A CONDITIONAL PROOF OF THE GOLDBACH CONJECTURE

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> *"The famous problem of which I propose to speak tonight is probably as difficult as any of the unsolved problems of mathematics, and my lecture cannot be entirely easy."*
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> —G. H. Hardy, *Goldbach's Theorem* [4, p. 1]

Abstract. We give what is, to our knowledge, the first conditional proof of the binary Goldbach conjecture for all sufficiently large even integers, assuming the generalized Riemann hypothesis for Dirichlet L-functions together with either of two additional spectral hypotheses. Our approach is fundamentally different from the classical sieve and circle methods. For each even integer N, odd Dirichlet characters modulo N give an exact projection of a smoothed Goldbach correlation. Subtracting the exact diagonal and a model built from Euler's beta function isolates an arithmetic residual, which under GRH defines a compact selfadjoint Hilbert–Schmidt operator on a Hardy space. An exact quadratic-form identity expresses the weighted Goldbach error in terms of this operator. Its spectrum defines a symmetric quaternionic function whose zero locations determine the operator norm. A prescribed zero-free region yields the weighted Hardy–Littlewood asymptotic $\mathcal{G}(N) \sim c_\phi \mathfrak{S}(N)$. Under the stronger hypothesis that all spectral zeros have real part $1/2$, we obtain

$$\mathcal{G}(N) = c_\phi \mathfrak{S}(N) + O_\phi\left(\sqrt{\frac{\mathfrak{S}(N)}{N}}(\log N)^{3/2}\right).$$

Here $\mathcal{G}(N)$ is the normalized weighted Goldbach correlation, $\mathfrak{S}(N)$ is the Goldbach singular series, and $c_\phi = \int_0^1 \phi(u)^2\,du$ for a fixed nonzero, nonnegative, symmetric smooth weight supported in $[1/3, 2/3]$. Under either spectral hypothesis together with GRH, every sufficiently large even integer N has $\gg_\phi \mathfrak{S}(N)N/(\log N)^2$ ordered representations as a sum of two primes in $[N/3, 2N/3]$. The method also extends naturally to other problems in additive number theory.

1. Introduction and statement of results

1.1. **Historical background.** Goldbach's conjecture, proposed in 1742 [3], states that every even integer $N \geqslant 4$ is a sum of two primes. The classical approaches to this problem are the circle method and sieve methods. For the related ternary problem, in 1923 Hardy and Littlewood [5] used the circle method to show, under a quasi-Riemann Hypothesis, that every sufficiently large odd integer is a sum of three primes; Vinogradov [14] removed this hypothesis in 1937. For the binary problem, Rényi's prime + almost-prime theorem [12] was followed by Chen's $(1 + 2)$ theorem [2] and theorem $(1 + 1.9)$ of the authors [9]. The latter gives $N = p_1 + rp_2$ for sufficiently large even N, with p_1, p_2 prime and $r = 1$ or prime satisfying $r \leqslant p_2^{0.9}$. Despite these advances, neither the circle method

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nor sieve methods have so far resolved the binary Goldbach conjecture. An excellent account of these classical methods and their principal results is given in the monograph of Chengdong Pan and Chengbiao Pan [11].

Beyond existence, Hardy and Littlewood [5] conjectured the following weighted asymptotic formula:

$$\frac{1}{N}\sum_{\substack{m,n\geqslant 1\\ m+n=N}}\Lambda(m)\Lambda(n)\sim\mathfrak{S}(N)\qquad(N\to\infty,\ N\text{ even}),\tag{1.1}$$

where the von Mangoldt function is defined by $\Lambda(n)=\log p$ if $n=p^k$ for a prime p and an integer $k\geqslant 1$, and $\Lambda(n)=0$ otherwise. The singular series is

$$\mathfrak{S}(N)=2C_2\prod_{\substack{p\mid N\\ p>2}}\frac{p-1}{p-2},\qquad C_2=\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).\tag{1.2}$$

The singular series records the local arithmetic densities of prime pairs.

No proof of the binary conjecture is known under the generalized Riemann Hypothesis (GRH) alone, even when GRH is assumed for all Dirichlet L-functions. Under this hypothesis, Hardy and Littlewood [6] obtained the classical exceptional-set bound $E(x)\ll_\varepsilon x^{1/2+\varepsilon}$ for every $\varepsilon>0$, where $E(x)$ counts the even integers $4\leqslant N\leqslant x$ that are not sums of two primes. This estimate does not exclude exceptions altogether.

1.2. **Our results.** We give a conditional proof of the binary Goldbach conjecture for all sufficiently large even integers by supplementing GRH with either of two spectral hypotheses. Our route is fundamentally different from the circle method and sieve methods. We draw directly on the classical ideas of Dirichlet, Euler, Hilbert and Pólya, Riemann, and Hardy. Starting from the Goldbach problem, we use odd characters to isolate the correlation and the Beta model to define its residual. Under GRH, the residual yields a selfadjoint operator. Zeros of its spectral function control the norm; a Hardy quadratic-form identity transfers this control to the Goldbach error.

More precisely, we establish conditional asymptotic formulas for a weighted form of the Hardy–Littlewood prediction. Fix a nonzero function $\phi\in C_c^\infty(\mathbb{R})$ supported in $[1/3,2/3]$, with $0\leqslant\phi\leqslant 1$ and $\phi(u)=\phi(1-u)$. For even $N\geqslant 4$, write

$$\mathcal{G}(N)=\frac{1}{N}\sum_{\substack{n+m=N\\ (nm,N)=1}}\Lambda(n)\Lambda(m)\phi(n/N)\phi(m/N),\tag{1.3}$$

$$c_\phi=\int_0^1\phi^2(u)\,\mathrm{d}u>0,\tag{1.4}$$

The sum runs over ordered pairs of positive integers and gives a normalized weighted count for the Goldbach equation $n+m=N$. Our aim is to prove, under our hypotheses, that $\mathcal{G}(N)>0$ for all sufficiently large even N, with a positive contribution from prime pairs remaining after the removal of proper prime powers. The expected main term is $c_\phi\mathfrak{S}(N)$, with the same singular series (1.2) as in (1.1).

Gauss's congruences gave arithmetic a unifying language. Here the decisive choice is to fix the modulus at the very integer N to be represented. Within this one modulus,

Dirichlet's odd characters reveal Goldbach, an arithmetic operator T_N becomes accessible, and its spectrum produces a quaternionic function in Riemann's spirit. Each N thus carries an arithmetic world of its own. This is our departure from the usual sieve and circle-method routes.

For each N, the construction uses the following part of GRH.

GRH(N). *Every nontrivial zero of every primitive Dirichlet L-function whose conductor divides N has real part $1/2$.*

The following five steps connect the arithmetic and spectral sides of the proof.

Dirichlet. The elementary sign $\chi(-1) = -1$ is decisive: odd-character orthogonality separates equal from opposite residue classes, and the support of the weight makes the latter exactly $n+m = N$. Logarithmic derivatives of L-functions turn our exact projection into a two-variable arithmetic kernel.

Euler. The Beta model encodes the off-diagonal geometry and recovers the Goldbach main term; subtracting the exact diagonal and this model isolates the residual. The theory of several complex variables reveals striking phenomena; natural arithmetic examples help make them clear. As Lebl [8, §1.1] writes, "There is something very special about the holomorphic category." Here the Beta model supplies an example dictated by arithmetic. After continuation and conjugation in the second variable, the model and its residual are Fueter regular. In coordinates $\omega = s + zj$, the residual satisfies

$$(\partial_{\bar{s}} + j\partial_z)\mathscr{E}_N(s, \bar{z}) = 0.$$

The four real coordinates retain the two-variable geometry of the prime-pair problem.

Hilbert–Pólya. The idea of Hilbert–Pólya seeks a selfadjoint operator to prove a Riemann Hypothesis. We reverse this approach: assuming GRH, we use such an operator to derive arithmetic consequences. For $N > e^{10} > 22026$ satisfying GRH(N), we realize the Hilbert–Pólya idea in the Goldbach setting by explicitly constructing the selfadjoint operator T_N from the residual in (2.17). This operator is compact and Hilbert–Schmidt; the construction is detailed in Subsection 2.3. We then use either of the additional spectral hypotheses below to bound its operator norm. The resulting bound makes the arithmetic error $|\mathcal{G}(N) - c_\phi \mathfrak{S}(N)|$ smaller than the positive main term $c_\phi \mathfrak{S}(N)$ for all sufficiently large even N.

Riemann. Riemann connected primes with a complex function. For prime pairs, our passage to a function of four real variables goes through the operator: the function is built from its spectrum. Write $\mu_j(N)$ for the nonzero real eigenvalues of T_N, with multiplicity, so that $\sum_j \mu_j^2(N) < \infty$. Set

$$\delta_N = \sqrt{\frac{\mathfrak{S}(N)}{N}}(\log N)^{3/2}. \tag{1.5}$$

Define the symmetric spectral function

$$\mathfrak{X}_N(q) = \prod_j \left(1 - \frac{4\mu_j^2(N)}{\delta_N^2} q(1-q)\right), \qquad q \in \mathbb{H}, \tag{1.6}$$

where $\mathbb{H}$ denotes the quaternions and $\Re q$ denotes the real part of q. The product converges locally uniformly; an empty product is 1. Its symmetry under $q \mapsto 1 - q$ is similar to

that of Riemann's completed function [13]. This is an intrinsic slice-regular function of the spectral variable q; the Fueter regularity above concerns the arithmetic coordinates. The locations of its zeros determine the magnitudes of the nonzero eigenvalues and hence $\|T_N\|$. Bounds on this norm control $|\mathcal{G}(N) - c_\phi \mathfrak{S}(N)|$ through the Hardy quadratic-form identity below.

Hardy. Linnik's dispersion method [10] exemplifies the power of quadratic estimates in additive problems. Here the key step is to express the exact signed error as a quadratic form in a Hardy space. The weight determines v_N with $\|v_N\|^2 \ll_\phi 1$. Before applying Cauchy–Schwarz, we have the exact identity

$$c_\phi \mathfrak{S}(N) - \mathcal{G}(N) = \langle T_N v_N, v_N \rangle.$$

Applying Cauchy–Schwarz then gives an upper bound for the absolute error:

$$|\langle T_N v_N, v_N \rangle| \leqslant \|T_N\| \, \|v_N\|^2 \ll_\phi \|T_N\|.$$

Appendix B tests the counting method numerically at $N = 30030$, recovering the exact number of Goldbach representations. Section 2 develops these five steps.

To turn the last inequality into arithmetic estimates, we now formulate two conditions on the zeros of $\mathfrak{X}_N$ that give bounds for $\|T_N\|$. The first, Hypothesis S, excludes zeros sufficiently close to 1 and yields an asymptotic formula.

Hypothesis S. *Fix beforehand a positive function $h(N) \to \infty$ such that, eventually,*

$$\eta_N = \frac{h^2(N)(\log N)^3}{N\mathfrak{S}(N)} \leqslant \frac{1}{4}. \tag{1.7}$$

For every sufficiently large even N satisfying GRH(N)*, the function $\mathfrak{X}_N(q)$ has no zeros in $\Re q > 1 - \eta_N$.*

In fact, GRH alone already gives a zero-free region for $\mathfrak{X}_N$ of the same form. Its width, however, falls short of that required by Hypothesis S, so the resulting estimate is not strong enough to make the Goldbach error $o(\mathfrak{S}(N))$.

Theorem 1.1. *Assume* GRH(N) *for every sufficiently large even N, and Hypothesis S with the prescribed function h. Then*

$$\mathcal{G}(N) = c_\phi \mathfrak{S}(N) + O_\phi\left(\frac{\mathfrak{S}(N)}{h(N)}\right). \tag{1.8}$$

In particular, $\mathcal{G}(N) \sim c_\phi \mathfrak{S}(N)$. The implied constant is independent of N and h.

Although Hypothesis S already suffices for our present purpose of obtaining Goldbach representations, we nevertheless venture the following stronger, ultimate conjecture on the spectral zeros. Together with GRH, it would yield what we expect to be the best possible remainder bound in the weighted Goldbach asymptotic formula.

Hypothesis R. *For every sufficiently large even N satisfying* GRH(N)*, every zero of $\mathfrak{X}_N(q)$ has real part $1/2$.*

Hypothesis R implies Hypothesis S for every admissible choice of h and gives the following sharper estimate.

Theorem 1.2. *Assume* GRH(N) *for every sufficiently large even* N*, and Hypothesis R. Then*

$$\mathcal{G}(N) = c_\phi \mathfrak{S}(N) + O_\phi\left(\sqrt{\frac{\mathfrak{S}(N)}{N}}(\log N)^{3/2}\right). \tag{1.9}$$

Either theorem gives $\gg_\phi \mathfrak{S}(N)N/(\log N)^2$ ordered representations $N = p_1 + p_2$, with both primes in $[N/3, 2N/3]$, for every sufficiently large even N.

The method developed here also extends naturally to other additive problems. These extensions will be discussed in future work.

Notation. We use standard notation from analytic number theory. In particular, p denotes a prime, (a, b) the greatest common divisor of a and b, and $d \mid n$ divisibility. The symbols Λ, φ, and μ denote the von Mangoldt, Euler totient, and Möbius functions, respectively. The notations $O, o, \ll, \gg, \asymp$, and $\sim$ have their standard asymptotic meanings; subscripts indicate the allowed dependence of implied constants. We write $\mathbf{1}_{\mathcal{A}}$ for the indicator of the condition $\mathcal{A}$, and $\int_{(\sigma)}$ for integration upwards along $\Re s = \sigma$. All logarithms are natural, and all limits in N are taken through even integers.

2. Idea and outline of the proof

The proof has two tasks: represent $c_\phi\mathfrak{S}(N) - \mathcal{G}(N)$ as a quadratic form of T_N at a vector of uniformly bounded norm, and obtain the required bound for $\|T_N\|$ from a condition on spectral zeros. The construction uses GRH(N); the additional norm bound comes from Hypothesis S or R.

The five steps below complete the argument. We will prove the analytic results used here in Sections 3–7. We introduce notation as it enters the construction. In the final Hardy step, we will establish the exact quadratic-form identity and a bound for $\|v_N\|$ that is uniform in N. These results convert the bound for $\|T_N\|$ into the required estimate for the Goldbach error. Removing the contribution of proper prime powers then yields representations of N as a sum of two primes for all sufficiently large even N. We retain the fixed weights and arithmetic notation of Section 1.

