THE RAMANUJAN JOURNAL 1, 75–90 (1997)

On the Ramanujan-Göllnitz-Gordon Continued Fraction

HENG HUAT CHAN hhchan@mthmp.math.ccu.edu.tw
School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540.
Current address: National Chung-Cheng University, Ming-Hsiung, Chia-Yi, Taiwan.

SEN-SHAN HUANG shuang@math.uiuc.edu
Department of Mathematics, University of Illinois, Urbana, Illinois 61801.

Dedicated to Professor B. Gordon

Received September 1, 1995; Accepted September 13, 1995

Abstract. We derive many new identities involving the Ramanujan-Göllnitz-Gordon continued fraction $H(q)$. These include relations between $H(q)$ and $H(q^n)$, which are established using modular equations of degree n. We also evaluate explicitly $H(q)$ at $q = e^{-\pi\sqrt{n}/2}$ for various positive integers n. Using results of M. Deuring, we show that $H(\pm e^{-\pi\sqrt{n}/2})$ are units for all positive integers n.

Key words: continued fraction, units, singular moduli, modular equations

1991 Mathematics Subject Classification: Primary—11A55, 11R04, 11Y60

1. Introduction

Let

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$$

and let the Ramanujan-Göllnitz-Gordon continued fraction be defined as

$$H(q) := \frac{q^{\frac{1}{2}}}{1+q} \, {}_{+} \, \frac{q^2}{1+q^3} \, {}_{+} \, \frac{q^4}{1+q^5} \, {}_{+} \, \frac{q^6}{1+q^7} \, {}_{+} \, \cdots, \qquad |q| < 1.$$

On page 229 of his second notebook [13], Ramanujan recorded a product representation of $H(q)$, namely,

$$H(q) = q^{\frac{1}{2}} \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty}. \tag{1.1}$$

Without any knowledge of Ramanujan's work, Göllnitz [10] and Gordon [11] rediscovered and proved (1.1) independently. Shortly thereafter, Andrews [1] proved (1.1) as a corollary of a more general result.

In addition to (1.1), Ramanujan offered two other identities [13, p. 229] for $H(q)$:

$$\frac{1}{H(q)} - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)} \tag{1.2}$$

and

$$\frac{1}{H(q)} + H(q) = \frac{\varphi(q)}{q^{1/2}\psi(q^4)}, \tag{1.3}$$

where

$$\varphi(q) := \sum_{k=-\infty}^{\infty} q^{k^2}$$

and

$$\psi(q) := \sum_{k=0}^{\infty} q^{k(k+1)/2}.$$

Proofs of identities (1.2) and (1.3) can be found in Berndt's book [3, p. 221].

In this paper, we will establish several identities which will give us a better understanding of $H(q)$. Some of these identities are motivated by identities involving the Rogers-Ramanujan continued fraction [4]

$$F(q) := \frac{q^{\frac{1}{5}}}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \frac{q^3}{1} \underset{+}{} \cdots$$

and Ramanujan's cubic continued fraction [7]

$$G(q) = \frac{q^{\frac{1}{3}}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+}{} \cdots.$$

In Section 2, we will develop some basic identities satisfied by $H(q)$. These identities are proved by using (1.1) and some elementary theta-functions identities. In particular, we will reprove (1.2) and (1.3) by an approach slightly different from that in [3]. Then, by using these identities, we are able to establish relations between $H(q)$ and $H(-q)$, and $H(q)$ and $H(q^2)$.

Let, as customary,

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(a)_k = (a)(a+1)\cdots(a+k-1)$, and $|z| < 1$. We say that the *modulus* β has degree n over the *modulus* α when

$$\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)} = n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}. \tag{1.4}$$

A *modular equation of degree n* is a relation between α and β which is induced by (1.4). In Section 3, we will prove an important theorem which will allow us to deduce relations between $H(q)$ and $H(q^n)$ from modular equations of degree n and vice versa. In particular, we will use this theorem to obtain relations between $H(q)$ and the two continued fractions $H(q^3)$ and $H(q^4)$.

In Section 4, we will establish some explicit formulas for evaluating $H(e^{-\pi\sqrt{n}/2})$ in terms of Ramanujan-Weber class invariants. Using these formulas, we derive many numerical continued fractions. One such example, which first appeared in [10, (2.32)], is

$$H(e^{-\pi}) = \sqrt{4+2\sqrt{2}} - \sqrt{3+2\sqrt{2}}.$$

Observe that $H(e^{-\pi})$ is a unit. In fact, in the final section, we will show that *$H(\pm e^{-\pi\sqrt{n}/2})$ is a unit when n is a positive integer*. This is an analogue of a result recently established by Berndt et al. [4], which states that *$F(e^{-\pi\sqrt{n}})$ is a unit when n is a positive rational number*.

