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An alternative transformation formula for the Dedekind η-function via the Chinese Remainder Theorem

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Dedicated to Professor T. M. Apostol

In this article, we will give a new proof of the reciprocity law for Dedekind sums, as well as a proof of the transformation formula for the Dedekind η-function using the Chinese Remainder Theorem.

Keywords: Dedekind sum; Dedekind η-function; Chinese Remainder Theorem.

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1. Introduction

One of the most important functions in the theory of modular forms is the Dedekind η-function defined by

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi in\tau}), \quad \operatorname{Im}\tau > 0.$$

The Dedekind η-function satisfies the following transformation formula [4, p. 163].

Theorem 1.1. *If a, b, c, d are integers with $ad - bc = 1$, $c > 0$, and $\operatorname{Im}\tau > 0$, then*

$$\eta\left(\frac{a\tau + b}{c\tau + d}\right) = \nu(a, b, c, d)\sqrt{\frac{c\tau + d}{i}}\eta(\tau),$$

where

$$\nu(a,b,c,d) = \begin{cases} \left(\frac{d}{c}\right) i^{(1-c)/2} \exp\left(\frac{\pi i}{12}(bd(1-c^2)+c(a+d))\right), & c \text{ odd and } c>0, \\ \left(\frac{c}{d}\right) \exp\left(\frac{\pi i}{12}(3d+ac(1-d^2)+d(b-c))\right), & c \text{ even and } d>0. \end{cases}$$

Here

$$\left(\frac{h}{k}\right)$$

is the Jacobi symbol and it is defined when $k>1$ is odd. When $k=1$, we set the value as 1 .

The expression for $\nu(a,b,c,d)$ in Theorem 1.1 is not unique. In this article, we will derive another expression for $\nu(a,b,c,d)$ using the Chinese Remainder Theorem. Our main result is stated as follows.

Theorem 1.2. *If a,b,c,d are integers with $ad-bc=1$, $c>0$, and $\operatorname{Im}\tau>0$, then*

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \xi(a,b,c,d)\sqrt{c\tau+d}\,\eta(\tau), \tag{1.1}$$

where

$$\xi(a,b,c,d) = \begin{cases} \left(\frac{d}{c}\right) \exp\left(\frac{\pi i}{12}(-3c+c(a+d)+(1-c^2)ab)\right), & c \text{ odd}, \\ \left(\frac{c}{d}\right) \exp\left(\frac{\pi i}{12}(3(d-1)+d(b-c)-(1-d^2)ab)\right), & c \text{ even and } d>0. \end{cases}$$

The proof of Theorem 1.2, as in the proof of Theorem 1.1, begins with the following transformation formula for $\eta(\tau)$.

Theorem 1.3. *Let a,b,c,d be integers with $c>0$ and $ad-bc=1$. Let the Dedekind sum be defined by*

$$s(h,k) = \sum_{r=0}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right). \tag{1.2}$$

If $\operatorname{Im}\tau>0$, then

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon(a,b,c,d)\sqrt{c\tau+d}\,\eta(\tau), \tag{1.3}$$

where

$$\epsilon(a,b,c,d) = \exp(\pi i\phi(a,b,c,d)),$$

with

$$\phi(a,b,c,d) = \frac{a+d}{12c} + s(-d,c) - \frac{1}{4}. \tag{1.4}$$

Theorem 1.3 follows from Iseki's transformation formula [4; 2, Theorem 3.4].

The Dedekind sum $s(h,k)$ in Theorem 1.3 is not easy to compute. It is for this reason that we need to replace $\epsilon(a,b,c,d)$ with $\nu(a,b,c,d)$ or $\xi(a,b,c,d)$. We discovered $\xi(a,b,c,d)$ when we were trying to simplify $\epsilon(a,b,c,d)$ using the Chinese Remainder Theorem. We had initially thought that $\xi(a,b,c,d)$ was new but was pleasantly surprised to find it (in the case when c is odd) in a famous article by Hardy and Ramanujan [2, p. 94]. It is likely that Hardy and Ramanujan derived Theorem 1.2 from Theorem 1.1 instead of Theorem 1.3.