2.1. **Dirichlet: an exact projection of the Goldbach correlation.**

2.1.1. *The finite projection.* The relation $n+m = N$ implies $n \equiv -m \bmod N$. Let X_N^- be all odd characters modulo N, including imprimitive characters, extended by 0 at nonunits. Their orthogonality gives

$$\frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} \chi(n)\overline{\chi(m)} = \mathbf{1}_{(nm,N)=1}\left(\mathbf{1}_{n\equiv m \bmod N} - \mathbf{1}_{n\equiv -m \bmod N}\right). \tag{2.1}$$

On the support of the weight, $|n - m| \leqslant N/3$ and $2N/3 \leqslant n + m \leqslant 4N/3$. The two congruences thus mean $n = m$ and $n + m = N$, respectively. Multiplying by $\Lambda(n)\Lambda(m)\phi(n/N)\phi(m/N)/N$ and summing gives

$$\mathcal{G}(N) = \mathcal{D}(N) - \mathcal{P}(N), \tag{2.2}$$

where

$$\begin{aligned}\mathcal{D}(N) &= \frac{1}{N}\sum_{(n,N)=1}\Lambda^2(n)\phi^2(n/N),\\ \mathcal{P}(N) &= \frac{2}{\varphi(N)}\sum_{\chi\in X_N^-}|P_\chi|^2,\end{aligned} \tag{2.3}$$

and

$$P_\chi = \sum_{n\geqslant 1}\frac{\Lambda(n)\chi(n)}{\sqrt{n}}\Phi(n/N) = \frac{1}{\sqrt{N}}\sum_{n\geqslant 1}\Lambda(n)\chi(n)\phi(n/N). \tag{2.4}$$

Here the companion weight used throughout is defined by

$$\Phi(u) = \sqrt{u}\,\phi(u). \tag{2.5}$$

The factor $1/\sqrt{N}$ gives the required normalization after squaring. We will prove the unconditional projection identity in Lemma 3.1. It reduces the problem to estimating the character moment relative to the exact diagonal.

2.1.2. *The analytic kernel and Mellin inversion.* For $\Re s, \Re z > 1/2$, define

$$\mathscr{P}_N(s,z) = \frac{2}{\varphi(N)}\sum_{\chi\in X_N^-}\frac{L'}{L}(1/2+s,\chi)\frac{L'}{L}(1/2+z,\bar{\chi}). \tag{2.6}$$

The variables s, z are independent, and the bar conjugates only the character. We write $w = s + z$. Use the Mellin convention

$$\widetilde{\Phi}(s) = \int_0^\infty \Phi(u)u^{s-1}\,\mathrm{d}u, \quad \Phi(u) = \frac{1}{2\pi i}\int_{(\sigma)}\widetilde{\Phi}(s)u^{-s}\,\mathrm{d}s \quad (\sigma\in\mathbb{R}). \tag{2.7}$$

The series $-L'/L(1/2+s,\chi) = \sum_n \Lambda(n)\chi(n)n^{-1/2-s}$ and Mellin inversion represent P_χ by an integral. Applying this to $P_\chi P_{\bar{\chi}} = |P_\chi|^2$ and averaging gives

$$\mathcal{P}(N) = \frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} N^{s+z}\widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{P}_N(s,z)\,\mathrm{d}s\,\mathrm{d}z. \tag{2.8}$$

The two minus signs cancel. This formula holds unconditionally for $\sigma > 1/2$ by absolute convergence.

Under GRH(N), all inducing primitive characters have their nontrivial zeros on $\Re s = 1/2$. Since odd characters are nonprincipal, $L'/L(1/2+s,\chi)$ is holomorphic for $\Re s > 0$. We will prove the joint holomorphy of $\mathscr{P}_N$ in Proposition 3.2 and extend (2.8) to $0 < \sigma \leqslant 1/9$ in Lemma 3.5, where we will justify the contour shift using the required growth estimates. GRH enters through this contour shift.

2.1.3. *The families to be modeled.* For $\Re s, \Re z > 1/2$, the unweighted analytic kernel has the absolutely convergent expansion

$$\mathscr{P}_N(s,z) = \sum_{\substack{n,m\geqslant 1\\(nm,N)=1}}\frac{\Lambda(n)\Lambda(m)}{n^{1/2+s}m^{1/2+z}}\left(\mathbf{1}_{n\equiv m \bmod N} - \mathbf{1}_{n\equiv -m \bmod N}\right).$$

We will prove this identity in Proposition 3.2 using the logarithmic-derivative series and odd-character orthogonality. The positive congruence gives the diagonal $n = m$ and two difference families; the negative congruence gives the sum family. The three off-diagonal families are therefore

relation, $k \geqslant 1$	sign in $\mathscr{P}_N$
$m = n + kN$	$+$
$n = m + kN$	$+$
$n + m = kN$	$-$

After Mellin inversion restores the weight, its support gives $|n - m| \leqslant N/3$ and $n + m \leqslant 4N/3$. Thus both difference families and the sum terms with $k \geqslant 2$ vanish; among the off-diagonal families, only $n+m = N$ remains. Before this inversion, however, the analytic model must retain all three families. In the next step, we will keep the diagonal exactly and model their signed contribution; subtracting this model will isolate the Goldbach error.

2.2. **Euler: the Beta model and the analytic residual.**

2.2.1. *The signed continuous model.* By (2.2), a model for $\mathscr{P}_N$ must have weighted integral $\mathcal{D}(N) - c_\phi \mathfrak{S}(N)$. We retain the exact diagonal and model the three off-diagonal families using their local densities and Euler's Beta integral.

Consider the three families in the table in Subsection 2.1.3. Fix $k \geqslant 1$ and set $h = kN$. In the first two rows, $h = |n - m|$ is the positive difference; in the third, $h = n + m$ is the sum. All three have the same local-density factor $\mathfrak{S}(h)$. For $0 < \Re s, \Re z < 1/2$ and $w = s + z$, their continuous integrals, in the order of the table, are

$$\int_0^\infty x^{-1/2-s}(x+h)^{-1/2-z}\,\mathrm{d}x = h^{-w}B(1/2-s,w),$$
$$\int_0^\infty x^{-1/2-z}(x+h)^{-1/2-s}\,\mathrm{d}x = h^{-w}B(1/2-z,w),$$
$$\int_0^h x^{-1/2-s}(h-x)^{-1/2-z}\,\mathrm{d}x = h^{-w}B(1/2-s,1/2-z).$$

We will prove these evaluations and derive the common density factor in Lemma 4.1. Combining the three terms with the signs $+, +, -$ from the table gives the model $\mathfrak{S}(h)h^{-w}K_-(s,z)$, where

$$K_-(s,z) = B(1/2-s,s+z) + B(1/2-z,s+z) - B(1/2-s,1/2-z). \tag{2.9}$$

Here $B(s,z) = \Gamma(s)\Gamma(z)/\Gamma(s+z)$ is Euler's Beta function. The local-density calculation specifies this continuous model without assuming an asymptotic formula for prime pairs.

To collect the scales $h = kN$, introduce

$$\mathscr{S}_N(w) = \sum_{k\geqslant 1} \frac{\mathfrak{S}(kN)}{k^w} \quad (\Re w > 1). \tag{2.10}$$

The exact diagonal, the density Euler product, and the resulting off-diagonal model are

$$\begin{aligned}
\mathscr{D}_N(w) &= \sum_{\substack{n\geqslant 1\\(n,N)=1}} \frac{\Lambda^2(n)}{n^{1+w}},\\
\mathscr{S}_N(w) &= \mathfrak{S}(N)\zeta(w)P_N(w), \quad P_N(w) = \prod_{\substack{p>2\\p\nmid N}}\left(1+\frac{p^{-w}}{p-2}\right),\\
\mathscr{K}_N(s,z) &= N^{-s-z}\mathscr{S}_N(s+z)K_-(s,z).
\end{aligned} \tag{2.11}$$

The factor N^{-w} comes from $(kN)^{-w}$; P_N is an Euler product, distinct from P_χ. We will prove the Euler product identity $\mathscr{S}_N(w) = \mathfrak{S}(N)\zeta(w)P_N(w)$ for the density series in Lemma 4.3. There we will also continue $\mathscr{S}_N$ meromorphically to $\Re w > 0$, allowing its use on the Beta contours.

2.2.2. *Continuation and the exact main term.* Gamma factorization and the zeta functional equation give

$$\mathscr{K}_N(s,z) = \mathfrak{S}(N)\left(\frac{\pi}{N}\right)^{s+z} P_N(s+z)\zeta(1-s-z)\mathcal{X}_1(s)\mathcal{X}_1(z), \tag{2.12}$$

where $\mathcal{X}_1(s) = \Gamma((3/2-s)/2)/\Gamma((3/2+s)/2)$. We will prove in Proposition 4.4 that this expression is holomorphic on $0 < \Re s, \Re z < 3/2$, unconditionally.

When the weight is restored, N^{s+z} cancels the model's N^{-s-z}. At integer scales, support and symmetry give

$$\int_0^\infty \phi(u)\phi(u+k)\,\mathrm{d}u = 0 \quad (k\geqslant 1), \qquad \int_0^k \phi(u)\phi(k-u)\,\mathrm{d}u = \begin{cases} c_\phi, & k=1,\\ 0, & k\geqslant 2.\end{cases}$$

Hence only the negative sum term at $k=1$ survives, with density $\mathfrak{S}(N)$. For the continued model this evaluation requires a contour shift: the combined continuous mass of the two difference terms and the negative sum term is zero, cancelling the density series pole at $w=1$. We will justify this contour shift and prove the following identity for $0<\sigma\leqslant 1/9$ in Lemma 7.2:

$$\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} N^{s+z}\widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{K}_N(s,z)\,\mathrm{d}s\,\mathrm{d}z = -c_\phi\mathfrak{S}(N).$$

We will also evaluate the diagonal integral as $\mathcal{D}(N)$ in the same lemma. Thus the model has exactly the required weighted integral.

2.2.3. *The arithmetic residual.* Define

$$\mathscr{E}_N(s,z) = \mathscr{P}_N(s,z) - \mathscr{D}_N(s+z) - \mathscr{K}_N(s,z).$$

We will prove in Proposition 4.5 that, under GRH(N), the residual is holomorphic on $0<\Re s, \Re z < 3/2$. Combining the kernels' three Mellin evaluations with (2.2) gives

$$\mathcal{G}(N) = c_\phi\mathfrak{S}(N) - \frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} N^{s+z}\widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{E}_N(s,z)\,\mathrm{d}s\,\mathrm{d}z.$$

Indeed, the residual integral equals $\mathcal{P}(N) - \mathcal{D}(N) + c_\phi \mathfrak{S}(N) = c_\phi \mathfrak{S}(N) - \mathcal{G}(N)$, fixing the sign. The task is now to bound this integral by $o(\mathfrak{S}(N))$. We next express it as a quadratic form of a compact selfadjoint operator, whose norm will control the error.

2.3. **Hilbert–Pólya: reversing the direction of dependence.**

2.3.1. *The spectral strategy and the role of GRH.* The Hilbert–Pólya programme seeks a selfadjoint realization whose real spectral parameters would place zeta zeros on the critical line. Here the direction is reversed: for even $N > e^{10}$ we assume GRH(N), construct an operator from the arithmetic residual, and ask how its spectrum controls Goldbach. GRH supplies holomorphy; the further spectral hypotheses will supply smallness.

Indeed, every odd character is nonprincipal, and GRH makes $g_\chi(s) = L'/L(1/2+s,\chi)$ holomorphic for $\Re s > 0$. Together with the diagonal and the continued Beta model, this makes $\mathscr{E}_N$ holomorphic on $0 < \Re s, \Re z < 3/2$. Conjugating its second variable gives a Hermitian kernel, since

$$\mathscr{P}_N(s,\bar{z}) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} g_\chi(s)\overline{g_\chi(z)}, \quad g_\chi(s) = \frac{L'}{L}(1/2+s,\chi), \tag{2.13}$$

and the diagonal and Beta model have the same symmetry. Thus

$$\mathscr{E}_N(s,\bar{z}) = \overline{\mathscr{E}_N(z,\bar{s})}.$$

Subtraction need not preserve positivity, so eigenvalues of either sign must be allowed.

2.3.2. *The Hardy strip and the operator.* Set

$$\alpha_N = \frac{1}{\log N}, \quad \beta = \frac{1}{8}, \quad \sigma_* = \frac{1}{9}, \quad \Delta_N = \beta - \alpha_N, \quad S_N = \{s : \alpha_N < \Re s < \beta\}, \tag{2.14}$$

and write

$$\mathcal{H}_N = H^2(S_N), \quad \Gamma_N = (\alpha_N + i\mathbb{R}) \sqcup (\beta + i\mathbb{R}). \tag{2.15}$$

Here $\mathcal{H}_N$ consists of holomorphic functions whose paired vertical L^2 means remain bounded as the two lines approach the boundaries. Their traces form a closed subspace of $L^2(\Gamma_N)$, with inner product

$$\langle h, g \rangle_{\mathcal{H}_N} = \int_{\Gamma_N} h(s)\overline{g(s)}\,|\mathrm{d}s|, \tag{2.16}$$

linear in the first variable. Both boundaries carry positive arc length, with no $1/(2\pi)$ factor. The choice $\alpha_N < 1/10$ leaves σ_* strictly inside the strip and keeps $N^{2\alpha_N} = e^2$, which will make the weight functional uniformly bounded.

Insert decay at large heights and define

$$(T_N h)(s) = \int_{\Gamma_N} \frac{\mathscr{E}_N(s,\bar{z})}{(1+s)(1+\bar{z})} h(z)\,|\mathrm{d}z|. \tag{2.17}$$

We will prove in Proposition 5.3 that the boundary kernel is square integrable and that the operator maps into the Hardy trace space. Hermitian symmetry then makes T_N self-adjoint; square integrability makes it Hilbert–Schmidt and hence compact. Consequently

its nonzero eigenvalues $\mu_j(N)$, counted with multiplicity, satisfy

$$\mu_j(N) \in \mathbb{R}, \qquad \sum_j \mu_j^2(N) = \|T_N\|_{\mathrm{HS}}^2 < \infty, \qquad \|T_N\| = \sup_j |\mu_j(N)|.$$

The last supremum is understood as zero when $T_N = 0$. These facts hold for fixed N; the arithmetic question is the size of this norm as N varies.

2.3.3. *The bridge to the arithmetic error.* For $\alpha_N < \sigma \leqslant 1/9$, we define the linear functional $\ell_N : \mathcal{H}_N \to \mathbb{C}$ using the Mellin transform $\widetilde{\Phi}$ of the companion weight Φ from (2.5):

$$\ell_N(h) = \frac{1}{2\pi i} \int_{(\sigma)} N^s \widetilde{\Phi}(s)(1+s)h(s)\,\mathrm{d}s.$$

It has a Riesz vector v_N, characterized by $\ell_N(h) = \langle h, v_N\rangle$, with $\mathcal{Q}_N := \|v_N\|^2 \ll_\phi 1$ uniformly in N. We will verify this bound and derive the following identity for the residual in Subsection 2.5.2:

$$\mathcal{G}(N) = c_\phi \mathfrak{S}(N) - \langle T_N v_N, v_N\rangle.$$

Now Cauchy–Schwarz gives

$$|\mathcal{G}(N) - c_\phi \mathfrak{S}(N)| \leqslant \|T_N v_N\|\,\|v_N\| \leqslant \mathcal{Q}_N \|T_N\|. \tag{2.18}$$

Thus $\|T_N\| = o(\mathfrak{S}(N))$ suffices for the relative asymptotic, while $\|T_N\| \leqslant \delta_N$ gives the error $O_\phi(\delta_N)$ in Theorem 1.2. Before applying the inequality, the quadratic form remains the exact arithmetic correction. We next express the required norm bounds as conditions on spectral zeros.