We conclude this introduction with the remark that the results established for $H(q)$ are also valid for

$$H_1(q) := \frac{q^{\frac{1}{2}}}{1} + \frac{q+q^2}{1} + \frac{q^4}{1} + \frac{q^3+q^6}{1} + \frac{q^8}{1} + \cdots.$$

This follows from the identities (1.1) and

$$H_1(q) = q^{\frac{1}{2}} \frac{(q;q^8)_\infty (q^7;q^8)_\infty}{(q^3;q^8)_\infty (q^5;q^8)_\infty}, \tag{1.5}$$

which was stated by Ramanujan [13, p. 290; 14, p. 44] and first proved by Selberg [15]. Other proofs of (1.5) have also been given by Andrews [2] and Ramanathan [12].

2. Some identities satisfied by $H(q)$

Let

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \qquad |ab| < 1.$$

By Jacobi's triple product identity, we have [3, p. 35, Entry 19]

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{2.1}$$

Using (2.1), we rewrite (1.1) as

$$H(q) = q^{\frac{1}{2}} \frac{f(-q,-q^7)}{f(-q^3,-q^5)}. \tag{2.2}$$

Theorem 2.1.

(i)
$$H^2(q) = \frac{\varphi(q) - \varphi(q^2)}{\varphi(q) + \varphi(q^2)},$$

(ii)
$$\frac{H^{-1}(q) - H(q)}{H^{-1}(q) + H(q)} = \frac{\varphi(q^2)}{\varphi(q)},$$

(iii)
$$H^{-2}(q) - H^2(q) = \frac{\varphi(q)\varphi(q^2)}{q\psi^2(q^4)},$$

(iv)
$$H^{-1}(q) - H(q) = \frac{\varphi(q^2)}{q^{1/2}\psi(q^4)},$$

and

(v)
$$H^2(q)H^2(-q) = -H^2(q^2).$$

Proof of (i): We recall from [3, p. 51] that

$$\varphi(-q) + \varphi(q^2) = 2\frac{f^2(q^3, q^5)}{\psi(q)} \tag{2.3}$$

and

$$\varphi(-q) - \varphi(q^2) = -2q\frac{f^2(q, q^7)}{\psi(q)}. \tag{2.4}$$

Dividing (2.4) by (2.3), and replacing q by $-q$, we find that

$$q\frac{f^2(-q, -q^7)}{f^2(-q^3, -q^5)} = \frac{\varphi(q) - \varphi(q^2)}{\varphi(q) + \varphi(q^2)}.$$

The result follows by (2.2). □

Proof of (ii): We first rewrite Theorem 2.1(i) as

$$H^2(q) = \frac{1 - \dfrac{\varphi(q^2)}{\varphi(q)}}{1 + \dfrac{\varphi(q^2)}{\varphi(q)}}. \tag{2.5}$$

Since $b = \frac{1-a}{1+a}$ is equivalent to $a = \frac{1-b}{1+b}$, we find that

$$\frac{1 - H^2(q)}{1 + H^2(q)} = \frac{\varphi(q^2)}{\varphi(q)},$$

by (2.5), and this clearly implies Theorem 2.1(ii). □

Proof of (iii): We recall from [3, p. 40, Entry 25(v), (vi)] that

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2) \tag{2.6}$$

and

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4). \tag{2.7}$$

By Theorem 2.1(i) and (2.6), we deduce that

$$\begin{aligned} H^{-2}(q) - H^2(q) &= \frac{\varphi(q)+\varphi(q^2)}{\varphi(q)-\varphi(q^2)} - \frac{\varphi(q)-\varphi(q^2)}{\varphi(q)+\varphi(q^2)} = \frac{4\varphi(q)\varphi(q^2)}{\varphi^2(q)-\varphi^2(q^2)} \\ &= \frac{4\varphi(q)\varphi(q^2)}{\varphi^2(q) - \dfrac{\varphi^2(q)+\varphi^2(-q)}{2}} = \frac{8\varphi(q)\varphi(q^2)}{\varphi^2(q)-\varphi^2(-q)}. \end{aligned} \tag{2.8}$$

Using (2.7), we simplify (2.8) to obtain

$$H^{-2}(q) - H^2(q) = \frac{\varphi(q)\varphi(q^2)}{q\psi^2(q^4)},$$

which is Theorem 2.1(iii). □

Proof of (iv): By Theorem 2.1(ii) and (iii), we have

$$\begin{aligned} (H^{-1}(q) - H(q))^2 &= (H^{-2}(q) - H^2(q))\frac{H^{-1}(q)-H(q)}{H^{-1}(q)+H(q)} \\ &= \frac{\varphi(q)\varphi(q^2)}{q\psi^2(q^4)}\frac{\varphi(q^2)}{\varphi(q)} = \frac{\varphi^2(q^2)}{q\psi^2(q^4)}. \end{aligned} \tag{2.9}$$

Taking square roots on both sides of (2.9), we deduce Theorem 2.1(iv). □

Proof of (v): From [3, p. 46, Entry 30(iv)], we have

$$f(a,b)f(-a,-b) = f(-a^2,-b^2)\varphi(-ab). \tag{2.10}$$

By (2.2) and (2.10), we deduce that

$$\begin{aligned} H^2(q)H^2(-q) &= -q^2\left(\frac{f(-q,-q^7)f(q,q^7)}{f(-q^3,-q^5)f(q^3,q^5)}\right)^2 \\ &= -q^2\left(\frac{f(-q^2,-q^{14})\varphi(-q^8)}{f(-q^6,-q^{10})\varphi(-q^8)}\right)^2 = -H^2(q^2). \end{aligned}$$

This completes the proof of Theorem 2.1(v). □

By using Theorem 2.1, we can find relations between $H(q)$ and $H(-q)$, and $H(q)$ and $H(q^2)$.