We conclude this section with a summary of this article. In Sec. 2, we discuss the reciprocity law for the Dedekind sum $s(h,k)$. In Sec. 3, we simplify

$$12ks(h,k) \pmod{\ell}$$

for $\ell = 3, c$ and 8. In Sec. 4, we use the congruences obtained in Sec. 3 and the Chinese Remainder Theorem to simplify

$$12ks(h,k) \pmod{24k}.$$

In Sec. 5, we complete our derivation of $\xi(a,b,c,d)$. We emphasize that the results we used in this article can all be found in [1].

2. Some Properties of Dedekind Sums

We will first establish the reciprocity law for Dedekind sum.

Theorem 2.1. *If $h > 0, k > 0$ and $(h,k) = 1$, then*

$$s(h,k) + s(k,h) = \frac{h^2 + k^2 - 3hk + 1}{12hk}. \tag{2.1}$$

Proof. From the definition of $s(h,k)$, for positive integer $h \geq 1$,

$$s(h,1) = 0.$$

Next, by direct computations, we find that

$$s(1,k) = \frac{k^2 - 3k + 2}{12k}.$$

This implies that (2.1) is true if at least one of h or k is 1.

Suppose now that $h > 1$ and $k > 1$. Observe that

$$\begin{aligned} s(h,k) &= \sum_{r=1}^{k-1} \frac{r}{k}\left(\frac{hr}{k} - \left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\right) \\ &= \frac{h}{k^2}\sum_{r=1}^{k-1} r^2 - \sum_{r=1}^{k-1}\frac{r}{k}\left\lfloor \frac{hr}{k} \right\rfloor - \frac{1}{2}\sum_{r=1}^{k-1}\frac{r}{k} \\ &= \frac{(k-1)(-3k+4hk-2h)}{12k} - \sum_{r=1}^{k-1}\frac{r}{k}\sum_{1\leq \ell \leq \frac{hr}{k}} 1. \end{aligned} \tag{2.2}$$

The last sum is given by

$$\sum_{r=1}^{k-1}\frac{r}{k}\sum_{1\le \ell\le \frac{hr}{k}}1=\sum_{1\le\ell\le\frac{h(k-1)}{k}}\frac{1}{k}\sum_{\frac{\ell k}{h}\le r\le k-1}r$$

$$=\sum_{\ell=1}^{h-1}\frac{1}{2k}\left(k(k-1)-\left\lfloor\frac{\ell k}{h}\right\rfloor\left(\left\lfloor\frac{\ell k}{h}\right\rfloor+1\right)\right)$$

$$=\frac{(h-1)(k-1)}{2}-\frac{1}{2k}\sum_{\ell=1}^{h-1}\left\lfloor\frac{\ell k}{h}\right\rfloor\left(\left\lfloor\frac{\ell k}{h}\right\rfloor+1\right). \tag{2.3}$$

Substituting (2.3) into (2.2) and simplifying, we find that

$$s(h,k)=\frac{(k-1)(3k-2h-2hk)}{12k}+\frac{1}{2k}\sum_{\ell=1}^{h-1}\left\lfloor\frac{\ell k}{h}\right\rfloor\left(\left\lfloor\frac{\ell k}{h}\right\rfloor+1\right).$$

Now, using the identity

$$\left\lfloor\frac{\ell k}{h}\right\rfloor=\frac{\ell k}{h}-\left\{\frac{\ell k}{h}\right\},$$

we find that

$$\frac{1}{2k}\sum_{\ell=1}^{h-1}\left\lfloor\frac{\ell k}{h}\right\rfloor\left(\left\lfloor\frac{\ell k}{h}\right\rfloor+1\right)$$

$$=\frac{1}{2k}\sum_{\ell=1}^{h-1}\left(\frac{\ell^2k^2}{h^2}+\frac{\ell k}{h}+\left\{\frac{\ell k}{h}\right\}^2-\left\{\frac{\ell k}{h}\right\}\right)-\frac{1}{k}\sum_{\ell=1}^{h-1}\frac{\ell k}{h}\left\{\frac{\ell k}{h}\right\}.$$