2.4. **Riemann: spectral zeros and operator norm bounds.** For even $N > e^{10}$ satisfying GRH(N), inequality (2.18), together with $\mathcal{Q}_N \ll_\phi 1$, gives the master estimate

$$|\mathcal{G}(N) - c_\phi \mathfrak{S}(N)| \ll_\phi \|T_N\|. \tag{2.19}$$

Thus bounding the operator norm directly controls the error in the weighted Goldbach count. We will express the required norm bounds as conditions on spectral zeros and derive the two asymptotic formulas under Hypotheses S and R, respectively.

2.4.1. *The spectral product and its zeros.* Recall $\delta_N^2 = \mathfrak{S}(N)(\log N)^3/N$ and the product (1.6):

$$\mathfrak{X}_N(q) = \prod_j \left(1 - \frac{4\mu_j^2(N)}{\delta_N^2} q(1-q)\right).$$

The squared eigenvalues are summable, so the product converges locally uniformly. On the usual complex slice, it has the Fredholm determinant representation

$$\mathfrak{X}_N(q) = \det\!\left(\mathrm{Id} - 4\delta_N^{-2} q(1-q) T_N^2\right).$$

Its real coefficients give a consistent intrinsic slice-regular extension to $q \in \mathbb{H}$. This spectral variable is independent of the arithmetic variables (s, z); its regularity is distinct from the Fueter regularity of $\mathscr{E}_N(s, \bar{z})$ in the coordinates $\omega = s + zj$.

The factors immediately give

$$\mathfrak{X}_N(0) = \mathfrak{X}_N(1) = 1, \tag{2.20}$$

$$\mathfrak{X}_N(1-q) = \mathfrak{X}_N(q), \tag{2.21}$$

and

$$\mathfrak{X}_N(\bar{q}) = \overline{\mathfrak{X}_N(q)}. \tag{2.22}$$

For real $q \notin (0,1)$ every factor is at least 1, so there are no real zeros there. To find all zeros, put $d_j = \delta_N^2/\mu_j^2(N)$ and write $q = x + yI$, where $x, y \in \mathbb{R}$ and I is a unit imaginary quaternion. The factor equation is

$$q(1-q) = x(1-x) + y^2 + (1-2x)yI = d_j/4.$$

Its imaginary part forces $x = 1/2$ whenever $y \neq 0$, and the remaining quadratic gives

$$\begin{cases} \Re q = \frac{1}{2}, \quad |\Im q| = \frac{1}{2}\sqrt{d_j - 1}, & d_j > 1, \\ q = \frac{1}{2} \quad \text{(a double zero of the factor)}, & d_j = 1, \\ q = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - d_j}, & 0 < d_j < 1. \end{cases}$$

The nonreal zeros form two-spheres on $\Re q = 1/2$; every zero off that hyperplane is real and comes from $|\mu_j| > \delta_N$. Compactness allows only finitely many such exceptional pairs. Absolute convergence excludes additional zeros of the infinite product. We will give the convergence argument and prove the full symmetry statements in Section 6.

2.4.2. *The zero–norm dictionary.* Fix $0 < \eta < 1/2$. Only the larger root of an exceptional pair can enter $\Re q > 1 - \eta$. Keeping it outside means

$$\frac{1}{2} + \frac{1}{2}\sqrt{1 - \delta_N^2/\mu_j^2} \leqslant 1 - \eta,$$

which is equivalent to

$$|\mu_j| \leqslant \frac{\delta_N}{2\sqrt{\eta(1-\eta)}}.$$

Nonexceptional eigenvalues already satisfy this inequality. Taking the supremum therefore gives the exact dictionary

$$\mathfrak{X}_N(q) \neq 0 \ (\Re q > 1 - \eta) \quad \Longleftrightarrow \quad \|T_N\| \leqslant \frac{\delta_N}{2\sqrt{\eta(1-\eta)}}.$$

The half-space is open, so boundary zeros and equality in the norm bound are allowed. In particular, a bound $\|T_N\| \leqslant M_N$ with $M_N > \delta_N$ guarantees width

$$\eta(M_N) = \frac{1 - \sqrt{1 - \delta_N^2/M_N^2}}{2}, \qquad \eta(M_N)(1 - \eta(M_N)) = \frac{\delta_N^2}{4M_N^2}. \tag{2.23}$$

Thus the width has order δ_N^2/M_N^2.

2.4.3. *What GRH alone supplies.* In Theorem 6.2, we will prove the uniform analytic estimate

$$\|T_N\| \leqslant \|T_N\|_{\mathrm{HS}} \ll (\log N)^2 \tag{2.24}$$

for sufficiently large even N satisfying GRH(N). Its proof splits the operator according to

$$\mathscr{E}_N = \mathscr{P}_N - \mathscr{D}_N - \mathscr{K}_N.$$

Writing $\mathcal{P}_N$, $\mathcal{D}_N$, and $\mathcal{K}_N$ for these three kernels with the weighting and integration of (2.17), one has

$$T_N = \mathcal{P}_N - \mathcal{D}_N - \mathcal{K}_N.$$

The character average controls the first component, positivity controls the diagonal, and the continued Beta formula controls the model. Their separate estimates each have size $O((\log N)^2)$ at $\alpha_N = 1/\log N$; they do not measure cancellation between the components.

Putting $M_N = C(\log N)^2$ in the dictionary gives a guaranteed zero-free region

$$\Re q > 1 - \eta_N^{\mathrm{GRH}}, \qquad \eta_N^{\mathrm{GRH}} \asymp \frac{\mathfrak{S}(N)}{N \log N}. \tag{2.25}$$

The constant is chosen sufficiently small using the absolute constant implied in (2.24). This available norm bound is too large to make the Goldbach error smaller than its main term; S and R impose the required improvement.

2.4.4. *Completing the implications of S and R.* For any zero-free width $0 < \eta_N \leqslant 1/4$, the dictionary and the master estimate (2.19) give

$$\frac{|\mathcal{G}(N) - c_\phi \mathfrak{S}(N)|}{c_\phi \mathfrak{S}(N)} \ll_\phi \sqrt{\frac{(\log N)^3}{N \eta_N \mathfrak{S}(N)}}. \tag{2.26}$$

It therefore suffices for the asymptotic that

$$\frac{N \eta_N \mathfrak{S}(N)}{(\log N)^3} \longrightarrow \infty. \tag{2.27}$$

Writing the left side as $h(N)^2$ explains precisely the scale chosen in Hypothesis S.

Hypothesis S. Write S($N; h$) for its zero-exclusion assertion at the fixed modulus, with the prescribed admissible h and width (1.7). Substitution gives

$$\mathrm{S}(N; h) \quad \Longleftrightarrow \quad \|T_N\| \leqslant \frac{\mathfrak{S}(N)}{2h(N)\sqrt{1-\eta_N}}. \tag{2.28}$$

Since $1-\eta_N \geqslant 3/4$, the master estimate (2.19) proves $\mathcal{G}(N) = c_\phi \mathfrak{S}(N) + O_\phi(\mathfrak{S}(N)/h(N))$, which is Theorem 1.1. The implied constant is independent of N and h; the required lower bound on N may depend on ϕ and the preassigned function h. The gain over the available GRH width is

$$\frac{\eta_N}{\eta_N^{\mathrm{GRH}}} \asymp h^2(N) \frac{(\log N)^4}{\mathfrak{S}^2(N)} \longrightarrow \infty, \tag{2.29}$$

using $\mathfrak{S}(N) \ll \log\log N$. The excluded zeros are real pairs near the endpoints; the Siegel-zero analogy concerns their location, without asserting uniqueness, simplicity, or repulsion.

Hypothesis R. Write $\mathrm{R}(N)$ for its assertion at the fixed modulus. The zero classification shows directly that

$$\mathrm{R}(N) \quad \Longleftrightarrow \quad \|T_N\| \leqslant \delta_N. \tag{2.30}$$

The master estimate (2.19) therefore gives $\mathcal{G}(N) = c_\phi \mathfrak{S}(N) + O_\phi(\delta_N)$, proving Theorem 1.2. All exceptional real pairs are excluded, so R also implies S for every admissible choice of h.

In the following Hardy subsection, we will establish the exact quadratic-form identity and the uniform bound for $\|v_N\|$ needed to convert the spectral estimates into these arithmetic conclusions.

2.5. **Hardy: the arithmetic error as an exact quadratic form.** The final step evaluates the residual at the vector selected by the Goldbach weight. We obtain an exact identity first; Cauchy–Schwarz then turns it into the estimate needed for the two theorems. Throughout, assume GRH(N) and $N > e^{10}$, with $\Phi(u) = \sqrt{u}\,\phi(u)$ as in (2.5).

2.5.1. *Mellin inversion and the arithmetic residual.* Mellin inversion of the prime polynomial gives

$$P_\chi = -\frac{1}{2\pi i}\int_{(\sigma)} N^s \widetilde{\Phi}(s) \frac{L'}{L}(1/2+s,\chi)\,\mathrm{d}s. \tag{2.31}$$

It holds first on $\sigma = 1$ by absolute convergence, and then on $0 < \sigma \leqslant 1/9$: GRH excludes poles between the contours, while the smooth weight makes the horizontal integrals vanish. Multiplying the formulas for χ and $\bar\chi$ and averaging over X_N^- proves (2.8) on these lines. The two subtracted kernels have the evaluations

$$\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} N^{s+z}\widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{D}_N(s+z)\;\mathrm{d}s\;\mathrm{d}z = \mathcal{D}(N), \tag{2.32}$$

$$\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} \widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{S}_N(s+z)K_-(s,z)\;\mathrm{d}s\;\mathrm{d}z = -c_\phi\mathfrak{S}(N). \tag{2.33}$$

We will prove both evaluations in Lemma 7.2.

The residual integral is therefore

$$\begin{aligned} r_\sigma(N) &= \frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} N^{s+z}\widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{E}_N(s,z)\,\mathrm{d}s\,\mathrm{d}z \\ &= \mathcal{P}(N) - \mathcal{D}(N) + c_\phi\mathfrak{S}(N). \end{aligned} \tag{2.34}$$

The exact projection (2.2) now yields

$$\mathcal{G}(N) = c_\phi\mathfrak{S}(N) - r_\sigma(N). \tag{2.35}$$

In particular, this signed discrepancy is independent of σ; write it as $r(N)$.

2.5.2. *The residual as a quadratic form.* For an interior line $\alpha_N < \sigma < \beta$, define

$$\ell_N(h) = \frac{1}{2\pi i}\int_{(\sigma)} N^s\widetilde{\Phi}(s)(1+s)h(s)\,\mathrm{d}s, \qquad h \in \mathcal{H}_N. \tag{2.36}$$

The factor $1+s$ compensates for the denominator inserted in the operator kernel. Put $V(u) = \Phi(u) - u\Phi'(u)$, so that $\widetilde{V}(s) = (1+s)\widetilde{\Phi}(s)$. We will prove the following formula

for the exact squared norm in Proposition 7.4, using the Fourier–Laplace representation of the strip Hardy space:

$$\mathcal{Q}_N := \|\ell_N\|^2 = \frac{1}{2\pi}\int_0^\infty \frac{(Nu)^{2\alpha_N}|V(u)|^2}{1+(Nu)^{-2\Delta_N}}\,\frac{\mathrm{d}u}{u}. \tag{2.37}$$

This representation also proves independence of the interior contour. Since $\alpha_N = 1/\log N$ and V is supported in $[1/3, 2/3]$,

$$\mathcal{Q}_N \leqslant \frac{e^2}{2\pi}\int_0^\infty |\Phi(u) - u\Phi'(u)|^2\,\frac{\mathrm{d}u}{u} \ll_\phi 1. \tag{2.38}$$

The choice of the moving strip thus prevents a loss growing with N.

Let v_N be the Riesz vector, with $\ell_N(h) = \langle h, v_N\rangle$ and $\|v_N\|^2 = \mathcal{Q}_N$; the inner product is linear in its first entry. Let k_z be the evaluation vector, defined by $h(z) = \langle h, k_z\rangle$. The operator kernel is

$$\langle T_N k_z, k_s\rangle = \frac{\mathscr{E}_N(s,\bar z)}{(1+s)(1+\bar z)}.$$

Apply ℓ_N to each kernel variable, conjugating the second. The factors $1+s$ and $1+\bar z$ cancel; the reality of Φ then converts the second contour to the holomorphic Mellin contour. This recovers precisely (2.34), and hence

$$r(N) = \langle T_N v_N, v_N\rangle. \tag{2.39}$$

Only now do we estimate. Cauchy–Schwarz gives

$$|r(N)| \leqslant \|T_N v_N\|\,\|v_N\| \leqslant \mathcal{Q}_N\|T_N\| \ll_\phi \|T_N\|. \tag{2.40}$$

Together with (2.35), this proves the master estimate (2.19) and hence Theorems 1.1 and 1.2 under S and R. Retaining the signed quadratic form gives the exact weighted Goldbach identity.

2.5.3. *Goldbach representations.* Goldbach representations follow readily from either Theorem 1.1 or Theorem 1.2: under its hypotheses, every sufficiently large even integer N can be written as $N = p_1 + p_2$, with p_1, p_2 prime, and the number of representations is $\gg_\phi N(\log N)^{-2}$.

3. Dirichlet: an exact projection of the Goldbach correlation

We express the weighted Goldbach correlation as the difference between a diagonal moment and a moment over odd characters modulo N. The identity is finite and unconditional. We then pass to logarithmic derivatives of Dirichlet L-functions and use GRH to move the Mellin contours into the region needed for the continuous model.

Fix the weight ϕ used in (1.3) and its companion Φ defined by (2.5). The symbol $\varphi(N)$ denotes Euler's totient, whereas ϕ always denotes the weight. With Λ the von Mangoldt function, the correlation to be studied is

$$\mathcal{G}(N) = \frac{1}{N}\sum_{\substack{n+m=N\\(nm,N)=1}} \Lambda(n)\Lambda(m)\phi(n/N)\phi(m/N),$$

as in (1.3). The sum runs over ordered pairs of positive integers. Its expected main term is $c_\phi \mathfrak{S}(N)$, where

$$c_\phi = \int_0^1 \phi^2(u)\, \mathrm{d}u,$$

as in (1.4), and

$$\mathfrak{S}(N) = 2C_2 \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2}, \qquad C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right),$$

as in (1.2). We write

$$X_N^- = \{\chi \bmod N : \chi(-1) = -1\}$$

for all odd characters modulo N, including imprimitive characters, and extend each character by zero when its argument is not coprime to N. These conventions account for the coprimality condition in the projection below.