Theorem 2.2. *Let $u = H(q)$, $v = H(-q)$, and $w = H(q^2)$. Then*

(i)
$$\left(\frac{1}{u} - u\right)^2 + \left(\frac{1}{v} - v\right)^2 = 0$$

and

(ii)
$$u^2 = w\frac{1-w}{1+w}.$$

Proof of (i): Squaring both sides of the identity (iv) in Theorem 2.1, we arrive at

$$\left(\frac{1}{u} - u\right)^2 = \frac{\varphi^2(q^2)}{q\psi^2(q^4)}. \tag{2.11}$$

Replacing q by $-q$ in (2.11), we find that

$$\left(\frac{1}{v} - v\right)^2 = -\frac{\varphi^2(q^2)}{q\psi^2(q^4)}. \tag{2.12}$$

Adding (2.11) and (2.12), we complete the proof. □

Proof of (ii): We first rewrite (i) as

$$\left(\frac{1}{u} - u\right)^2 = -\frac{1}{v^2} + 2 - v^2. \tag{2.13}$$

From (2.13) and Theorem 2.1(v), we have

$$\left(\frac{1}{u} - u\right)^2 = \left(\frac{u}{w}\right)^2 + 2 + \left(\frac{w}{u}\right)^2. \tag{2.14}$$

Simplifying (2.14), we obtain

$$\frac{1}{u} - u = \frac{u}{w} + \frac{w}{u},$$

which yields the desired result, after some simple manipulations. □

3. Modular equations of degree n and relations between $H(q)$ and $H(q^n)$

Theorem 3.1. *If*

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right), \tag{3.1}$$

then

$$\left(\frac{H^{-1}(\sqrt{q}) - H(\sqrt{q})}{2}\right)^4 = \frac{1}{\alpha}. \tag{3.2}$$

Proof: If we replace q by $\sqrt{q}$ in Theorem 2.1(iv), divide both sides by 2, and then raise both sides to the fourth power, we find that

$$\left(\frac{H^{-1}(\sqrt{q}) - H(\sqrt{q})}{2}\right)^4 = \frac{\varphi^4(q)}{16q\psi^4(q^2)}. \tag{3.3}$$

On the other hand, we recall from [3, p. 40, Entry 25(vii)] that

$$\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2). \tag{3.4}$$

Substituting (3.4) into (3.3), we find that

$$\left(\frac{H^{-1}(\sqrt{q}) - H(\sqrt{q})}{2}\right)^4 = \frac{\varphi^4(q)}{\varphi^4(q) - \varphi^4(-q)} = \frac{1}{1 - \dfrac{\varphi^4(-q)}{\varphi^4(q)}}. \tag{3.5}$$

From [3, p. 100, Entry 5], we know that the identity (3.1) implies that

$$\alpha = 1 - \frac{\varphi^4(-q)}{\varphi^4(q)}. \tag{3.6}$$

Combining (3.6) and (3.5), we complete the proof. □

Now, let α and q be related by (3.1). If β has degree n over α, then Theorem 3.1 gives us

$$\left(\frac{H^{-1}(q^{n/2}) - H(q^{n/2})}{2}\right)^4 = \frac{1}{\beta}. \tag{3.7}$$

Hence, by (3.2), (3.7), and any given modular equation of degree n, we can obtain a relation between $H(\sqrt{q})$ and $H(q^{n/2})$. Replacing q by q^2 will then yield a relation between $H(q)$ and $H(q^n)$. We illustrate these ideas with $n = 3$ and 4.

Corollary 3.2. *Let* $u = H(q)$, $v = H(q^3)$, *and* $w = H(q^4)$. *Then*

(i) $$3uv(1-uv)(u+v) + (u^3 - v)(1 + uv^3) = 0$$

and

(ii) $$u=\sqrt{\sqrt{\left(\frac{2w(1-w)}{1+w^2}\right)^2+\frac{w(1-w)}{1+w}}-\frac{2w(1-w)}{1+w^2}}.$$

Proof of (i): Let

$$x := H(\sqrt{q}) \qquad \text{and} \qquad y := H(q^{3/2}),$$

where q is given by (3.1). When β has degree 3 over α, we have [3, p. 231, Entry 5(xiii)]

$$\left(\frac{\beta}{\alpha}\right)^{1/4}-\left(\frac{\alpha}{\beta}\right)^{1/4}=2((\alpha\beta)^{1/8}-(\alpha\beta)^{-1/8}). \tag{3.8}$$