Let

$$\ell k=q_\ell h+\lambda_\ell, \tag{2.4}$$

where $1\le\lambda_\ell\le h-1$. Since $(k,h)=1$, we conclude that

$$\{\lambda_\ell\in\mathbf{Z}\,|\,1\le\ell\le h-1\}=\{s\in\mathbf{Z}\,|\,1\le s\le h-1\}. \tag{2.5}$$

Now, by (2.4), we find that for any positive integer j,

$$\sum_{\ell=1}^{h-1}\left\{\frac{\ell k}{h}\right\}^j=\frac{1}{h^j}\sum_{\ell=1}^{h-1}\lambda_\ell^j=\frac{1}{h^j}\sum_{s=1}^{h-1}s^j, \tag{2.6}$$

by (2.5).

Using the above observation, we find that

$$\frac{1}{2k}\sum_{\ell=1}^{h-1}\left\lfloor\frac{\ell k}{h}\right\rfloor\left(\left\lfloor\frac{\ell k}{h}\right\rfloor+1\right)=\frac{(h-1)(2k^2h-h+3kh-k^2-1)}{12kh}-\sum_{\ell=1}^{h-1}\frac{\ell}{h}\left\{\frac{\ell k}{h}\right\}.$$

But

$$-\sum_{\ell=1}^{h-1}\frac{\ell}{h}\left\{\frac{\ell k}{h}\right\}=-\sum_{\ell=1}^{h-1}\frac{\ell}{h}\left(\frac{\ell k}{h}-\left\lfloor\frac{\ell k}{h}\right\rfloor\right)=-\frac{(h-1)}{4}-s(k,h).$$

Therefore,

$$\begin{aligned} s(h,k) &= \frac{(k-1)(3k-2h-2hk)}{12k} + \frac{(h-1)(2k^2h-h+3kh-k^2-1)}{12kh} \\ &\quad - \frac{(h-1)}{4} - s(k,h) \\ &= \frac{h^2+k^2-3hk+1}{12hk} - s(k,h). \end{aligned}$$

This completes the proof of (2.1). □

Remark 2.2. For other proofs of Theorem 2.1, see [5, pp. 148–150; 2, pp. 62–65; 6, Chap. 2].

We end this section by deriving another simple property of $s(h,k)$ that will be needed in the subsequent sections. By replacing h by $-h$, we find that

$$\begin{aligned} s(-h,k) &= \sum_{r=0}^{k-1} \frac{r}{k}\left(-\frac{hr}{k} - \left\lfloor \frac{-hr}{k} \right\rfloor - \frac{1}{2}\right) \\ &= \sum_{r=0}^{k-1} \frac{r}{k}\left(-\frac{hr}{k} + \left\lfloor \frac{hr}{k} \right\rfloor + 1 - \frac{1}{2}\right), \end{aligned}$$

where we have used the identity for $x \notin \mathbf{Z}$,

$$\lfloor -x \rfloor = -\lfloor x \rfloor - 1.$$

This implies that

$$s(-h,k) = -s(h,k). \tag{2.7}$$

3. Congruences Satisfied by the Dedekind Sum

In this section, we state and prove two lemmas satisfied by the Dedekind sum. The proofs of these lemmas are included with the aim of having a self-contained article. Lemma 3.1 can be found in [1, pp. 64–65] while Lemma 3.2 can be found in [1, (40)].

Lemma 3.1. *Let c and d be integers with $c > 0$.*

(a) *Suppose $3 \nmid c$. Then $6ds(c,d)$ is an integer and*

$$12cs(d,c) \equiv 0 \pmod{3}. \tag{3.1}$$

If $(c,d) = 1$ and d' is an integer satisfying $dd' \equiv 1 \pmod{c}$, then

$$12cs(d,c) \equiv d + d' \pmod{c}. \tag{3.2}$$

(b) *Suppose $3 \mid c$ and $(c,d) = 1$. Let d^* be an integer satisfying $dd^* \equiv 1 \pmod{3c}$. Then*