3.1. **The finite projection and its size.** We will study the two moments

$$\mathcal{D}(N) = \frac{1}{N} \sum_{(n,N)=1} \Lambda^2(n)\phi^2(n/N),$$

$$\mathcal{P}(N) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} |P_\chi|^2,$$

as in (2.3), where P_χ is the character polynomial

$$P_\chi = \sum_{n \geqslant 1} \frac{\Lambda(n)\chi(n)}{\sqrt{n}} \Phi(n/N) = \frac{1}{\sqrt{N}} \sum_{n \geqslant 1} \Lambda(n)\chi(n)\phi(n/N),$$

as in (2.4). Thus $\mathcal{D}(N)$ retains equal indices, while $\mathcal{P}(N)$ averages the squared character sums with the same normalization.

Lemma 3.1 (Finite projection)**.** *For every even $N \geqslant 4$, the moments just defined satisfy the exact identity*

$$\mathcal{G}(N) = \mathcal{D}(N) - \mathcal{P}(N)$$

as in (2.2)*, and the bound*

$$0 \leqslant \mathcal{P}(N) \leqslant \mathcal{D}(N) \ll_\phi \log N. \tag{3.1}$$

Proof. The selector $(1 - \chi(-1))/2$ in the full character sum gives

$$\frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} \chi(n)\overline{\chi(m)} = \frac{1}{\varphi(N)} \sum_{\chi \bmod N} (1 - \chi(-1))\chi(n)\overline{\chi(m)}.$$

Full character orthogonality gives, with the coprimality restriction coming from the zero extension,

$$\frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} \chi(n)\overline{\chi(m)} = \mathbf{1}_{(nm,N)=1} \left(\mathbf{1}_{n \equiv m \bmod N} - \mathbf{1}_{n \equiv -m \bmod N}\right),$$

as in (2.1). If the weight is nonzero, $N/3 \leqslant n, m \leqslant 2N/3$, so $|n-m| \leqslant N/3$ and $2N/3 \leqslant n+m \leqslant 4N/3$. The only permitted multiples of N are therefore 0 for $n-m$ and N for $n+m$. Thus

$$\mathbf{1}_{(nm,N)=1}\mathbf{1}_{n+m=N} = \mathbf{1}_{(n,N)=1}\mathbf{1}_{n=m} - \frac{2}{\varphi(N)}\sum_{\chi\in X_N^-}\chi(n)\overline{\chi(m)}$$

on this support. Multiply by $\Lambda(n)\Lambda(m)\phi(n/N)\phi(m/N)/N$ and sum. The diagonal is $\mathcal{D}(N)$, while the character term factors as $P_\chi P_{\bar\chi} = |P_\chi|^2$, since ϕ is real. This proves the finite projection with its exact normalization. Since both $\mathcal{G}(N)$ and $\mathcal{P}(N)$ are nonnegative, $0 \leqslant \mathcal{P}(N) \leqslant \mathcal{D}(N)$. Chebyshev's estimate $\sum_{n\leqslant x}\Lambda(n) \ll x$ gives

$$\mathcal{D}(N) \leqslant \frac{\log N}{N}\sum_{n\leqslant N}\Lambda(n) \ll \log N.$$

This proves the lemma. □

The arithmetic problem is to estimate the difference between these two moments at the smaller scale $\mathfrak{S}(N)$.

3.2. **Holomorphy of the character kernel.** To insert the Mellin weight into both character polynomials, we use the two-variable kernel

$$\mathscr{P}_N(s,z) = \frac{2}{\varphi(N)}\sum_{\chi\in X_N^-} g_\chi(s)g_{\bar\chi}(z),$$

as in (2.6), where g_χ is the logarithmic derivative

$$g_\chi(s) = \frac{L'}{L}(1/2+s,\chi),$$

as in (2.13). The variables s, z are independent; conjugation here acts only on the character. Thus $\mathscr{P}_N(s,z)$ is holomorphic in both variables wherever the logarithmic derivatives are holomorphic. The Hermitian kernel used later is $\mathscr{P}_N(s,\bar z)$: it is holomorphic in the first variable and antiholomorphic in the second, as is usual for kernels on spaces of holomorphic functions. We keep these two forms distinct when applying Mellin inversion.

Write χ^* for the primitive character inducing χ and $d_\chi \mid N$ for its conductor. In this section, GRH(N) means that every nontrivial zero of every primitive Dirichlet L-function whose conductor divides N has real part 1/2. On $\Re s > 1$, $L(s,\chi) = \sum_{n\geqslant 1}\chi(n)n^{-s}$, and its logarithmic derivative is

$$-\frac{L'}{L}(1/2+s,\chi) = \sum_{n\geqslant 1}\frac{\Lambda(n)\chi(n)}{n^{1/2+s}} \qquad (\Re s > 1/2).$$

Proposition 3.2 (The character kernel)**.** *For $\Re s, \Re z > 1/2$, the kernel has the absolutely convergent expansion*

$$\mathscr{P}_N(s,z) = \sum_{\substack{n,m\geqslant 1\\(nm,N)=1}}\frac{\Lambda(n)\Lambda(m)}{n^{1/2+s}m^{1/2+z}}\left(\mathbf{1}_{n\equiv m \bmod N} - \mathbf{1}_{n\equiv -m \bmod N}\right). \tag{3.2}$$

The kernel $\mathscr{P}_N$ is meromorphic on $\mathbb{C}^2$, with possible poles only where one of its logarithmic derivatives has a pole. Under GRH(N) *it is jointly holomorphic for $\Re s, \Re z > 0$ and satisfies*

$$\mathscr{P}_N(s,z) = \mathscr{P}_N(z,s), \qquad \overline{\mathscr{P}_N(s,z)} = \mathscr{P}_N(\bar{s},\bar{z}). \tag{3.3}$$

The Hermitian kernel $\mathscr{P}_N(s,\bar{z})$ is positive semidefinite, of rank at most $\varphi(N)/2$.

Proof. Expand both logarithmic derivatives in their absolutely convergent Dirichlet series and apply the odd-character orthogonality formula proved above. The two signs from the logarithmic derivatives cancel; the remaining minus sign comes from the odd-character projection. This proves (3.2).

Every odd character is nonprincipal, so its L-function is entire and its logarithmic derivative is meromorphic, with poles at the zeros. Under GRH the nontrivial zeros lie on $\Re s = 1/2$; the primitive trivial zeros are negative odd integers, and the additional Euler-factor zeros of imprimitive characters lie on $\Re s = 0$. Hence each $g_\chi(s)$ is holomorphic for $\Re s > 0$. The character sum is finite, so products of these one-variable functions are jointly holomorphic on the product domain. No limiting interchange is needed to pass from the individual logarithmic derivatives to the kernel.

Conjugation gives $g_{\bar{\chi}}(\bar{z}) = \overline{g_\chi(z)}$. Consequently the Hermitian form of the kernel is

$$\mathscr{P}_N(s,\bar{z}) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} g_\chi(s)\overline{g_\chi(z)},$$

as in (2.13). This Gram representation proves positivity and the rank bound, since $|X_N^-| = \varphi(N)/2$. Reindexing by $\chi \mapsto \bar{\chi}$ gives (3.3). This proves the proposition. □

3.3. **Bounds for logarithmic derivatives under GRH.** The contour shifts require vertical bounds on fixed positive strips. We derive them from the primitive zero expansion, retaining the Euler factors that distinguish a character modulo N from its inducing character.

For a primitive odd character χ^* of conductor d_χ, write

$$\mathcal{L}(u,\chi^*) = (d_\chi/\pi)^{(u+1)/2}\Gamma((u+1)/2)L(u,\chi^*).$$

For the following results, see [7].

Lemma 3.3. *For a primitive odd character χ^*, the nontrivial zeros ρ, counted with multiplicity, satisfy*

$$\frac{\mathcal{L}'}{\mathcal{L}}(u,\chi^*) - \frac{\mathcal{L}'}{\mathcal{L}}(u_0,\chi^*) = \sum_\rho \left(\frac{1}{u-\rho} - \frac{1}{u_0-\rho}\right), \tag{3.4}$$

$$\#\{\rho : t \leqslant \Im\rho < t+1\} \ll \log\big(d_\chi(|t|+2)\big), \tag{3.5}$$

where the first identity holds away from zeros and its series converges absolutely and locally uniformly in u, u_0.

Under GRH(N), the nontrivial zeros of $L(s,\chi^*)$ have the form $1/2+i\gamma_\chi$, where χ^* is the primitive character inducing χ. Set

$$Z_\chi(s)=\sum_{\gamma_\chi}\left(\frac{1}{s-i\gamma_\chi}-\frac{1}{3/2-i\gamma_\chi}\right), \qquad \Re s>0, \tag{3.6}$$

$$e_\chi(u)=\sum_{\substack{p\mid N\\ p\nmid d_\chi}}\frac{\chi^*(p)\log p}{p^u-\chi^*(p)}, \qquad \Re u>0. \tag{3.7}$$

Lemma 3.4. *Assume* GRH(N). *The series* (3.6) *converges absolutely and locally uniformly for* $\Re s>0$, *and*

$$\begin{aligned} g_\chi(s)=Z_\chi(s)+\frac{L'}{L}(2,\chi)-\frac{1}{2}\left\{\frac{\Gamma'}{\Gamma}(3/4+s/2)-\frac{\Gamma'}{\Gamma}(3/2)\right\}\\ +e_\chi(1/2+s)-e_\chi(2). \end{aligned} \tag{3.8}$$

For every compact interval $I\subset(0,\infty)$,

$$g_\chi(\sigma+it)\ll_I \log^2\big(N(|t|+2)\big) \qquad (\sigma\in I,\ t\in\mathbb{R},\ \chi\in X_N^-). \tag{3.9}$$

Proof. The paired summand in (3.6) is $(3/2-s)/((s-i\gamma_\chi)(3/2-i\gamma_\chi))$. The local zero count therefore proves locally uniform absolute convergence. Apply (3.4) at $u=1/2+s$ and $u_0=2$. The conductor terms cancel. Subtracting the Gamma terms and using

$$L(u,\chi)=L(u,\chi^*)\prod_{\substack{p\mid N\\ p\nmid d_\chi}}(1-\chi^*(p)p^{-u})$$

gives (3.8). On a compact subset of the right half-plane, the real parts stay uniformly positive. Thus the finitely many low zeros cannot spoil the local uniform convergence of the paired series.

For the bound, let $I=[\delta,M]$ with $\delta>0$ and put $T=|t|+2$. For $|\gamma_\chi|\leqslant 2T$, group each fraction in (3.6) by unit distance from its imaginary part. Equation (3.5) gives

$$\ll_I \log\big(N(T+2)\big)\left(1+\sum_{1\leqslant j\leqslant 3T+2}\frac{1}{j}\right)\ll_I \log^2\big(N(T+2)\big).$$

The denominators nearest t are bounded below by δ. For $|\gamma_\chi|>2T$, keep the paired fractions together: their difference is $O_I(T/|\gamma_\chi|^2)$. Dyadic summation and (3.5) give a total $O_I(\log(N(T+2)))$ from this range. Here $d_\chi\leqslant N$ makes the zero-count bound uniform in the inducing conductor, so the implied constant depends only on I.

The remaining terms in (3.8) are smaller. The series at $u=2$ is bounded absolutely, Stirling's formula bounds the Gamma difference by $O_I(\log T)$, and

$$|e_\chi(u)|\leqslant\sum_{p\mid N}\frac{\log p}{p^{1/2+\delta}-1}\leqslant\frac{\log N}{2^{1/2+\delta}-1} \qquad (\Re u\geqslant 1/2+\delta).$$

The same estimate at $u=2$ gives $e_\chi(2)\ll\log N$. This proves (3.9). This proves the lemma. □

3.4. **Mellin inversion and the signed families.** We connect the unconditional finite projection to the contours used in the model calculation and identify the three signed families.

For the companion weight Φ defined in (2.5), our Mellin transform and inversion conventions are

$$\widetilde{\Phi}(s) = \int_0^\infty \Phi(u)u^{s-1}\,\mathrm{d}u, \qquad \Phi(u) = \frac{1}{2\pi i}\int_{(\sigma)} \widetilde{\Phi}(s)u^{-s}\,\mathrm{d}s \quad (\sigma \in \mathbb{R}),$$

as in (2.7), where $\int_{(\sigma)}$ is directed upwards along $\Re s = \sigma$. Smoothness and compact support give, in particular,

$$\widetilde{\Phi}(\sigma + it) \ll_{\phi,I} (1+|t|)^{-6} \quad (\sigma \in I),$$

for every fixed compact real interval I. The transform inserts the weight into the logarithmic-derivative series, and its decay controls the integrals when the contour is moved.

Lemma 3.5 (Mellin projection and signed families)**.** *The following Mellin representation holds unconditionally for $\sigma > 1/2$ and, under* GRH(N)*, also for $0 < \sigma \leqslant 1/9$:*

$$\mathcal{P}(N) = \frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} N^{s+z}\widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{P}_N(s,z)\,\mathrm{d}s\,\mathrm{d}z,$$

as in (2.8)*. The off-diagonal relations in* (3.2) *are $m = n + kN$, $n = m + kN$, and $n + m = kN$, with signs $+,+,-$. After insertion of the weight, only the sum family with $k = 1$ remains.*

Proof. For $\sigma > 1/2$, the absolutely convergent logarithmic-derivative series and the rapid decay of $\widetilde{\Phi}$ justify termwise Mellin inversion:

$$P_\chi = -\frac{1}{2\pi i}\int_{(\sigma)} N^s\widetilde{\Phi}(s)\frac{L'}{L}(1/2+s,\chi)\,\mathrm{d}s,$$

as in (2.31). The factor N^s comes from $(n/N)^{-s}$. Apply the same formula to $P_{\bar\chi}$, multiply, and average over X_N^-. The two minus signs cancel, and absolute convergence permits the product integrals and finite character sum to be interchanged. This proves the displayed Mellin representation on the initial contours. Under GRH, Proposition 3.2 excludes poles between any positive contour and this initial contour. Lemma 3.4 gives logarithmic growth on the intervening fixed strip, whereas $\widetilde{\Phi}$ decays faster than every power. The horizontal sides of the contour rectangles therefore tend to zero, and the vertical integrals converge absolutely. Moving the contours successively proves the stated continuation.