By Theorem 3.1, (3.8) is equivalent to

$$\frac{x^{-1}-x}{y^{-1}-y}-\frac{y^{-1}-y}{x^{-1}-x}=2\left(\frac{2}{\sqrt{(x^{-1}-x)(y^{-1}-y)}}-\frac{\sqrt{(x^{-1}-x)(y^{-1}-y)}}{2}\right). \tag{3.9}$$

Using *Mathematica*, we simplify (3.9) to arrive at

$$\begin{aligned}&(x^4y^3-3x^3y^2+x^3-3x^2y^3+3x^2y-xy^4+3xy^2-y)\\ &\quad\times(x^3y^4-3x^2y^3+y^3-3x^3y^2+3xy^2-x^4y+3x^2y-x)=0.\end{aligned}$$

The second factor in the product is a non-zero function when $|q|\to 0$, since there is only one term x with leading term $q^{1/4}$. Hence, we conclude that

$$x^4y^3-3x^3y^2+x^3-3x^2y^3+3x^2y-xy^4+3xy^2-y=0,$$

i.e.,

$$3xy(1-xy)(x+y)+(x^3-y)(1+xy^3)=0.$$

Replacing q by q^2, we complete the proof. □

Proof of (ii): When β has degree 4 over α, we have [3, p. 215, Eq. (24.22)]

$$\sqrt{\beta}=\left(\frac{1-(1-\alpha)^{1/4}}{1+(1-\alpha)^{1/4}}\right)^2.$$

Replacing α by $1-\beta$ and β by $1-\alpha$, we arrive at another modular equation of degree 4, which is

$$\sqrt{1-\alpha}=\left(\frac{1-\beta^{1/4}}{1+\beta^{1/4}}\right)^2, \tag{3.10}$$

by [3, p. 216, Entry 24(v)].

If

$$q^2 = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right),$$

then, by Theorem 3.1, we deduce that

$$\left(\frac{u^{-1}-u}{2}\right)^4 = \frac{1}{\alpha} \tag{3.11}$$

and

$$\left(\frac{w^{-1}-w}{2}\right)^4 = \frac{1}{\beta}. \tag{3.12}$$

Squaring both sides of (3.10) and combining it with (3.11) and (3.12), we find that

$$1 - \left(\frac{2}{u^{-1}-u}\right)^4 = \left(\frac{w^{-1}-w-2}{w^{-1}-w+2}\right)^4. \tag{3.13}$$

Upon simplifying (3.13) using *Mathematica*, we arrive at

$$\begin{aligned}&(u^4w^3 + u^4w^2 + u^4w + u^4 - 4u^2w^3 + 4u^2w + w^4 - w^3 + w^2 - w)\\ &\quad \times (1 + u^4w^4 - u^4w^3 + u^4w^2 - u^4w - 4u^2w^3 + 4u^2w + w^3 + w^2 + w) = 0.\end{aligned}$$

Clearly, the second factor in the product does not vanish when $|q| \to 0$. Therefore,

$$u^4w^3 + u^4w^2 + u^4w + u^4 - 4u^2w^3 + 4u^2w + w^4 - w^3 + w^2 - w = 0. \tag{3.14}$$

Rearranging (3.14), we find that

$$(w+1)(w^2+1)u^4 - 4w(w-1)(w+1)u^2 + w(w-1)(w^2+1) = 0. \tag{3.15}$$

By regarding (3.15) as a quadratic equation of u^2, we conclude that

$$u^2 = \frac{-2w(1-w)}{1+w^2} + \sqrt{\left(\frac{2w(1-w)}{1+w^2}\right)^2 + \frac{w(1-w)}{1+w}}, \tag{3.16}$$

where the $+$ sign in front of the radical is verified by letting $0 < q < 1$. Finally, taking the square roots on both sides of (3.16) yields the result. □

4. Explicit formulas for the evaluations of *H*(*q*)

Let $q_n := e^{-\pi\sqrt{n}}$, and let the corresponding value of α in (3.1) be denoted by α_n; $\sqrt{\alpha_n}$ is called a singular modulus. Then, by applying Theorem 3.1 and solving (3.2) for $H(q_n)$, we have

$$H(e^{-\pi\sqrt{n}/2}) = \sqrt{\sqrt{\frac{1}{\alpha_n}}+1} - \sqrt{\sqrt{\frac{1}{\alpha_n}}}. \tag{4.1}$$

It is known from [3, p. 97] that $\alpha_1 = 1/2$, $\alpha_2 = (\sqrt{2}-1)^2$, and $\alpha_4 = (\sqrt{2}-1)^4$. Hence, by using (4.1), we deduce that

$$H(e^{-\pi/2}) = \sqrt{\sqrt{2}+1} - \sqrt{\sqrt{2}}, \tag{4.2}$$

$$H(e^{-\pi/\sqrt{2}}) = \sqrt{\sqrt{2}+2} - \sqrt{\sqrt{2}+1}, \tag{4.3}$$

and

$$H(e^{-\pi}) = \sqrt{4+2\sqrt{2}} - \sqrt{3+2\sqrt{2}}. \tag{4.4}$$

In fact, more values of $H(q)$ can be obtained simply by using (4.1) and known values of α_n [5]. But this process does not always give us elegant radicals for the values of $H(q)$. Hence, we require another expression for the right hand side of (4.1).