$$12cs(d,c) \equiv d + d^* \pmod{3c}. \tag{3.3}$$

Proof. We will first prove (a). From the definition of the Dedekind sum, we find that

$$\begin{aligned} 6cs(d,c) &= \frac{6d}{c}\sum_{r=0}^{c-1} r^2 - 6\sum_{r=0}^{c-1} r\left\lfloor \frac{dr}{c}\right\rfloor - 3\sum_{r=0}^{c-1} r \\ &= d(c-1)(2c-1) - 3\frac{c(c-1)}{2} - 6\sum_{r=0}^{c-1} r\left\lfloor \frac{dr}{c}\right\rfloor . \end{aligned} \tag{3.4}$$

Hence, $6cs(d,c)$ is an integer and

$$12cs(d,c) \equiv 2d(c-1)(2c-1) \equiv d(c-1)(c+1) \pmod 3. \tag{3.5}$$

Since $3 \nmid c$, (3.5) implies that

$$12cs(d,c) \equiv 0 \pmod 3.$$

This completes the proof of (3.1).

Next, by (2.1), we find that for $d > 0$,

$$12cds(d,c) + 12cds(c,d) = c^2 + d^2 - 3cd + 1. \tag{3.6}$$

From the proof of (3.1) with (c,d) replaced by (d,c), we conclude that $6ds(c,d)$ is an integer. Hence,

$$12cds(c,d) \equiv 0 \pmod c$$

and we deduce from (3.6) that

$$12cds(d,c) \equiv d^2 + 1 \pmod c. \tag{3.7}$$

Since

$$dd' \equiv 1 \pmod c,$$

we deduce that

$$12cs(d,c) \equiv d + d' \pmod c.$$

Hence the proof of (3.2) for positive integer d is complete.

Now, if $d < 0$, then from (3.7), we deduce that

$$12c(-d)s(-d,c) \equiv d^2 + 1 \pmod c.$$

Multiplying both sides by $-d'$ we deduce that

$$12cs(-d,c) \equiv -d - d' \pmod c.$$

Using (2.7), we deduce that

$$12cs(d,c) \equiv d + d' \pmod c$$

and this proves (3.2) for a negative integer d and the proof of (a) is complete.

To prove (b), we suppose d is a positive integer and that $3 \mid c$. By (3.6), we deduce that

$$12cds(d,c) + 12cds(c,d) \equiv d^2 + 1 \pmod{3c}.$$

But (3.4), with (c, d) replaced by (d, c), shows that

$$12cds(c,d) \equiv 2c^2(d-1)(2d-1) \equiv 0 \pmod{3c},$$

since

$$c^2 \equiv 0 \pmod{3c}.$$

Therefore,

$$12cs(d,c) \equiv d + d^* \pmod{3c},$$

where $dd^* \equiv 1 \pmod{3c}$. Using a similar argument as in the proof of (3.2), we find that (3.3) is also valid for $d < 0$. This completes the proof of the lemma. □

Lemma 3.2. *Let c and d be integers such that $(c, d) = 1$. If c is odd, then*

$$12cs(d,c) \equiv c - 1 + 4 \sum_{r=0}^{(c-1)/2} \left\lfloor \frac{2rd}{c} \right\rfloor \pmod{8}. \tag{3.8}$$

Proof. To prove the congruence, we first observe that by (2.6),

$$\sum_{r=0}^{c-1} \left\lfloor \frac{dr}{c} \right\rfloor = \sum_{r=0}^{c-1} \frac{dr}{c} - \sum_{r=0}^{c-1} \left\{ \frac{dr}{c} \right\}$$
$$= \frac{(d-1)(c-1)}{2}. \tag{3.9}$$

Next, from (3.4), we find using $4c(d-1) \equiv 0 \pmod{8}$ that

$$12cs(d,c) \equiv (c-1)(2d(2c-1) - 3c) - 4 \sum_{r=0}^{c-1} r \left\lfloor \frac{dr}{c} \right\rfloor \pmod{8}$$
$$\equiv (c-1)(c-2d) - 4 \sum_{r=0}^{c-1} \left\lfloor \frac{dr}{c} \right\rfloor + 4 \sum_{\substack{r=0 \\ r \text{ even}}}^{c-1} \left\lfloor \frac{dr}{c} \right\rfloor \pmod{8}.$$

