1-\breve{\alpha}}}}\breve{F}(\breve{\alpha}) + \breve{\alpha}(1-\breve{\alpha})\breve{F}'(\breve{\alpha})\sqrt{1+3\sqrt{1-\breve{\alpha}}}.\end{aligned} \tag{4.6}$$

The key now is to choose a suitable $\breve{G}(\breve{\alpha})$ to play the role of $G(\alpha)$ and $\tilde{G}(\tilde{\alpha})$ in the previous sections. According to [6, p. 178, (5.5.5)], the Borweins suggested

$$E_{1/4}(\breve{\alpha}) = (1-\breve{\alpha})\breve{F}(\breve{\alpha}) + \frac{2}{3}\breve{\alpha}(1-\breve{\alpha})\breve{F}'(\breve{\alpha})$$

but we propose to let

$$\breve{G}(\breve{\alpha}) = (1-\breve{\alpha})\breve{F}(\breve{\alpha}) + 4\breve{\alpha}(1-\breve{\alpha})\breve{F}'(\breve{\alpha}).$$

This function is chosen to clear denominators that appear in (4.6), and the final result is

$$\breve{G}(\breve{\alpha}) = \sqrt{1+3\sqrt{1-\breve{\alpha}}}\,\breve{G}(\breve{\beta}) - \sqrt{1-\breve{\alpha}}\,\breve{F}(\breve{\alpha}). \tag{4.7}$$

Now, let

$$\breve{A}_n = A(e^{-2^n\cdot\sqrt{2}\pi}),\ \breve{B}_n = B(e^{-2^n\cdot\sqrt{2}\pi}),\ \breve{C}_n = C(e^{-2^n\cdot\sqrt{2}\pi}),$$

and

$$\breve{\alpha}_n = \frac{(\breve{C}_n)^2}{(\breve{A}_n)^2}.$$

By (4.7), we find that

$$\sqrt{\breve{A}_n}\breve{G}(\breve{\alpha}_n) = 2\sqrt{\breve{A}_{n+1}}\breve{G}(\breve{\alpha}_{n+1}) - \breve{B}_n.$$

As before, we have used

$$\breve{F}(\breve{\alpha}_n)/\sqrt{\breve{A}_n} = 1.$$

Next, observe that

$$\breve{B}_n = 2\breve{A}_{n+1} - \breve{A}_n - \frac{\breve{B}_n - \breve{A}_n}{2}.$$

Using this relation and simplifying as before, we conclude that

$$\sqrt{\breve{A}_0}\breve{G}(\breve{\alpha}_0) = \breve{A}_0 - \sum_{j=0}^{\infty} 2^{j-1}\left(\breve{A}_j - \breve{B}_j\right).$$

Following the previous cases, we have the following iteration:

Iteration 4.2 *Let* $a_0 = 1$ *and* $b_0 = \frac{1}{\sqrt{2}}$. *Let*

$$a_{n+1} = \frac{a_n + 3b_n}{4}, \; b_{n+1} = \sqrt{\frac{b_n(a_n + b_n)}{2}}$$

and

$$\pi_N = \frac{2\sqrt{2}a_N}{1 - \sum_{j=0}^{N} 2^j (a_j - b_j)}.$$

Then, $\pi_N \to \pi$.

The numbers π_N for $N = 1, 2, 3$ and 4 in Iteration 4.2 coincide with π for up to 1, 5, 13 and 28 decimal places, respectively.

It is clear that general result similar to Iteration 2.7 exists for quartic theory of elliptic functions and we leave this as an exercise for the reader.

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