Theorem 4.1. *Let the Ramanujan-Weber class invariants be defined by*

$$G_n := 2^{-1/4} q_n{}^{-1/24}\left(-q_n; q_n^2\right)_\infty \tag{4.5}$$

and

$$g_n := 2^{-1/4} q_n{}^{-1/24}\left(q_n; q_n^2\right)_\infty, \tag{4.6}$$

where $q_n = e^{-\pi\sqrt{n}}$, *and set* $p = G_n^{12}$ *and* $p_1 = g_n^{12}$. *Then*

(i) $$H(e^{-\pi\sqrt{n}/2}) = \sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}+1} - \sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}}$$
$$= \sqrt{p_1+\sqrt{p_1^2+1}+1} - \sqrt{p_1+\sqrt{p_1^2+1}}$$

and

(ii) $$H(e^{-\pi\sqrt{n}}) = \sqrt{\left(\sqrt{p+1}+\sqrt{p}\right)^2\left(\sqrt{p}+\sqrt{p-1}\right)^2+1}$$
$$- \left(\sqrt{p+1}+\sqrt{p}\right)\left(\sqrt{p}+\sqrt{p-1}\right)$$
$$= \left(\sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}+1} - \sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}}\right)$$
$$\times \left(\sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}} - \sqrt{\sqrt{p(p+1)}+\sqrt{p(p-1)}-1}\right)$$

$$= \left(\sqrt{p_1+\sqrt{p_1^2+1}+1}-\sqrt{p_1+\sqrt{p_1^2+1}}\right)$$
$$\times \left(\sqrt{p_1+\sqrt{p_1^2+1}}-\sqrt{p_1+\sqrt{p_1^2+1}-1}\right).$$

Proof of (i): Since [5]

$$G_n = (4\alpha_n(1-\alpha_n))^{-1/24}, \tag{4.7}$$

we deduce that

$$\frac{1}{\alpha_n} = \left(\sqrt{p(p+1)}+\sqrt{p(p-1)}\right)^2. \tag{4.8}$$

Using (4.8) in (4.1), we obtain the first equality of (i).

Next, from [5],

$$\frac{1}{\sqrt{\alpha_n}} - \sqrt{\alpha_n} = 2g_n^{12}. \tag{4.9}$$

Hence,

$$\frac{1}{\sqrt{\alpha_n}} = p_1 + \sqrt{p_1^2+1}. \tag{4.10}$$

From (4.10) and (4.1), we deduce the second equality. □

Proof of (ii): By (4.7) and (4.8), we deduce that

$$\frac{1}{1-\alpha_n} = \left(\sqrt{p(p+1)}-\sqrt{p(p-1)}\right)^2. \tag{4.11}$$

From [3, p. 213, Eq. (24.12)], (4.8) and (4.11),

$$\begin{aligned}\alpha_{4n} &= \left(\sqrt{\sqrt{\frac{1}{\alpha_n}}}-\sqrt{\sqrt{\frac{1}{\alpha_n}}-1}\right)^4\\ &= \left(\sqrt{p+1}+\sqrt{p-1}\right)^4\left(\sqrt{p}-\frac{\sqrt{p+1}+\sqrt{p-1}}{2}\right)^4\\ &= \left(\sqrt{p+1}-\sqrt{p}\right)^4\left(\sqrt{p}-\sqrt{p-1}\right)^4.\end{aligned} \tag{4.12}$$

Using (4.12) in (4.1), we obtain the first equality of (ii). Next, from Theorem 2.2(i) and (3.2), we observe that

$$H(-e^{-\pi\sqrt{n}/2}) = -i\left(\sqrt{\sqrt{\frac{1}{\alpha_n}}}-\sqrt{\sqrt{\frac{1}{\alpha_n}}-1}\right). \tag{4.13}$$

Using (4.1), (4.13) and Theorem 2.1(v), we obtain

$$H(e^{-\pi\sqrt{n}}) = \left(\sqrt{\sqrt{\frac{1}{\alpha_n}}+1} - \sqrt{\sqrt{\frac{1}{\alpha_n}}}\right)\left(\sqrt{\sqrt{\frac{1}{\alpha_n}}} - \sqrt{\sqrt{\frac{1}{\alpha_n}}-1}\right). \tag{4.14}$$

Substituting (4.8) and (4.10) into (4.14), respectively, we arrive at the second and third equalities of (ii). □

Examples. Let $n = 1$. Since $G_1 = 1$, Theorem 4.1 yields (4.2) and (4.4). Let $n = 2$. Since $g_2 = 1$ [16, p. 721], Theorem 4.1 gives (4.3) and

$$H(e^{-\pi\sqrt{2}}) = \left(\sqrt{\sqrt{2}+2} - \sqrt{\sqrt{2}+1}\right)\left(\sqrt{\sqrt{2}+1} - \sqrt{\sqrt{2}}\right).$$