Rewriting the last sum by letting $r = 2s$ and using (3.9), we conclude that

$$12cs(d,c) \equiv (c-1)(c-2d) + 4 \sum_{s=0}^{(c-1)/2} \left\lfloor \frac{2sd}{c} \right\rfloor - 2(d-1)(c-1) \pmod{8}$$
$$\equiv c^2 - 3c + 2 + 4 \sum_{s=0}^{(c-1)/2} \left\lfloor \frac{2sd}{c} \right\rfloor \pmod{8}. \tag{3.10}$$

Since c is odd, we find that

$$c^2 \equiv 1 \pmod{8}$$

and we deduce that

$$c^2 - 3c + 2 \equiv 3(1-c) \equiv (c-1) \pmod{8}$$

since $4(1-c) \equiv 0 \pmod{8}$. This completes the proof of Lemma 3.2. □

4. The Chinese Remainder Theorem and $s(d, c)$

If $(m, n) = 1$ then by the Chinese Remainder Theorem, we know that the integer

$$x = anu + bmv,$$

with

$$nu \equiv 1 \pmod{m} \quad \text{and} \quad mv \equiv 1 \pmod{n}$$

satisfies the congruences

$$x \equiv a \pmod{m} \quad \text{and} \quad x \equiv b \pmod{n}.$$

In order to apply the Chinese Remainder Theorem to obtain an expression for

$$12cs(d, c) \pmod{24c},$$

we need to find u and v with suitable m and n. This is usually done using the Euclidean Algorithm. We will show that we can avoid the use of the Euclidean Algorithm in the proof of the following theorem.

Theorem 4.1. *Let a, b, c, d be integers such that $ad - bc = 1$. If $c > 0$ and odd, then*

$$\xi(a, b, c, d) = (-1)^{g(d,c)} \exp\left(\frac{\pi i}{12}\left(-3c + c(a + d) + (1 - c^2)ab\right)\right),$$

where

$$g(h, k) = \sum_{r=0}^{(k-1)/2} \left\lfloor \frac{2rh}{k} \right\rfloor.$$

Proof. To prove Theorem 4.1, it suffices to show that

$$12c\phi(a, b, c, d) \equiv c(-3c + c(a + d) + (1 - c^2)ab - 36cg(d, c)) \pmod{24c}, \tag{4.1}$$

where $\phi(a, b, c, d)$ is defined in (1.4). We first consider the case where $3 \nmid c$. Since

$$8 \cdot 2 \equiv 1 \pmod{3} \quad \text{and} \quad 3 \cdot 3 \equiv 1 \pmod{8},$$

by the Chinese Remainder Theorem, we conclude by (3.1) and (3.8) that

$$12cs(d, c) \equiv 9(c - 1) + 36g(d, c) \pmod{24}. \tag{4.2}$$

Next, we need to find u and v such that

$$24u \equiv 1 \pmod{c} \quad \text{and} \quad cv \equiv 1 \pmod{24}. \tag{4.3}$$

For an odd integer c that are relatively prime to 3,

$$c^2 \equiv 1 \pmod{24}.$$

This means that we can choose $v = c$ to be a solution of the second congruence in (4.3). Once c is chosen, we find that

$$c^2 - 1 = 24j \quad \text{for some integer } j.$$

Therefore, we can choose $u = -j$, or

$$u = \frac{1-c^2}{24}$$

to be a solution of the first congruence in (4.3). By the Chinese Remainder Theorem, (3.2) and (4.2), we conclude that

$$12cs(d,c) \equiv (1-c^2)(d+d') + 9c^2((c-1)+4g(d,c)) \pmod{24c}.$$

Now, suppose a,b,c,d are integers such that $ad-bc=1$. Then we may choose $d' = a$ and conclude that