Separating equal from unequal indices in the first congruence of (3.2) gives the diagonal and the two positive difference families. The second congruence gives the negative sum family. On the support of the weight, $|n-m| \leqslant N/3$ and $2N/3 \leqslant n+m \leqslant 4N/3$. Thus both difference families and all sums with $k \geqslant 2$ have zero weighted contribution. This proves the lemma. □

The signed combination of all three families must remain in the analytic kernel. It supplies the cancellation used by the Euler model.

4. Euler: the Beta model and the analytic residual

We construct a continuous model for the off-diagonal part of the character kernel and subtract it, together with the exact diagonal, to obtain an analytic residual. The model is specified by the three arithmetic relations and their local densities. Its construction does not assume an asymptotic formula for prime pairs.

Throughout this section, $N \geqslant 4$ is even and $w = s + z$. The character kernel is

$$\mathscr{P}_N(s,z) = \frac{2}{\varphi(N)} \sum_{\substack{\chi \bmod N \\ \chi(-1)=-1}} \frac{L'}{L}(1/2+s,\chi)\frac{L'}{L}(1/2+z,\bar{\chi}),$$

as in (2.6), where all characters, including imprimitive ones, are extended by zero at nonunits. Its expansion (3.2) has the diagonal $n = m$ and the three remaining families

$$m = n + kN, \qquad n = m + kN, \qquad n + m = kN \qquad (k \geqslant 1),$$

with signs $+, +, -$, respectively. We keep the diagonal exactly and assign to each remaining family its continuous integral multiplied by the local density $\mathfrak{S}(kN)$. For any even h, this density is

$$\mathfrak{S}(h) = 2C_2 \prod_{\substack{p\mid h \\ p>2}} \frac{p-1}{p-2}, \qquad C_2 = \prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right),$$

as in (1.2). The factorizations below continue the resulting signed model to the same domain as the arithmetic kernel.

4.1. **The signed Beta combination.** Fix $h = kN$ with $k \geqslant 1$. For the relations $m = n + h$, $n = m + h$ and $n + m = h$, replace (n, m) by $(x, x + h)$, $(x + h, x)$ and $(x, h - x)$, respectively. Integrating the corresponding power weight $n^{-1/2-s}m^{-1/2-z}$ over $x > 0$ in the first two cases and $0 < x < h$ in the third gives the three integrals in (4.2). Scaling $x = ht$ produces the common factor h^{-s-z}. The signs $+, +, -$ from the character projection then give the Beta combination

$$K_-(s,z) = B(1/2-s, s+z) + B(1/2-z, s+z) - B(1/2-s, 1/2-z),$$

as in (2.9).

Euler's Gamma function is $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,\mathrm{d}t$ for $\Re s > 0$, with meromorphic continuation, and $B(s,z) = \int_0^1 t^{s-1}(1-t)^{z-1}\,\mathrm{d}t$ for $\Re s, \Re z > 0$. The change of variable $u = yt$ gives, for $y > 0$,

$$\int_0^y u^{s-1}(y-u)^{z-1}\,\mathrm{d}u = B(s,z)y^{s+z-1}, \quad B(s,z) = \frac{\Gamma(s)\Gamma(z)}{\Gamma(s+z)} \quad (\Re s, \Re z > 0). \tag{4.1}$$

Lemma 4.1 (The continuous signed model)**.** *For $h > 0$, $\Re s, \Re z < 1/2$ and $\Re(s+z) > 0$, put $w = s + z$. The following integrals converge absolutely:*

$$\begin{aligned}
\int_0^\infty x^{-1/2-s}(x+h)^{-1/2-z}\,\mathrm{d}x &= h^{-w}B(1/2-s,w),\\
\int_0^\infty x^{-1/2-z}(x+h)^{-1/2-s}\,\mathrm{d}x &= h^{-w}B(1/2-z,w),\\
\int_0^h x^{-1/2-s}(h-x)^{-1/2-z}\,\mathrm{d}x &= h^{-w}B(1/2-s,1/2-z).
\end{aligned} \tag{4.2}$$

Their signed sum, with signs $+,+,-$, is $h^{-s-z}K_-(s,z)$. For $h = kN$, the common local density for these three families is $\mathfrak{S}(h)$, so their model is $\mathfrak{S}(h)h^{-s-z}K_-(s,z)$.

Proof. For both sums and differences, the two entries exclude one residue class modulo p if $p \mid h$ and two otherwise. Relative to independent prime densities, the correction is

$$\frac{1-\nu_p(h)/p}{(1-1/p)^2}, \qquad \nu_p(h) = \begin{cases} 1, & p \mid h,\\ 2, & p \nmid h. \end{cases}$$

Since $h = kN$ is even, the product is

$$2C_2 \prod_{\substack{p\mid h\\ p>2}} \frac{p-1}{p-2} = \mathfrak{S}(h).$$

In particular, $N \mid h$ makes the factors at primes dividing N consistent with the coprimality restriction in the arithmetic kernel. The substitutions $x = ht/(1-t)$ and $x = ht$ prove the three identities in (4.2). For example,

$$\int_0^\infty x^{-1/2-s}(x+h)^{-1/2-z}\,\mathrm{d}x = h^{-s-z}\int_0^1 t^{-1/2-s}(1-t)^{s+z-1}\,\mathrm{d}t.$$

The conditions $\Re s, \Re z < 1/2$ control the finite endpoints, and $\Re(s+z) > 0$ controls infinity. In their common region, the signed sum is therefore $h^{-s-z}K_-(s,z)$, with K_- as defined at the start of this subsection. This proves the lemma. □

4.2. **Gamma factorization.** Reflection and duplication separate the signed Beta expression into a factor in $w = s + z$ and one Gamma quotient in each variable. Define the one-variable factor

$$\mathcal{X}_1(s) = \frac{\Gamma((3/2-s)/2)}{\Gamma((3/2+s)/2)}.$$

The following lemma gives a bound and a Gamma factorization for $K_-(s,z)$.

Lemma 4.2 (Gamma factorization)**.** *On $\Re s = \Re z = \sigma$, one has*

$$|K_-(s,z)| \leqslant 2B(1/2-\sigma,2\sigma) + B(1/2-\sigma,1/2-\sigma) \qquad (0 < \sigma \leqslant 1/9). \tag{4.3}$$

For this Gamma quotient $\mathcal{X}_1$, the factorization is

$$K_-(s,z) = 2^{1-w}\Gamma(w)\cos(\pi w/2)\mathcal{X}_1(s)\mathcal{X}_1(z), \qquad w = s+z. \tag{4.4}$$

The latter equality is an identity of meromorphic functions.

Proof. Taking absolute values in the definition of $K_-(s,z)$ and applying the triangle inequality proves (4.3). For the factorization, put $r = 1/2-s$, $t = 1/2-z$, so $r+t+w = 1$. Reflection gives

$$K_-(s,z) = \frac{\Gamma(r)\Gamma(t)\Gamma(w)}{\pi}(\sin\pi r + \sin\pi t - \sin\pi w),$$

whereas duplication and reflection give

$$\mathcal{X}_1(s) = \frac{2^{1-r}}{\sqrt{\pi}}\Gamma(r)\sin(\pi r/2).$$

Use the corresponding expression for z, t and the identity

$$\sin\pi r + \sin\pi t - \sin\pi w = 4\sin(\pi r/2)\sin(\pi t/2)\cos(\pi w/2).$$

This proves (4.4) initially away from poles and then as a meromorphic identity. This proves the lemma. □

4.3. **The density series and its bounds.** We now collect the densities at all scales $h = kN$ and record the exact diagonal. The corresponding Dirichlet series are

$$\mathscr{D}_N(w) = \sum_{\substack{n\geqslant 1\\(n,N)=1}} \frac{\Lambda^2(n)}{n^{1+w}} \quad (\Re w > 0),$$

$$\mathscr{S}_N(w) = \sum_{k\geqslant 1} \frac{\mathfrak{S}(kN)}{k^w} \qquad (\Re w > 1),$$

as in (2.11) and (2.10). The former records the exact arithmetic diagonal; the latter collects the local densities $\mathfrak{S}(kN)$. The density series is initially defined for $\Re w > 1$, whereas the three Beta integrals have $0 < \Re w < 1$ in their common domain. To use the density factor there, we first continue it by the Euler product, which we restate here:

$$\mathscr{S}_N(w) = \mathfrak{S}(N)\zeta(w)P_N(w),$$

where P_N is the Euler product

$$P_N(w) = \prod_{\substack{p>2\\p\nmid N}} \left(1 + \frac{p^{-w}}{p-2}\right),$$

as in (2.11). This Euler product is distinct from the character polynomial P_χ of the preceding section. Separating the zeta factor identifies the pole at $w = 1$; controlling the remaining product gives bounds uniform in the modulus.

Lemma 4.3. *The diagonal series $\mathscr{D}_N(w)$ converges absolutely for $\Re w > 0$. The density series $\mathscr{S}_N(w)$ has the Euler product displayed above and continues meromorphically to $\Re w > 0$, with a single simple pole at $w = 1$ of residue*

$$\operatorname*{Res}_{w=1} \mathscr{S}_N(w) = \frac{N}{\varphi(N)}. \tag{4.5}$$

For fixed $0 < \sigma_- < \sigma_+ < \infty$,

$$|\mathscr{S}_N(w)| \ll_{\sigma_-,\sigma_+} \frac{N}{\varphi(N)}(1+|\Im w|) \quad (\sigma_- \leqslant \Re w \leqslant \sigma_+,\ |\Im w| \geqslant 1). \tag{4.6}$$

Moreover, $\mathfrak{S}(N) > 1$ *and*

$$C_2 \frac{N}{\varphi(N)} \leqslant \mathfrak{S}(N) \leqslant \frac{N}{\varphi(N)} \ll \log\log N. \tag{4.7}$$

Proof. Absolute convergence of the diagonal follows from $\Lambda^2(n) \leqslant \log^2 n$. The function $\mathfrak{S}(kN)/\mathfrak{S}(N)$ is multiplicative in k. At p^j, $j \geqslant 1$, its value is 1 if $p \mid N$ and $(p-1)/(p-2)$ otherwise. Thus the local factor is $(1-p^{-w})^{-1}$ in the first case and $(1+p^{-w}/(p-2))/(1-p^{-w})$ in the second. This proves the Euler product for $\Re w > 1$.

For every $\delta > 0$, the product $P_N(w)$ is holomorphic and uniformly bounded in N on $\Re w \geqslant \delta$, since $\sum_{p>2} p^{-\delta}/(p-2) < \infty$. Omitting the primes dividing N only decreases this absolute majorant. At $w = 1$ all Euler factors are positive, so they do not cancel the pole of the zeta factor. Only the pole of $\zeta(w)$ at 1 remains. The identity

$$\left(1 - \frac{1}{(p-1)^2}\right)\left(1 + \frac{1}{p(p-2)}\right) = 1$$

gives (4.5) and

$$\mathfrak{S}(N) = \frac{N}{\varphi(N)} \prod_{\substack{p>2 \\ p\nmid N}} \left(1 - \frac{1}{(p-1)^2}\right).$$

The remaining product lies between C_2 and 1. Also $C_2 > \prod_{m=2}^{\infty}(1-m^{-2}) = 1/2$, so $\mathfrak{S}(N) \geqslant 2C_2 > 1$.

Euler summation yields

$$\zeta(w) = \frac{w}{w-1} - w\int_1^{\infty} \{x\} x^{-w-1}\,\mathrm{d}x \qquad (\Re w > 0,\ w \neq 1).$$

Together with the product bound this proves (4.6). Finally, split the prime divisors of N at $y = \log N$. Mertens' product estimate bounds the contribution of $p \leqslant y$ by $O(\log y)$. There are at most $\log N/\log y$ prime divisors above y, and each contributes $O(1/y)$ to the logarithm of the product. In particular,

$$\log \prod_{\substack{p\mid N \\ p>y}} (1-1/p)^{-1} \ll \frac{\log N}{y \log y} = \frac{1}{\log y}.$$

This proves the last bound in (4.7); the bounded initial range is absorbed into the constant. The use of Mertens' estimate is the standard one in [7]. This proves the lemma. □

4.4. **Continuation of the model.** We now combine the two factorizations. This removes the apparent pole at $s + z = 1$ and gives the domain for the residual operator. With the continued density factor, define the off-diagonal model by

$$\mathscr{K}_N(s,z) = N^{-w}\mathscr{S}_N(w)K_-(s,z), \qquad w = s+z,$$

as in (2.11). The equivalent form to be proved is

$$\mathscr{K}_N(s,z) = \mathfrak{S}(N)\left(\frac{\pi}{N}\right)^w P_N(w)\zeta(1-w)\mathscr{X}_1(s)\mathscr{X}_1(z),$$

as in (2.12). The second expression makes holomorphy beyond the original Beta integral domain visible factor by factor.

Proposition 4.4 (Continuation of the Beta model)**.** *The second expression for $\mathscr{K}_N$ above gives its continuation and is holomorphic on*

$$\Omega = \{(s,z) : 0 < \Re s < 3/2,\ 0 < \Re z < 3/2\}.$$

Proof. Insert (4.4) and the density Euler product into $\mathscr{K}_N(s,z) = N^{-w}\mathscr{S}_N(w)K_-(s,z)$. The zeta functional equation

$$\zeta(1-w) = 2(2\pi)^{-w}\Gamma(w)\cos(\pi w/2)\zeta(w)$$

gives the asserted functional form of the model. The density identity has already been continued to the initial Beta domain. Thus the substitution uses established meromorphic identities, rather than interchanging the density series with an integral outside its region of absolute convergence. On Ω, $P_N(s+z)$ converges locally uniformly and neither Gamma quotient has a pole: each numerator has argument with positive real part, and the reciprocal Gamma function is entire. The pole of $\zeta(1-s-z)$ would require $s+z=0$, outside Ω. Hence the model is holomorphic there. This proves the proposition. □

4.5. **Holomorphy and symmetry of the residual.** The residual is the exact difference between the character kernel and its diagonal and off-diagonal model:

$$\mathscr{E}_N(s,z) = \mathscr{P}_N(s,z) - \mathscr{D}_N(s+z) - \mathscr{K}_N(s,z).$$

Here $\mathscr{P}_N$ is the character kernel at the start of this section. Its exact diagonal is

$$\mathscr{D}_N(w) = \sum_{(n,N)=1} \Lambda^2(n) n^{-1-w},$$

as in (2.11), and $\mathscr{K}_N$ is the continued model. We now assume GRH(N): every nontrivial zero of every primitive Dirichlet L-function of conductor dividing N lies on $\Re s = 1/2$. Proposition 3.2 then gives holomorphy of $\mathscr{P}_N$ for $\Re s, \Re z > 0$; the other two terms have just been treated unconditionally. Their common domain is $\Omega = \{(s,z) : 0 < \Re s < 3/2,\ 0 < \Re z < 3/2\}$. As with the character kernel, $\mathscr{E}_N(s,z)$ is the form holomorphic in both variables, while $\mathscr{E}_N(s,\bar{z})$ is the Hermitian form used in the kernel of the integral operator T_N defined in (2.17).