When $n = 3$, $G_3^{12} = 2$ [16, p. 721] and hence,

$$H(e^{-\pi\sqrt{3}/2}) = \sqrt{\sqrt{6}+\sqrt{2}+1} - \sqrt{\sqrt{6}+\sqrt{2}}$$

and

$$\begin{aligned} H(e^{-\pi\sqrt{3}}) &= \sqrt{(\sqrt{3}+\sqrt{2})^2(\sqrt{2}+1)^2+1} - (\sqrt{3}+\sqrt{2})(\sqrt{2}+1) \\ &= \left(\sqrt{\sqrt{6}+\sqrt{2}+1} - \sqrt{\sqrt{6}+\sqrt{2}}\right)\left(\sqrt{\sqrt{6}+\sqrt{2}} - \sqrt{\sqrt{6}+\sqrt{2}-1}\right). \end{aligned}$$

It is clear that the continued fractions we discovered are all units. It is therefore natural to conjecture that $H(e^{-\pi\sqrt{n}/2})$ is a unit when n is a positive integer. To prove this, by (4.1), it suffices to show that $1/\alpha_n$ is an algebraic integer for each positive integer n.

5. $H(\pm e^{-\pi\sqrt{n}/2})$ is a unit when n is a positive integer

We first recall some basic definitions from algebraic number theory. An *order* $\mathfrak{O}_f$ with *conductor* f in a quadratic field K is a subset $\mathfrak{O}_f \subset K$ such that

(i) $\mathfrak{O}_f$ is a subring of K containing 1,
(ii) $\mathfrak{O}_f$ is a finitely generated $\mathbb{Z}$-module,
(iii) $\mathfrak{O}_f$ contains a $\mathbb{Q}$-basis of K, and
(iv) $[\mathfrak{O}_K : \mathfrak{O}_f] = f$,

where $\mathfrak{O}_K = \mathfrak{O}_1$ is the ring of integers of K.

If α_1 and α_2 generate an $\mathfrak{O}_f$-ideal $\mathfrak{a}_{\mathfrak{O}_f}$ over $\mathbb{Z}$, we say that $[\alpha_1, \alpha_2]$ is a basis of $\mathfrak{a}_{\mathfrak{O}_f}$. An ideal $\mathfrak{a}_{\mathfrak{O}_f}$ is said to be *proper* if $\mathfrak{a}_{\mathfrak{O}_f}$ is coprime to f. For more details on orders of an imaginary quadratic field, we refer the reader to [8].

If η is an algebraic integer and $\mathfrak{a}$ is an $\mathfrak{O}_K$ ideal, we write $\eta \approx \mathfrak{a}$ to mean that $\eta\mathfrak{O}_\Omega = \mathfrak{a}\mathfrak{O}_\Omega$ in some large number field Ω. Similarly, if η_1 and η_2 are algebraic integers, we write $\eta_1 \approx \eta_2$ to mean that η_1/η_2 is a unit.

Theorem 5.1. *Suppose p is a prime such that $p^t \parallel f$, where f is a positive integer. Let a, b, c and d be integers, and $P := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix with determinant p and $\Delta(\tau) = q(q;q)_\infty^{24}$, where $q = e^{2\pi i \tau}$, with $\mathrm{Im}(\tau) > 0$. Define*

$$\varphi_P(\tau) = p^{12} \frac{\Delta\left(P\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}\right)}{\Delta\begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}}, \tag{5.1}$$

where $\tau = \frac{\tau_1}{\tau_2}$ and $\Delta\left(\begin{smallmatrix}\tau_1\\ \tau_2\end{smallmatrix}\right) = \tau_2^{-12}\Delta(\tau)$. Let $[\alpha_1, \alpha_2]$ be a basis of a proper $\mathfrak{O}_f$-ideal $\mathfrak{a}_{\mathfrak{O}_f}$ and set $\alpha = \frac{\alpha_1}{\alpha_2}$. The action of P on the basis $[\alpha_1, \alpha_2]$ is defined as