$$12cs(d,c) \equiv (1-c^2)(a+d) + 9c^2((c-1)+4g(d,c)) \pmod{24c}. \tag{4.4}$$

Using (1.4), (2.7) and (4.4), we deduce that

$$\begin{aligned} 12c\phi(a,b,c,d) &= a+d-12cs(d,c)-3c \\ &\equiv c^2(a+d) - 9c^2(c-1) - 3c - 36c^2g(d,c) \pmod{24c} \\ &\equiv c(-3c + c(a+d) - 36cg(d,c)) \pmod{24c}, \end{aligned} \tag{4.5}$$

since $c^2 \equiv 1 \pmod{24}$ and

$$-9c^2+9c-3 \equiv 12(c-1) - 3c \equiv -3c \pmod{24}.$$

The last congruence follows since $12(c-1)$ is divisible by 24 for odd c. Using

$$c^2 \equiv 1 \pmod{24},$$

we rewrite (4.5) as

$$12c\phi(a,b,c,d) \equiv c(-3c + c(a+d) + (1-c^2)ab - 36cg(d,c)) \pmod{24c}$$

and this completes the proof of (4.1) when $3 \nmid c$.

Next, let us consider the case when $3 \mid c$. We will need to find u and v such that

$$8u \equiv 1 \pmod{3c} \quad \text{and} \quad 3cv \equiv 1 \pmod{8}. \tag{4.6}$$

We observe that if t is an odd integer,

$$t^2 \equiv 1 \pmod{8}.$$

Since $3c$ is odd, we find that

$$(3c)^2 \equiv 1 \pmod{8},$$

and we may choose $v = 3c$ to be a solution of the second congruence in (4.6). Now write

$$9c^2 - 1 = 8j \quad \text{for some integer } j.$$

The integer

$$u = -j = \frac{1-9c^2}{8}$$

can be chosen as a solution of the first congruence in (4.6). By the Chinese Remainder Theorem, (3.3) and (3.8), we conclude that

$$12cs(d,c) \equiv 9c^2((c-1)+4g(d,c)) + (1-9c^2)(d+d^*) \pmod{24c}.$$

If $ad-bc=1$, then

$$ad-1=bc.$$

Since $3\,|\,c$, we have

$$(ad-1)^2 \equiv 0 \pmod{3c}.$$

This implies that

$$d(2a-a^2d) \equiv 1 \pmod{3c},$$

and we may choose

$$d^* = 2a-a^2d.$$

Using

$$ad-bc=1,$$

we find that

$$d+2a-a^2d = d+a-abc.$$

Therefore,

$$12cs(d,c) \equiv 9c^2(c-1) + (1-9c^2)(a+d-abc) + 36c^2g(d,c) \pmod{24c}$$

and we deduce that

$$\begin{aligned} &12c\phi(a,b,c,d) \\ &\quad \equiv c\left(-9c(c-1) + (1-9c^2)ab + 9c(a+d) - 3 - 36cg(d,c)\right) \pmod{24c}. \end{aligned} \tag{4.7}$$

Now, observe that when $3\,|\,c$,

$$c^2 \equiv 9 \pmod{24} \quad \text{and} \quad 9c \equiv c \pmod{24}. \tag{4.8}$$

Using (4.8) and the fact that

$$12(1-c) \equiv 0 \pmod{24},$$

we find that

$$-9c^2+9c-3 \equiv -81+9c-3 \equiv 12(c-1)-3c \equiv -3c \pmod{24}. \tag{4.9}$$

Substituting (4.8) and (4.9) in (4.7), we complete the proof of (4.1) when $3\,|\,c$. □

To complete the proof of Theorem 1.2 with odd positive c, it remains to study

$$(-1)^{g(h,k)}.$$

5. Jacobi Symbol and $g(h,k)$

In this section, we will establish the following theorem.

Theorem 5.1. *Let h be any integer and k be an odd positive integer with $(h,k)=1$. Then*

$$(-1)^{g(h,k)} = \left(\frac{h}{k}\right).$$

Rademacher and Whiteman [6, p. 397] first established the above using the Generalized Gauss Lemma. In [4, pp. 158–159], Rademacher presented another proof using the Quadratic Reciprocity Law. The proof we present here is similar to the second approach.