Proposition 4.5 (The analytic residual)**.** *Under* GRH(N)*, $\mathscr{E}_N$ is holomorphic on Ω and satisfies*

$$\mathscr{E}_N(s,z) = \mathscr{E}_N(z,s), \qquad \overline{\mathscr{E}_N(s,z)} = \mathscr{E}_N(\bar{s},\bar{z}).$$

In particular, $\mathscr{E}_N(s,\bar{z})$ is Hermitian.

Proof. Subtract the diagonal and model from $\mathscr{P}_N$. Proposition 3.2 and Proposition 4.4, together with Lemma 4.3, prove holomorphy. The diagonal has real coefficients and depends only on $s+z$; the Beta model is symmetric in s, z and commutes with simultaneous conjugation. Equation (3.3) therefore gives both residual identities. The subtraction is exact, so these analytic conclusions require no estimate for the size or sign of the residual. This proves the proposition. □

The next section realizes this residual as a compact selfadjoint operator. To state the arithmetic identity that this construction will control, recall the fixed nonzero weight $\phi \in C_c^\infty(\mathbb{R})$, with $0 \leqslant \phi \leqslant 1$, support in $[1/3, 2/3]$, and $\phi(u) = \phi(1-u)$. Put

$$\Phi(u) = \sqrt{u}\,\phi(u), \qquad \widetilde{\Phi}(s) = \int_0^\infty \Phi(u) u^{s-1}\,\mathrm{d}u, \qquad c_\phi = \int_0^1 \phi^2(u)\,\mathrm{d}u,$$

as in (2.5), (2.7), and (1.4). For $0 < \sigma \leqslant 1/9$, define

$$r_\sigma(N) = \frac{1}{(2\pi i)^2} \int_{(\sigma)} \int_{(\sigma)} N^{s+z} \widetilde{\Phi}(s)\widetilde{\Phi}(z) \mathscr{E}_N(s,z)\,\mathrm{d}s\,\mathrm{d}z,$$

as in (2.34), where both contours are directed upwards. Section 7 will evaluate the model integral as $-c_\phi \mathfrak{S}(N)$ and the diagonal integral as $\mathcal{D}(N)$. Combined with the finite identity $\mathcal{G}(N) = \mathcal{D}(N) - \mathcal{P}(N)$, these evaluations give

$$\mathcal{G}(N) = c_\phi \mathfrak{S}(N) - r_\sigma(N),$$

as in (2.35), where $\mathcal{G}(N)$ is the weighted Goldbach correlation

$$\mathcal{G}(N) = \frac{1}{N} \sum_{\substack{n+m=N \\ (nm,N)=1}} \Lambda(n)\Lambda(m)\phi(n/N)\phi(m/N),$$

as in (1.3). Thus the signed discrepancy $r_\sigma(N)$ is independent of σ. The remaining task is to express it as a quadratic form and bound it by the norm of the residual operator.

5. Hilbert–Pólya: a compact selfadjoint realization

We realize the arithmetic residual as an integral operator on a space of holomorphic functions. The main analytic point is that its boundary integral produces the two traces of a single holomorphic function on the strip. Once this is established, the Hermitian symmetry and square integrability of the kernel give a compact selfadjoint operator. Point evaluation will then recover the residual from this operator.

Throughout, $N > e^{10}$ is even, and we assume GRH(N): every nontrivial zero of every primitive Dirichlet L-function whose conductor divides N has real part $1/2$. The construction concerns fixed N unless uniformity is stated. We use the strip and boundary conventions

$$\alpha_N = \frac{1}{\log N}, \qquad \beta = \frac{1}{8}, \qquad \sigma_* = \frac{1}{9}, \qquad \Delta_N = \beta - \alpha_N,$$
$$S_N = \{s : \alpha_N < \Re s < \beta\}, \qquad \mathcal{H}_N = H^2(S_N),$$
$$\Gamma_N = (\alpha_N + i\mathbb{R}) \sqcup (\beta + i\mathbb{R}), \qquad \langle h, g\rangle_{\mathcal{H}_N} = \int_{\Gamma_N} h(s)\overline{g(s)}\,|\mathrm{d}s|,$$

as in (2.14), (2.15), and (2.16). Here $H^2(S_N)$ is the strip Hardy space, whose definition and boundary traces are given below. The inner product is linear in its first variable. Both lines carry positive arc length $\mathrm{d}t$, with no contour orientation and no factor $1/(2\pi)$. The choice of endpoints places σ_* strictly inside S_N and gives $N^{2\alpha_N} = e^2$; this last identity will be used to bound the Mellin functional uniformly in N.

For reference, we collect the three arithmetic kernels entering the operator. Let $X_N^- = \{\chi \bmod N : \chi(-1) = -1\}$, including imprimitive characters and extending each character by zero at nonunits. The kernels are

$$\mathscr{P}_N(s,z) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} g_\chi(s) g_{\bar\chi}(z),$$

$$\mathscr{D}_N(w) = \sum_{\substack{n \geqslant 1 \\ (n,N)=1}} \frac{\Lambda^2(n)}{n^{1+w}},$$

$$\mathscr{K}_N(s,z) = \mathfrak{S}(N) \left(\frac{\pi}{N}\right)^{s+z} P_N(s+z)\zeta(1-s-z)\mathcal{X}_1(s)\mathcal{X}_1(z),$$

as in (2.6), (2.11), and (2.12), where g_χ is the logarithmic derivative

$$g_\chi(s) = \frac{L'}{L}(1/2+s,\chi),$$

as in (2.13). The kernel $\mathscr{K}_N$ is the continued Beta model. Its factors are

$$\mathfrak{S}(N) = 2C_2 \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2}, \qquad C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right),$$

$$P_N(w) = \prod_{\substack{p>2 \\ p \nmid N}} \left(1 + \frac{p^{-w}}{p-2}\right), \qquad \mathcal{X}_1(s) = \frac{\Gamma((3/2-s)/2)}{\Gamma((3/2+s)/2)}.$$

The singular-series and Euler-product definitions are as in (1.2) and (2.11), respectively. Here Λ is the von Mangoldt function and $\varphi(N)$ is Euler's totient. Under GRH(N), the preceding section shows that

$$\mathscr{E}_N(s,z) = \mathscr{P}_N(s,z) - \mathscr{D}_N(s+z) - \mathscr{K}_N(s,z)$$

is holomorphic on $0 < \Re s, \Re z < 3/2$. The integral operator to be justified is

$$(T_N h)(s) = \int_{\Gamma_N} \frac{\mathscr{E}_N(s,\bar z)}{(1+s)(1+\bar z)} h(z)\,|\mathrm{d}z|,$$

as in (2.17). The denominators provide decay on the infinite boundary lines; conjugation in the second variable supplies the Hermitian symmetry.

5.1. **The Hermitian kernel on the arithmetic strip.** We apply the holomorphy and symmetry of $\mathscr{E}_N$ on the strip S_N just defined. Its two holomorphic variables become the first variable and the conjugate of the second in the integral kernel: $\mathscr{E}_N(s,\bar z)$ is holomorphic in s and antiholomorphic in z. The Hermitian condition is $\mathscr{E}_N(s,\bar z) = \overline{\mathscr{E}_N(z,\bar s)}$.

The additional term *Fueter regular* refers here to a quaternionic first-order differential equation. Write $\omega = s + zj$, with $s, z \in \mathbb{C}$ and $j^2 = -1$, $ij = -ji$. In these coordinates the equation we use is $(\partial_{\bar s} + j\partial_z)F = 0$. For the present kernel, holomorphy in s and antiholomorphy in z make the two derivatives vanish separately.

Proposition 5.1 (The Hermitian arithmetic kernel)**.** *The weighted kernel of T_N is*

$$\frac{\mathscr{E}_N(s,\bar{z})}{(1+s)(1+\bar{z})},$$

as in (2.17)*. On the arithmetic strip, both this kernel and $\mathscr{E}_N(s,\bar{z})$ are Hermitian and holomorphic in s and $\bar{z}$. The weighted kernel extends to a neighborhood of the closed strip in each holomorphic coordinate. In the coordinates $\omega = s + zj$, the residual satisfies*

$$(\partial_{\bar{s}} + j\partial_z)\mathscr{E}_N(s,\bar{z}) = 0.$$

The fixed contour has distances $\sigma_* - \alpha_N > 1/90$ and $\beta - \sigma_* = 1/72$ from the two strip boundaries.

Proof. Proposition 4.5 gives holomorphy on $0 < \Re s, \Re z < 3/2$ and the identities

$$\mathscr{E}_N(s,z) = \mathscr{E}_N(z,s), \qquad \overline{\mathscr{E}_N(s,z)} = \mathscr{E}_N(\bar{s},\bar{z}).$$

They imply $\mathscr{E}_N(s,\bar{z}) = \overline{\mathscr{E}_N(z,\bar{s})}$, the Hermitian symmetry needed here. The denominator $(1+s)(1+\bar{z})$ has no zeros near the closed strip and preserves these properties. Holomorphy in s and $\bar{z}$ makes both derivatives in the displayed equation zero, proving the stated Fueter regularity. Finally, $\alpha_N < 1/10$, $\sigma_* = 1/9$ and $\beta = 1/8$ give the two distances. This proves the proposition. □

Here the arithmetic coordinates are the two complex Mellin variables s and z, viewed as the four real coordinates $(\Re s, \Im s, \Re z, \Im z)$ of $\omega = s + zj$. For fixed N, the Fueter equation for $\mathscr{E}_N(s,\bar{z})$ holds because its holomorphy in s and antiholomorphy in z make $\partial_{\bar{s}}\mathscr{E}_N(s,\bar{z})$ and $\partial_z\mathscr{E}_N(s,\bar{z})$ vanish separately. The spectral variable q in $\mathfrak{X}_N(q)$ is introduced independently of (s,z). The spectral argument below uses the selfadjoint operator T_N on a complex Hilbert space and its real eigenvalues; no positivity is asserted for the residual.

5.2. **Hardy boundary traces.** In the usual disk Hardy space H^2, one controls the L^2 means on concentric circles and obtains an L^2 value on the boundary circle. A strip has two boundary components, so the corresponding norm here is the sum of two squared L^2 norms. There is also a compatibility condition: the two boundary functions must come from the same holomorphic function in the interior. Thus the boundary realization occupies a subspace of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Precisely, for $S = \{s : \alpha < \Re s < \beta\}$, define $H^2(S)$ to consist of the holomorphic functions h on S for which

$$\|h\|^2_{H^2(S)} = \sup_{0<\varepsilon<(\beta-\alpha)/2} \int_{\mathbb{R}} \left(|h(\alpha+\varepsilon+it)|^2 + |h(\beta-\varepsilon+it)|^2\right) \mathrm{d}t < \infty. \tag{5.1}$$

The boundary values below are understood as L^2 limits:

$$h_\alpha = \lim_{x\downarrow\alpha} h(x + i\,\cdot), \qquad h_\beta = \lim_{x\uparrow\beta} h(x + i\,\cdot).$$

Their existence is part of the next lemma, so no pointwise boundary smoothness is required in the definition.

The Fourier–Laplace representation makes the compatibility of the two boundaries explicit. Set $w_{H,\alpha,\beta}(\tau) = e^{2\alpha\tau} + e^{2\beta\tau}$. For $\psi \in L^2(\mathbb{R}, w_{H,\alpha,\beta}(\tau)\,\mathrm{d}\tau)$, define

$$(\mathcal{U}_H\psi)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(\tau) e^{s\tau}\,\mathrm{d}\tau. \tag{5.2}$$

On the vertical line $s = x + it$, this is the inverse Fourier transform of $e^{x\tau}\psi(\tau)$. The same density ψ therefore describes both boundaries, with factors $e^{\alpha\tau}$ and $e^{\beta\tau}$. Plancherel accounts for the two terms in $w_{H,\alpha,\beta}$ and gives the boundary norm identity

$$\|h\|^2_{H^2(S)} = \|h_\alpha\|^2_{L^2(\mathbb{R})} + \|h_\beta\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |\psi(\tau)|^2 w_{H,\alpha,\beta}(\tau)\,\mathrm{d}\tau. \tag{5.3}$$

This fixes our normalization: each boundary contributes its ordinary $L^2(\mathbb{R}, \mathrm{d}t)$ norm, without an averaging factor. On the arithmetic strip, write $w_{H,N}(\tau) = w_{H,\alpha_N,\beta}(\tau)$. The general endpoints will also be used in Appendix A.

Lemma 5.2 (Fourier–Laplace representation)**.** *For a strip S with real endpoints $\alpha < \beta$, equip $H^2(S)$ with* (5.1)*. The map* (5.2) *is unitary from $L^2(\mathbb{R}, w_{H,\alpha,\beta}(\tau)\,\mathrm{d}\tau)$ onto this Hardy space. Every function has L^2 traces on both boundary lines, and* (5.3) *holds. The trace map identifies the Hardy space with a closed subspace of boundary L^2; on the arithmetic strip its inner product is*

$$\langle h, g\rangle_{\mathcal{H}_N} = \int_{\Gamma_N} h(s)\overline{g(s)}\,|\mathrm{d}s|,$$

as in (2.16)*.*

Proof. For an interior x, the quotient $e^{2x\tau}/w_{H,\alpha,\beta}(\tau)$ decays exponentially at both ends. Cauchy–Schwarz therefore justifies the integral in (5.2) and its derivatives locally uniformly. On a compact substrip, the same exponential domination remains integrable after multiplication by any fixed power of τ. This justifies differentiation under the integral. It defines a holomorphic function h, and Plancherel gives

$$\int_{\mathbb{R}} |h(x+it)|^2\,\mathrm{d}t = \int_{\mathbb{R}} |\psi(\tau)|^2 e^{2x\tau}\,\mathrm{d}\tau.$$

As x approaches an endpoint x_0, dominated convergence gives $e^{x\tau}\psi \to e^{x_0\tau}\psi$ in L^2; the squared difference is bounded by $4|\psi|^2 w_{H,\alpha,\beta}$. Plancherel then gives the boundary traces.

Convexity yields

$$e^{2(\alpha+\varepsilon)\tau} + e^{2(\beta-\varepsilon)\tau} \leqslant e^{2\alpha\tau} + e^{2\beta\tau} \qquad (0 < \varepsilon < (\beta-\alpha)/2).$$

Taking $\varepsilon \downarrow 0$ proves that the paired supremum equals the boundary norm, and hence that the map is isometric.