$$P[\alpha_1, \alpha_2] := [a\alpha_1 + b\alpha_2, c\alpha_1 + d\alpha_2].$$

1. *When p splits completely in K, namely $p = \mathfrak{p}\bar{\mathfrak{p}}$, then*
   (1.1) *$\varphi_P(\alpha)$ is a unit if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp}$-ideal,*
   (1.2) *$\varphi_P(\alpha) \approx p^{12}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp^{-1}}$-ideal, and*
   (1.3) *if $p \nmid f$, then $\varphi_P(\alpha) \approx \bar{\mathfrak{p}}^{12}$ and $\varphi_P(\alpha) \approx \mathfrak{p}^{12}$ when $P[\alpha_1, \alpha_2]$ is a basis of $\mathfrak{a}_{\mathfrak{O}_f}\mathfrak{p}_{\mathfrak{O}_f}$ and $P[\alpha_1, \alpha_2]$ is a basis of $\mathfrak{a}_{\mathfrak{O}_f}\bar{\mathfrak{p}}_{\mathfrak{O}_f}$, respectively.*
2. *When p ramifies in K, namely $p = \mathfrak{p}^2$, then*
   (2.1) *$\varphi_P(\alpha) \approx p^{6/p^{t+1}}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp}$-ideal,*
   (2.2) *$\varphi_P(\alpha) \approx p^{12-6/p^t}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp^{-1}}$-ideal, and*
   (2.3) *$\varphi_P(\alpha) \approx p^6$ if $P[\alpha_1, \alpha_2]$ is a basis of $\mathfrak{a}_{\mathfrak{O}_f}\mathfrak{p}_{\mathfrak{O}_f}$.*
3. *When p is inert in K, then*
   (3.1) *$\varphi_P(\alpha) \approx p^{12/p^t(p+1)}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp}$-ideal, and*
   (3.2) *$\varphi_P(\alpha) \approx p^{12(1-1/p^{t-1}(p+1))}$ if $P[\alpha_1, \alpha_2]$ is a basis of a proper $\mathfrak{O}_{fp^{-1}}$-ideal.*

Proof: See [9, p. 43]. □

Corollary 5.2. *Let the Ramanujan-Weber class invariants be defined by* (4.5) *and* (4.6). *Then*
(i) *for $n \equiv 1 \pmod 4$, G_n is a unit,*
(ii) *for $n \equiv 3 \pmod 8$, $2^{-1/12}G_n$ is a unit,*
(iii) *for $n \equiv 7 \pmod 8$, $2^{-1/4}G_n$ is a unit, and*
(iv) *for $n \equiv 2 \pmod 4$, g_n is a unit.*

Proof: Throughout the proof, we will assume that $n = f^2 d$, where d is squarefree. We will also let $K = \mathbb{Q}(\sqrt{-d})$.

(i) When $n \equiv 1 \pmod 4$, $d \equiv 1 \pmod 4$, $\mathfrak{O}_K = \mathbb{Z}[\sqrt{-d}]$ and $(2) = \mathfrak{p}^2$ ramifies in $\mathfrak{O}_K$. Let $[\sqrt{-n}, 1]$ be a basis of $\mathfrak{O}_f$ and $P = \left(\begin{smallmatrix} 1 & f \\ 0 & 2 \end{smallmatrix}\right)$. Since $P[\sqrt{-n}, 1] = [\sqrt{-n} + f, 2]$ is a basis of $\mathfrak{p}\mathfrak{O}_f$, we deduce from Theorem 5.1(2.3) that

$$\varphi_P(\sqrt{-n}) \approx 2^6. \tag{5.2}$$

By (5.2), (5.1) and observing that f is odd, we conclude that

$$2^{-6} \frac{\Delta\left(\frac{\sqrt{-n}+1}{2}\right)}{\Delta\left(\sqrt{-n}\right)} \tag{5.3}$$

is a unit, which implies the desired result, since (5.3) reduces to $-G_n^{24}$ by using the definition of Δ appeared in Theorem 5.1 and (4.5).

(ii) When $n \equiv 3 \pmod 8$, $d \equiv 3 \pmod 8$, $\mathfrak{O}_K = \mathbb{Z}[\frac{1+\sqrt{-d}}{2}]$ and (2) is inert in $\mathfrak{O}_K$. Let $[\sqrt{-n} + f, 1]$ be a basis of a $\mathfrak{O}_{2f}$-ideal and $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Since $P[\sqrt{-n} + f, 1]$ is a basis of a $\mathfrak{O}_f$-ideal, we conclude that

$$\varphi_P(\sqrt{-n} + f) \approx 2^8,$$

by Theorem 5.1(3.2). Now, by a similar argument as in (i), we obtain

$$(2^{1/4} G_n)^{24} \approx 2^8,$$

which implies that $2^{-1/12} G_n$ is a unit.

(iii) When $n \equiv 7 \pmod 8$, $d \equiv 7 \pmod 8$, $\mathfrak{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{-d}}{2}\right]$ and (2) splits completely in $\mathfrak{O}_K$. Let $[\sqrt{-n} + f, 1]$ be a basis of a $\mathfrak{O}_{2f}$-ideal and $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Since $P[\sqrt{-n} + f, 1]$ is a basis of a $\mathfrak{O}_f$-ideal, we find that

$$\varphi_P(\sqrt{-n} + f) \approx 2^{12},$$

by Theorem 5.1(1.2). Therefore, $2^{-1/4} G_n$ is a unit by (4.5) and (5.1).