We first observe that if $k=1$, then $g(h,1)=0$ and since

$$\left(\frac{h}{1}\right) = 1 = (-1)^{g(h,1)},$$

Theorem 5.1 is true for $k=1$. So from now on, we may assume that $k>1$.

We will establish Theorem 5.1 in several steps. But first, we need two lemmas.

Lemma 5.2. *Let h and k be odd positive integers with $(h,k)=1$. Then*

$$g(h,k)+g(k,h) \equiv \frac{(h-1)(k-1)}{4} \pmod 2.$$

Proof. By (3.8), we find that

$$12hks(h,k) \equiv hk - h + 4hg(h,k) \pmod 8$$

and

$$12khs(k,h) \equiv hk - k + 4kg(k,h) \pmod 8.$$

Adding the two congruences and using (2.1), we conclude that

$$4hg(h,k)+4kg(k,h) \equiv 3-5hk+h+k \equiv 4-4hk+h+k-hk-1 \pmod 8.$$

The congruence in the lemma follows after dividing both sides by 4 and observing that $1-hk$ is even. □

It is possible to prove Lemma 5.2 directly without using (2.1). One needs to show that

$$\sum_{a=1}^{(k-1)/2} \left\lfloor \frac{2ah}{k} \right\rfloor \equiv \sum_{a=1}^{(k-1)/2} \left\lfloor \frac{ah}{k} \right\rfloor \pmod 2$$

and

$$\sum_{a=1}^{(k-1)/2} \left\lfloor \frac{ah}{k} \right\rfloor + \sum_{b=1}^{(h-1)/2} \left\lfloor \frac{bk}{h} \right\rfloor = \frac{(h-1)(k-1)}{4}.$$

We leave the details to the reader.

Lemma 5.3. *Let k be an odd positive integer and let h be any integer. Then*

$$(-1)^{g(-h,k)} = (-1)^{(k-1)/2}(-1)^{g(h,k)}. \tag{5.1}$$

Proof. By (3.8),

$$g(h,k) \equiv \frac{12ks(h,k)-k+1}{4} \pmod 2. \tag{5.2}$$

By replacing h by $-h$ and using (2.7) and (5.2), we have

$$g(-h,k) \equiv \frac{-12ks(h,k)-k+1}{4} \equiv -g(h,k) - \frac{k-1}{2} \pmod 2.$$

Hence

$$(-1)^{(k-1)/2}(-1)^{g(h,k)} = (-1)^{g(-h,k)},$$

and this proves the lemma. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. We first establish Theorem 5.1 when h is an odd positive integer. This is done by mathematical induction. First, we observe that if

$$h \equiv h' \pmod k$$

then

$$\sum_{a=1}^{(k-1)/2} \left\lfloor \frac{2ha}{k} \right\rfloor \equiv \sum_{a=1}^{(k-1)/2} \left\lfloor \frac{2h'a}{k} \right\rfloor \pmod 2. \tag{5.3}$$

Therefore, we may assume $h < k$. For $k = 3$ and $h < k$ with $(h,k) = 1$, it is possible to verify the claim directly. Suppose the claim is true for all odd positive integers less than k. Now,

$$\begin{aligned}(-1)^{g(h,k)+g(k,h)} &= (-1)^{(h-1)(k-1)/4} = \left(\frac{h}{k}\right)\left(\frac{k}{h}\right) \\ &= \left(\frac{h}{k}\right)(-1)^{g(k,h)} \quad \text{(by mathematical induction)},\end{aligned}$$

where we have used Lemma 5.2 and quadratic reciprocity. This completes the proof of Theorem 5.1 when h is an odd positive integer.

Next, we will prove Theorem 5.1 when h is an odd negative integer. Assume that h is an odd negative integer. We have just shown that Theorem 5.1 holds for positive odd integer $-h$. Therefore, by (5.1),

$$\left(\frac{-h}{k}\right) = (-1)^{g(-h,k)} = (-1)^{(k-1)/2}(-1)^{g(h,k)}.$$

Since

$$\left(\frac{-1}{k}\right) = (-1)^{(k-1)/2},$$

we deduce that Theorem 5.1 holds when h is an odd negative integer.