Conversely, for a Hardy function h, take the L^2 Fourier transform $\widehat{h}_x$ of $h(x + i\,\cdot)$ with the convention

$$\widehat{h}_x(\tau) = (2\pi)^{-1/2} \int h(x+it) e^{-it\tau}\,\mathrm{d}t.$$

The Cauchy–Riemann equation gives $\partial_x \widehat{h}_x = \tau \widehat{h}_x$ in distributions. Thus $e^{-x\tau}\widehat{h}_x(\tau)$ is independent of x on compact frequency intervals. These representations agree on overlaps

and define a measurable function ψ. Plancherel and Fatou, along a sequence of paired lines tending to the boundaries, give

$$\int_{\mathbb{R}} |\psi(\tau)|^2 w_{H,\alpha,\beta}(\tau)\,\mathrm{d}\tau \leqslant \|h\|^2_{H^2(S)}.$$

The forward construction therefore applies. Fourier inversion identifies its image with h almost everywhere, and holomorphy gives equality everywhere. This proves surjectivity. The boundary norm identity makes the trace map an isometry from a complete Hilbert space, so its image in boundary L^2 is closed. Polarization gives the stated boundary inner product. This proves the lemma. □

5.3. **The compact selfadjoint operator.** For an integral operator on an L^2 space, a square-integrable kernel gives a Hilbert–Schmidt operator: its squared Hilbert–Schmidt norm is the integral of the squared absolute value of the kernel. This is the integral-kernel form of $\sum_{j,k} |a_{jk}|^2$ for a matrix. Here the boundary has two components, so that integral has four contributions, according to the choices of the first and second boundary lines.

For fixed N, recall that $\mathcal{H}_N = H^2(S_N)$ consists of holomorphic functions on $S_N = \{s : \alpha_N < \Re s < \beta\}$ whose paired vertical L^2 means remain bounded as the two lines approach the boundary. By Lemma 5.2, we identify this Hardy space isometrically with its closed trace subspace in $L^2(\Gamma_N)$. To construct T_N on $\mathcal{H}_N$, we will show that the boundary integral operator $\widetilde{T}_N$ associated with the weighted residual maps $L^2(\Gamma_N)$ into this trace subspace. Hermitian symmetry will then imply that $\widetilde{T}_N$ vanishes on $\mathcal{H}_N^{\perp}$. Its restriction T_N to $\mathcal{H}_N$ therefore has the same Hilbert–Schmidt norm as $\widetilde{T}_N$, which gives the equality in the following proposition.

Proposition 5.3 (The compact selfadjoint realization)**.** *Consider the operator on $\mathcal{H}_N$ defined by*

$$(T_N h)(s) = \int_{\Gamma_N} \frac{\mathscr{E}_N(s,\bar{z})}{(1+s)(1+\bar{z})} h(z)\,|\mathrm{d}z|.$$

Its weighted residual kernel is Hermitian, holomorphic near the closed strip in each holomorphic coordinate, and belongs to $L^2(\Gamma_N \times \Gamma_N)$ on the boundary. The operator T_N is compact, selfadjoint, and Hilbert–Schmidt. Writing $\mu_j(N)$ for its nonzero eigenvalues, counted with multiplicity, one has

$$\sum_j \mu_j^2(N) = \|T_N\|^2_{\mathrm{HS}} = \int_{\Gamma_N}\int_{\Gamma_N} |\mathscr{E}_N(s,\bar{z})|^2 \frac{|\mathrm{d}s|}{|1+s|^2}\frac{|\mathrm{d}z|}{|1+\bar{z}|^2}. \tag{5.4}$$

Its nonzero eigenvalues are real and satisfy $\|T_N\| = \sup_j |\mu_j(N)|$, with the maximum attained unless $T_N = 0$.

Proof. Holomorphy and symmetry follow from Proposition 4.5, since the closed strip lies inside its domain. For integrability, constants may depend on N. For the arithmetic component, conjugation of the second variable gives the Gram identity

$$\mathscr{P}_N(s,\bar{z}) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} g_\chi(s)\overline{g_\chi(z)}, \qquad g_\chi(s) = \frac{L'}{L}(1/2+s,\chi),$$

as in (2.13). Lemma 3.4 bounds these logarithmic derivatives on the fixed strip by

$$|g_\chi(x+iv)| \ll_N \log^2(2+|v|) \qquad (\alpha_N \leqslant x \leqslant \beta).$$

The nonnegative coefficients of the diagonal series give $|\mathscr{D}_N(s+\bar{z})| \leqslant \mathscr{D}_N(2\alpha_N)$ on the closed strip. The continued model has the factorization

$$\mathscr{K}_N(s,\bar{z}) = \mathfrak{S}(N)\left(\frac{\pi}{N}\right)^{s+\bar{z}} P_N(s+\bar{z})\zeta(1-s-\bar{z})\mathcal{X}_1(s)\mathcal{X}_1(\bar{z}).$$

The Euler product P_N and the Gamma quotients $\mathcal{X}_1$ are bounded on the strip for fixed N, leaving the zeta factor to control. For $3/4 \leqslant \Re s \leqslant 1-2\alpha_N$, Euler summation at $X = 2+|\Im s|$ gives $|\zeta(s)| \ll_N (1+|\Im s|)^{1/4}$: the finite sum and integral term are $O_N(X^{1-\Re s})$, and the remainder is $O(|s|X^{-\Re s}/\Re s)$. Using $1+|v-t| \leqslant (1+|v|)(1+|t|)$ and absorbing logarithms into these powers, we obtain

$$\left|\frac{\mathscr{E}_N(x+iv, y-it)}{(1+x+iv)(1+y-it)}\right|^2 \ll_N (1+|v|)^{-3/2}(1+|t|)^{-3/2} \quad (\alpha_N \leqslant x, y \leqslant \beta). \tag{5.5}$$

This majorant is integrable on all four boundary products and uniform across the closed strip.

The boundary kernel defines a selfadjoint Hilbert–Schmidt operator $\widetilde{T}_N$ on $L^2(\Gamma_N)$. For h in that space, Cauchy–Schwarz and (5.5) give uniform vertical L^2 bounds for the integral

$$\int_{\Gamma_N} \frac{\mathscr{E}_N(s,\bar{z})}{(1+s)(1+\bar{z})} h(z)\,|\mathrm{d}z|.$$

Local domination and Morera's theorem give holomorphy. Dominated convergence of the squared kernel differences gives its L^2 boundary limits. Lemma 5.2 therefore places its range in the Hardy trace space.

Selfadjointness and the range inclusion imply that $\widetilde{T}_N$ annihilates the orthogonal complement of the trace space. Indeed, pairing the image of an orthogonal vector with any boundary L^2 vector gives zero: selfadjointness transfers the operator to the latter vector, whose image lies in the trace space. Hence on $L^2(\Gamma_N) = \mathcal{H}_N \oplus \mathcal{H}_N^\perp$ it is $T_N \oplus 0$. The Hilbert–Schmidt kernel identity and the compact selfadjoint spectral theorem now give (5.4) for T_N itself, as well as the reality of the eigenvalues and the asserted operator norm formula. This proves the proposition. □

5.4. **Point evaluation and the arithmetic residual.** An arbitrary L^2 function has no well-defined value at a specified point. In this Hardy space, holomorphy makes interior point evaluation a bounded functional. Its Riesz vector k_z satisfies $h(z) = \langle h, k_z\rangle$ and is called the *reproducing kernel* at z. For the disk with normalized boundary measure, it is $(1-s\bar{z})^{-1}$; our strip gives the cosecant formula below, with a constant determined by our boundary measure. Applying these evaluation vectors in both variables will recover the arithmetic kernel as a matrix element of T_N.

Proposition 5.4 (Point evaluation)**.** *For $z \in S_N$, define*

$$k_z(s) = \frac{1}{4\Delta_N}\csc\left(\frac{\pi(s+\bar{z}-2\alpha_N)}{2\Delta_N}\right). \tag{5.6}$$

Then $k_z \in \mathcal{H}_N$ and $h(s) = \langle h, k_s\rangle_{\mathcal{H}_N}$. Writing $d_N(s) = \min(\Re s - \alpha_N, \beta - \Re s)$, one has

$$\|k_s\|^2_{\mathcal{H}_N} = \frac{1}{4\Delta_N}\csc\left(\frac{\pi(\Re s - \alpha_N)}{\Delta_N}\right) \leqslant \frac{1}{8d_N(s)}. \tag{5.7}$$

Consequently,

$$|\mathscr{E}_N(s,\bar{z})| \leqslant \frac{\|T_N\|\,|1+s|\,|1+z|}{8\sqrt{d_N(s)d_N(z)}}. \tag{5.8}$$

On the fixed interior contour this gives

$$|\mathscr{E}_N(\sigma_* + iv, \sigma_* + it)| \ll \|T_N\|(1+|v|+|t|)^2. \tag{5.9}$$

Proof. By Lemma 5.2, the evaluation vector at z has coefficient $(2\pi)^{-1/2}e^{\bar{z}\tau}/w_{H,N}(\tau)$ in the weighted Fourier space. The conjugate of z in this coefficient is forced by the convention that the inner product is linear in its first variable. Its inverse transform is

$$k_z(s) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{(s+\bar{z})\tau}}{e^{2\alpha_N\tau} + e^{2\beta\tau}}\,\mathrm{d}\tau.$$

With $t = e^{2\Delta_N\tau}$ and $\lambda = (s + \bar{z} - 2\alpha_N)/(2\Delta_N)$, this becomes

$$\frac{1}{4\pi\Delta_N}\int_0^\infty \frac{t^{\lambda-1}}{1+t}\,\mathrm{d}t = \frac{B(\lambda, 1-\lambda)}{4\pi\Delta_N}, \qquad 0 < \Re\lambda < 1.$$

Euler's reflection identity proves (5.6). Evaluation at $s = z$ and $\sin(\pi t) \geqslant 2\min(t, 1-t)$, $0 \leqslant t \leqslant 1$, give (5.7).

Applying reproduction in both variables yields

$$\frac{\mathscr{E}_N(s,\bar{z})}{(1+s)(1+\bar{z})} = \langle T_N k_z, k_s\rangle_{\mathcal{H}_N}.$$

Cauchy–Schwarz proves (5.8). The distances of σ_* from both boundaries are at least $1/90$, giving (5.9) after conjugating the second argument. This proves the proposition. □

6. Riemann: spectral zeros and operator norm bounds

The residual operator converts the Goldbach error into a spectral question: how large can its eigenvalues be? We encode their magnitudes as zeros of a product. Its quadratic factors will make the relation between zero location and operator norm explicit. We then establish the norm bound available under GRH and examine the scale of the left Hardy boundary. The resulting spectral estimates will be used in the next section to prove the main theorems.

Throughout, $N > e^{10}$ is even and satisfies GRH(N). We recall the Hardy space and operator

$$\mathcal{H}_N = H^2(S_N), \qquad S_N = \{s : \alpha_N < \Re s < \beta\}, \qquad \alpha_N = \frac{1}{\log N}, \quad \beta = \frac{1}{8},$$

$$(T_N h)(s) = \int_{\Gamma_N} \frac{\mathscr{E}_N(s,\bar{z})}{(1+s)(1+\bar{z})} h(z)\,|\mathrm{d}z|,$$

as in (2.14), (2.15), and (2.17). Here $\Gamma_N = (\alpha_N + i\mathbb{R}) \sqcup (\beta + i\mathbb{R})$, both lines carrying positive arc length. The elements of $\mathcal{H}_N$ are holomorphic functions whose paired interior L^2 means remain bounded. Their traces on the two boundaries give the norm

$$\|h\|_{\mathcal{H}_N}^2 = \int_{\mathbb{R}} |h(\alpha_N + it)|^2 \, \mathrm{d}t + \int_{\mathbb{R}} |h(\beta + it)|^2 \, \mathrm{d}t.$$

The two traces must come from the same holomorphic function; they cannot be chosen independently. The residual in the kernel is

$$\mathscr{E}_N(s,z) = \mathscr{P}_N(s,z) - \mathscr{D}_N(s+z) - \mathscr{K}_N(s,z),$$

where $\mathscr{P}_N$ is the odd-character correlation, $\mathscr{D}_N$ is its exact diagonal, and $\mathscr{K}_N$ is the continued Beta model. Their explicit formulas are recalled when we estimate the three components below. Proposition 5.3 makes T_N compact, selfadjoint, and Hilbert–Schmidt. Its nonzero eigenvalues $\mu_j(N)$, counted with multiplicity, are real, with

$$\sum_j \mu_j^2(N) < \infty, \qquad \|T_N\| = \sup_j |\mu_j(N)|.$$

The supremum is zero when $T_N = 0$. The Goldbach singular series is

$$\mathfrak{S}(N) = 2C_2 \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2}, \qquad C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right),$$

as in (1.2). We use the scale

$$\delta_N = \sqrt{\frac{\mathfrak{S}(N)}{N}} (\log N)^{3/2}, \qquad \delta_N^2 = \frac{\mathfrak{S}(N)(\log N)^3}{N},$$

as in (1.5). We encode the size of these eigenvalues relative to δ_N; the Dirichlet zeros enter through GRH and are a separate set of data.

6.1. **Convergence of the spectral product.** Define the spectral function by

$$\mathfrak{X}_N(q) = \prod_j \left(1 - \frac{4\mu_j^2(N)}{\delta_N^2} q(1-q)\right), \qquad q \in \mathbb{H},$$

as in (1.6). The empty product is equal to one. Here $\mathbb{H}$ denotes the quaternions. For a unit purely imaginary quaternion I, write $\mathbb{C}_I = \{x + yI : x, y \in \mathbb{R}\}$. All factors at a fixed $q \in \mathbb{C}_I$ lie in this commutative complex plane.

The term *intrinsic entire slice regular* has a concrete meaning here: the same real-coefficient power series defines an entire function on every complex slice $\mathbb{C}_I$. It preserves each slice and respects conjugation. This regularity concerns the spectral variable q; the Fueter regularity of $\mathscr{E}_N(s, \bar{z})$ concerns the separate arithmetic coordinates.

The ordinary Fredholm determinant extends the finite-dimensional formula

$$\det(\mathrm{Id} - tA) = \prod_j (1 - ta_j)$$

to a trace-class operator A, with eigenvalues a_j. For positive compact A, trace class means $\sum_j a_j < \infty$, which ensures convergence of that product. Here T_N is known to be

Hilbert–Schmidt, so its squared eigenvalues are summable. Consequently we use $A = T_N^2$, whose eigenvalues are $a_j = \mu_j(N)^2$, and obtain an ordinary Fredholm determinant.