(iv) When $n \equiv 2 \pmod 4$, $d \equiv 2 \pmod 4$, $\mathfrak{O}_K = \mathbb{Z}[\sqrt{-d}]$ and $(2) = \mathfrak{p}^2$ ramifies in $\mathfrak{O}_K$. Let $[\sqrt{-n}, 1]$ be a basis of $\mathfrak{O}_f$ and $P = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Then $P[\sqrt{-n}, 1]$ is a basis of $\mathfrak{p}\mathfrak{O}_f$. By Theorem 5.1(2.3), we deduce that

$$\varphi_P(\sqrt{-n}) \approx 2^6. \tag{5.4}$$

Hence, by (5.4) and (5.1), we find that

$$\frac{\Delta\left(\frac{\sqrt{-n}}{2}\right)}{\Delta\left(\sqrt{-n}\right)} \approx 2^6, \tag{5.5}$$

which implies that g_n is a unit, since the left hand side of (5.5) reduces to $2^6 g_n^{24}$ by the definition of Δ and (4.6). This completes the proof of the corollary. □

Remarks. The results given in Corollary 5.2 were briefly mentioned in [6] without any proofs. There is a misprint on page 291, line 14 of [6]. The statement "...$\sigma^3(\omega)$ is a real unit..." should be replaced by "...$2^{-1}\sigma^3(\omega)$ is a real unit...."

From Corollary 5.2, we deduce the following theorem.

Theorem 5.3. *Let*

$$\sqrt{n} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha_n\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha_n\right)}.$$

Then ${\alpha_n}^{-1}$ *is an algebraic integer for every positive integer* n.

Proof: From (4.7) we deduce that

$$\left(\frac{1}{\alpha_n}\right)^2 - 4G_n^{24}\left(\frac{1}{\alpha_n}\right) + 4G_n^{24} = 0. \tag{5.6}$$

By Corollary 5.2(i), (ii) and (iii), we find that G_n^{24} is an algebraic integer when n is odd. Hence, by (5.6), α_n^{-1} is an algebraic integer when n is odd.

By Corollary 5.2(iv) and (4.9), we conclude that α_n^{-1} is a unit when $n \equiv 2 \pmod 4$. In order to complete the proof of the theorem, it remains to show that α_n^{-1} is a unit when $n \equiv 0 \pmod 4$.

Let $m \equiv \pm 1,\ 2 \pmod 4$ and k be a positive integer. We will show by induction that α_n^{-1} is a unit when $n = 4^k m$. Since α_m^{-1} is an algebraic integer when $4 \nmid m$, we conclude from (4.12) that α_{4m} is a unit. This proves the case $k = 1$.

Now, suppose $\alpha_{4^{k-1}m}^{-1}$ is an algebraic integer. Then applying (4.12) again, we conclude that $\alpha_{4^k m}^{-1}$ is an algebraic integer. This completes the proof of the theorem. □

Combining (4.1) and Theorems 5.3 and 2.1(v), we deduce the following corollary.

Corollary 5.3. *For every positive integer* n, $H(\pm e^{-\pi\sqrt{n}/2})$ *is a unit.*

Acknowledgments

The authors would like to thank Professor Berndt for many of his suggestions and encouragement. We thank Professor Alladi for providing information on Göllnitz's work. The first author would also like to thank the Department of Mathematics at National University of Singapore for the facilities provided during his stay in Singapore.

References

1. G.E. Andrews, "On q-difference equations for certain well-poised basic hypergeometric series," *Quart. J. Math.* (Oxford) **19** (1968), 433–447.

2. G.E. Andrews, "An introduction to Ramanujan's '*lost*' notebook," *Amer. Math.* (Monthly) **86** (1979), 89–108.
3. B.C. Berndt, *Ramanujan Notebooks Part III*, Springer-Verlag, New York, 1991.
4. B.C. Berndt, H.H. Chan, and L.-C. Zhang, "Explicit evaluations of the Rogers-Ramanujan continued fraction," *J. Reine Angew. Math.* (to appear).
5. B.C. Berndt, H.H. Chan, and L.-C. Zhang, "Ramanujan's singular moduli," *The Ramanujan Journal* (to appear).
6. B.J. Birch, "Weber's class invariants," *Mathematika* **16** (1969), 283–294.
7. H.H. Chan, "On Ramanujan's cubic continued fraction," *Acta Arith.* **73** (1995), 343–355.
8. D.A. Cox, *Primes of the Form* $x^2 + ny^2$, Wiley, New York, 1989.
9. M. Deuring, "Die Klassenkörper der komplexes multiplication," *Enz. Math. Wiss. Band* I_2, Heft 10, Teil II (Stuttgort) 1958.
10. H. Göllnitz, "Partitionen mit Differenzenbedingungen," *J. Reine Angew. Math.* **225** (1967), 154–190.
11. B. Gordon, "Some continued fractions of the Rogers-Ramanujan type," *Duke Math. J.* **32** (1965), 741–748.
12. K.G. Ramanathan, "Ramanujan's continued fraction," *Indian J. Pure Appl. Math.* **16** (1985), 695–724.
13. S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
14. S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.
15. A. Selberg, "Über einige arithmetische Identitäten," *Avh. Norske Vid.-Akad. Oslo I. Mat.-Naturv.*, Kl. No. **8** (1936), 3–23.
16. H. Weber, *Lehrbuch der Algebra, dritter band*, Chelsea, New York, 1961.