Finally, to complete the proof of Theorem 5.1, we assume that h is an even integer. From (5.3), we find that

$$g(-h,k) \equiv g(k-h,k) \pmod 2.$$

Since $k-h$ is odd and Theorem 5.1 is true for odd integer $k-h$, we deduce that

$$(-1)^{g(-h,k)} = (-1)^{g(k-h,k)} = \left(\frac{k-h}{k}\right) = \left(\frac{-h}{k}\right).$$

Hence by (5.1),

$$(-1)^{g(h,k)}(-1)^{(k-1)/2} = \left(\frac{-h}{k}\right) = (-1)^{(k-1)/2}\left(\frac{h}{k}\right).$$

By canceling $(-1)^{(k-1)/2}$ on both sides of the above identity, we conclude that Theorem 5.1 holds when h is an even integer.

This completes the proof of Theorem 5.1. □

Remark 5.4. We have identified

$$(-1)^{g(h,k)} \quad \text{with} \quad \left(\frac{h}{k}\right)$$

when $h \neq 0$. When $h = 0, k = 1$ is the only positive integer satisfying $(h,k) = 1$. In this case,

$$(-1)^{g(0,1)} = \left(\frac{0}{1}\right) = 1,$$

and this agrees with Theorem 5.1 when $h = 0$ and $k = 1$.

With the above remark, we may now restate Theorem 4.1 as follows.

Theorem 5.5. *Let a, b, c, d be integers such that $ad - bc = 1$. If $c > 0$ and odd, then*

$$\xi(a,b,c,d) = \left(\frac{d}{c}\right) \exp\left(\frac{\pi i}{12}(-3c + c(a+d) + (1-c^2)ab)\right),$$

where

$$\left(\frac{h}{k}\right)$$

is the Jacobi symbol.

6. Completion of the Proof of Theorem 1.2

We observed that by Theorem 5.5, we have established Theorem 1.2 when c is an odd positive integer. To complete the proof of Theorem 1.2, we now suppose that c is even.

By Theorem 5.5, with $a = 0, b = -1, c = 1, d = 0$, we find that

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau). \tag{6.1}$$

Now, suppose a, b, c, d are integers with $ad - bc = 1$ and c even positive. Since $ad - bc = 1$, d must be odd.

By Theorem 5.5, we conclude that

$$\eta\left(\frac{a(-1/\nu)+b}{c(-1/\nu)+d}\right)$$
$$=\eta\left(\frac{b\nu-a}{d\nu-c}\right)=\left(\frac{-c}{d}\right)\exp\left(\frac{\pi i}{12}(-3d+d(b-c)-(1-d^2)ab)\right)\sqrt{d\nu-c}\,\eta(\nu)$$
$$=\left(\frac{c}{d}\right)(-1)^{(d-1)/2}\exp\left(\frac{\pi i}{12}(-3d+d(b-c)-(1-d^2)ab)\right)\sqrt{d\nu-c}\,\eta(\nu).$$

Setting $\nu=-1/\tau$ and using (6.1), we deduce that

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)$$
$$=\left(\frac{-c}{d}\right)\exp\left(\frac{\pi i}{12}(-3d+d(b-c)-(1-d^2)ab)\right)\sqrt{-\frac{c\tau+d}{\tau}}\,\eta(-1/\tau)$$
$$=\left(\frac{c}{d}\right)(-1)^{(d-1)/2}\exp\left(\frac{\pi i}{12}(-3d+d(b-c)-(1-d^2)ab)\right)$$
$$\times\sqrt{-\frac{c\tau+d}{\tau}}\sqrt{-i\tau}\,\eta(\tau)$$
$$=\left(\frac{c}{d}\right)\exp\left(\frac{\pi i}{12}(3(d-1)+d(b-c)-(1-d^2)ab)\right)\sqrt{c\tau+d}\,\eta(\tau).$$

This completes the proof of Theorem 1.2.

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