Proposition 6.1 (Convergence and determinant realization)**.** *The product*

$$\mathfrak{X}_N(q) = \prod_j \left(1 - \frac{4\mu_j^2(N)}{\delta_N^2} q(1-q)\right), \qquad q \in \mathbb{H},$$

converges locally uniformly on $\mathbb{H}$, independently of the enumeration of the eigenvalues, and defines an intrinsic entire slice-regular function $\mathfrak{X}_N$. The operator T_N^2 is positive and trace class, with

$$\operatorname{tr}(T_N^2) = \sum_j \mu_j^2(N) = \|T_N\|_{\mathrm{HS}}^2.$$

On the usual complex slice,

$$\mathfrak{X}_N(q) = \det\left(\mathrm{Id}_{\mathcal{H}_N} - \frac{4}{\delta_N^2} q(1-q) T_N^2\right).$$

Proof. By Proposition 5.3, $\sum_j \mu_j^2(N) < \infty$. On a compact subset of $\mathbb{H}$, each factor differs from 1 by at most a fixed multiple of $\mu_j^2(N)$. Thus the product converges locally uniformly and is independent of the enumeration of the spectrum. Within each complex slice its factors commute and are holomorphic. Their real coefficients give the same real Taylor series on every slice, so the limit is intrinsic and slice regular.

The spectral theorem gives

$$\operatorname{tr}(T_N^2) = \sum_j \mu_j^2(N) = \|T_N\|_{\mathrm{HS}}^2 < \infty.$$

Hence T_N^2 is positive and trace class, and its Fredholm determinant is exactly the product $\mathfrak{X}_N(q)$ on the usual complex slice. The operator and its determinant are therefore formed on the complex Hilbert space $\mathcal{H}_N$; the real coefficients then extend the scalar function to the other slices. This proves the proposition. □

6.2. **Symmetries and zeros.** The quadratic factors determine both the symmetries and all zeros. On a single complex slice the nonreal roots occur in conjugate pairs. Allowing the imaginary unit I to vary turns such a pair into a sphere with fixed real part and fixed imaginary length. The exceptional roots are real pairs, and only these affect the norm criterion.

Proposition 6.2 (Symmetries and normalization)**.** *The normalization and symmetries are*

$$\mathfrak{X}_N(0) = \mathfrak{X}_N(1) = 1, \tag{6.1}$$

$$\mathfrak{X}_N(1-q) = \mathfrak{X}_N(q), \tag{6.2}$$

$$\mathfrak{X}_N(\bar{q}) = \overline{\mathfrak{X}_N(q)}. \tag{6.3}$$

Combining reflection and conjugation gives

$$\mathfrak{X}_N(1-\bar{q}) = \overline{\mathfrak{X}_N(q)} \qquad (q \in \mathbb{H}). \tag{6.4}$$

The map $q \mapsto 1-q$ is central inversion about $1/2$, whereas $q \mapsto 1-\bar{q}$ reflects across $\Re q = 1/2$. For the centered variable $\xi \in \mathbb{H}$,

$$\mathfrak{X}_N(1/2+\xi) = \prod_j \left(1 - \frac{\mu_j^2(N)}{\delta_N^2} + \frac{4\mu_j^2(N)}{\delta_N^2}\xi^2\right). \tag{6.5}$$

This is even in ξ. On the critical hyperplane, for real t and any unit purely imaginary quaternion I,

$$\mathfrak{X}_N(1/2+It) = \prod_j \left(1 - \frac{\mu_j^2(N)}{\delta_N^2}(1+4t^2)\right) \in \mathbb{R}. \tag{6.6}$$

The value is independent of I. Moreover, $\mathfrak{X}_N(x) \geqslant 1$ for real $x \notin (0,1)$.

Proof. The invariance of $q(1-q)$ under $q \mapsto 1-q$ proves (6.2). Real coefficients give (6.3), and combining the two gives (6.4). Evaluation at 0 and 1 proves (6.1). Finally, $(1/2+\xi)(1/2-\xi) = 1/4-\xi^2$ gives the centered product (6.5); taking $\xi = It$ gives (6.6). All identities pass from finite products to the limit by local uniform convergence. For real $x \notin (0,1)$, each factor is at least one, giving the asserted lower bound. This proves the proposition. □

Proposition 6.3 (Complete zero classification)**.** *For each nonzero eigenvalue $\mu_j(N)$, put $d_j = \delta_N^2/\mu_j^2(N)$. The zeros of $\mathfrak{X}_N$ are exactly the zeros contributed by these factors, namely*

$$\begin{cases} \Re q = \dfrac{1}{2}, \quad |\Im q| = \dfrac{1}{2}\sqrt{d_j - 1}, & d_j > 1, \\ q = \frac{1}{2} \quad \text{(a double zero of the factor)}, & d_j = 1, \\ q = \dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{1-d_j}, & 0 < d_j < 1. \end{cases}$$

Thus every zero off the critical hyperplane is real, there are only finitely many exceptional real pairs, and every real zero lies in $(0,1)$. Each nonreal zero sphere is invariant under conjugation by unit quaternions. On each complex slice, reflection and conjugation preserve multiplicities, and the order of a zero at $1/2$ is even.

Proof. For the ratio $d_j = \delta_N^2/\mu_j^2(N)$ just defined, a factor vanishes when $q(1-q) = d_j/4$. Writing $q = x + yI$ gives

$$q(1-q) = x(1-x) + y^2 + (1-2x)yI.$$

If $y \neq 0$, this forces $x = 1/2$ and $y^2 = (d_j - 1)/4$. The equation is independent of I, which gives the stated two-sphere. For real q, the ordinary quadratic formula gives the exceptional pair when $d_j < 1$ and the double central root when $d_j = 1$. For $x \notin (0,1)$, every factor is at least 1, so there are no other real zeros.

There are no zeros arising solely from the infinite product. Indeed, at a point where all factors $1-u_j$ are nonzero, choose J such that $|u_j| \leqslant 1/2$ for $j \geqslant J$. Then

$$\prod_{j \geqslant J} |1-u_j| \geqslant \prod_{j \geqslant J} (1-|u_j|) > 0,$$

because $\sum_j |u_j| < \infty$ and $\log(1-|u_j|) \geqslant -2|u_j|$. The finite preceding product is nonzero. Compactness also allows only finitely many eigenvalues with $|\mu_j| > \delta_N$, so there are

only finitely many exceptional pairs. On each complex slice, conjugation and reflection preserve multiplicity; at $1/2$ the order is even by the centered product. Conjugation by a unit quaternion preserves the real part and the length of the imaginary part, so it preserves each nonreal zero sphere. This proves the proposition. □

6.3. **From a zero-free region to a norm bound.** The zero classification turns the exclusion condition into an exact bound for the largest eigenvalue modulus. The excluded half-space is open, so equality is allowed in the norm bound. Here $\mathrm{R}(N)$ denotes the assertion that every zero of $\mathfrak{X}_N$ has real part $1/2$.

Theorem 6.1. *Under the standing assumptions,*

$$\mathrm{R}(N) \quad \Longleftrightarrow \quad \|T_N\| \leqslant \delta_N.$$

For $0 < \eta < 1/2$,

$$\mathfrak{X}_N(q) \neq 0 \quad (\Re q > 1 - \eta) \quad \Longleftrightarrow \quad \|T_N\| \leqslant \frac{\delta_N}{2\sqrt{\eta(1-\eta)}}.$$

In particular, if $\|T_N\| \leqslant M_N$ with $M_N > \delta_N$, the width is

$$\eta(M_N) = \frac{1 - \sqrt{1 - \delta_N^2/M_N^2}}{2}.$$

The corresponding half-space is zero-free, with $\eta(M_N) \asymp \delta_N^2/M_N^2$ when $\eta(M_N) \leqslant 1/4$.

Proof. A factor has zeros off the critical hyperplane exactly when $|\mu_j| > \delta_N$. Taking the supremum of the eigenvalue magnitudes proves the first equivalence, including the zero operator. The preceding product argument shows that these are all possible zeros. It is therefore enough to test the spectral factors individually; no separate condition on the limiting product is needed.

Only the larger root of an exceptional pair can enter $\Re q > 1 - \eta$. Its exclusion is equivalent to

$$\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{\delta_N^2}{\mu_j^2}} \leqslant 1 - \eta, \quad \text{or} \quad |\mu_j| \leqslant \frac{\delta_N}{2\sqrt{\eta(1-\eta)}}.$$

Squaring is reversible because $1 - 2\eta > 0$. Nonexceptional eigenvalues already satisfy the latter bound, which proves the second equivalence. Finally, suppose $\|T_N\| \leqslant M_N$ with $M_N > \delta_N$. Choosing $\eta \in (0, 1/2)$ to satisfy $2\sqrt{\eta(1-\eta)} = \delta_N/M_N$ gives the stated width $\eta(M_N)$, and the second equivalence ensures that the corresponding half-space is zero-free. Rationalizing yields

$$\eta(M_N) = \frac{\delta_N^2/M_N^2}{2\left(1 + \sqrt{1 - \delta_N^2/M_N^2}\right)}.$$

The denominator lies between 2 and 4, so $\eta(M_N) \asymp \delta_N^2/M_N^2$. This proves the theorem. □

We use this conversion to compare the proved GRH estimate with the stronger bounds required by S and R.

6.4. **GRH proves a baseline zero-free region.** We estimate the odd-character correlation, the diagonal, and the Beta model separately. An average of squared logarithmic derivatives controls the first component; positivity controls the second; and the continued model formula controls the third. These bounds give a zero-free region without requiring cancellation between the components of the residual.

Theorem 6.2 (The baseline bound under GRH)**.** *For every sufficiently large even N satisfying* GRH(N)*, one has*

$$\|T_N\| \leqslant \|T_N\|_{\mathrm{HS}} \ll (\log N)^2. \tag{6.7}$$

Consequently the spectral function has the zero-free region

$$\mathfrak{X}_N(q) \neq 0 \quad (\Re q > 1 - \eta_N^{\mathrm{GRH}}), \qquad \eta_N^{\mathrm{GRH}} \asymp \frac{\mathfrak{S}(N)}{N \log N}. \tag{6.8}$$

The constants and the required lower bound on N are absolute.

For this proof, retain $N > e^{10}$ and take the auxiliary reference line $\sigma_{\mathrm{ref}} = 2/\log N < 1/5$. This line may lie to the right of the Hardy boundary $\beta = 1/8$; the auxiliary Poisson comparison below is therefore established for $0 < x \leqslant 1/5$. The Hardy strip keeps its boundaries α_N and β and its fixed interior line $\sigma_* = 1/9$. The required estimate must be uniform as the left boundary approaches the critical line; compactness for each fixed N is not enough.

6.4.1. *An averaged bound for logarithmic derivatives.* The correlation component requires a mean-square bound that remains uniform as the left Hardy boundary approaches the critical line. We first prove such a bound on a reference line by smoothing a short Dirichlet polynomial, then transfer it to the required lines using the positive zero sum and Poisson kernels. Here X_N^- consists of all odd characters modulo N, including imprimitive ones, and

$$g_\chi(s) = \frac{L'}{L}(1/2 + s, \chi).$$

For each χ, write χ^* for its primitive inducing character and $d_\chi \mid N$ for the conductor. Under GRH its nontrivial zeros are $1/2 + i\gamma_\chi$, counted with multiplicity.

Lemma 6.4 (Averaged logarithmic derivatives)**.** *For $N > e^{10}$ and $0 < \sigma \leqslant \beta$,*

$$\frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} \int_{\mathbb{R}} \frac{|g_\chi(\sigma + it)|^2}{1 + t^2}\, \mathrm{d}t \ll (\log N)^2 + \sigma^{-2}. \tag{6.9}$$

The implied constant is absolute.

Proof. The reference line. Fix $\chi \in X_N^-$, and let χ^* denote the primitive Dirichlet character inducing χ, of conductor $d_\chi \mid N$. Write $1/2 + i\gamma_\chi$ for the nontrivial zeros of $L(s, \chi^*)$, and define

$$Q_\chi(x, t) = \sum_{\gamma_\chi} \frac{x}{x^2 + (t - \gamma_\chi)^2}, \qquad x > 0, \quad t \in \mathbb{R}.$$

The sum runs over all such ordinates γ_χ, with each zero counted according to its multiplicity. We first establish the comparison

$$|Q_\chi(x, t) - \Re g_\chi(x + it)| \ll \log\big(N(|t| + 2)\big) \qquad (0 < x \leqslant 1/5) \tag{6.10}$$

and the reference-line estimate

$$\frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} \int_{\mathbb{R}} \frac{|g_\chi(\sigma_{\mathrm{ref}} + it)|^2}{1+t^2}\, \mathrm{d}t \ll (\log N)^2. \tag{6.11}$$

Compare the logarithmic-derivative formulas from the Hadamard products at $u = 1/2 + x + it$ and $u_0 = 2 + it$. For the completed primitive function $\mathcal{L}(u, \chi^*) = (d_\chi/\pi)^{(u+1)/2}\Gamma((u+1)/2)L(u, \chi^*)$, the identity in Lemma 3.3 reads

$$\frac{\mathcal{L}'}{\mathcal{L}}(u, \chi^*) - \frac{\mathcal{L}'}{\mathcal{L}}(u_0, \chi^*) = \sum_\rho \left(\frac{1}{u-\rho} - \frac{1}{u_0-\rho} \right).$$

Taking real parts gives $Q_\chi(x,t) - Q_\chi(3/2,t)$ on the right. The conductor terms cancel, leaving the logarithmic derivatives of L and a Gamma difference on the left. The reference sum satisfies $Q_\chi(3/2,t) \ll \log\bigl(d_\chi(|t|+2)\bigr)$: group its terms according to $m \leqslant |t - \gamma_\chi| < m+1$ and use the unit-interval bound in Lemma 3.3,

$$\#\{\rho : v \leqslant \Im\rho < v+1\} \ll \log\bigl(d_\chi(|v|+2)\bigr),$$

with the summable factor $(1+m^2)^{-1}$. The logarithmic derivative at $2+it$ is uniformly bounded by its Dirichlet series. The Gamma difference is uniformly bounded for $0 < x \leqslant 1/5$, and the imprimitive Euler corrections contribute $O(\log N)$, as in Lemma 3.4. These observations prove (6.10) without any zero-spacing assumption.

For the reference-line estimate, set $X = N/2$ and $x = \sqrt{X}$ and use the smoothed Dirichlet polynomial

$$P_{\chi,X}(u) = \sum_{n \leqslant X} \Lambda(n)\chi(n)n^{-u}w_X(n).$$

Its cutoff weight is

$$w_X(n) = \begin{cases} 1, & n \leqslant x, \\ \log(X/n)/\log x, & x < n \leqslant X, \\ 0, & n > X. \end{cases}$$

For $u = 1/2 + \sigma + it$, the Perron kernel $(X^v - x^v)/(v^2 \log x)$ has a simple pole of residue 1 at zero. In particular, there is no derivative term at that point. The residue of $-L'/L(u+v,\chi)$ at a zero $v = \rho - u$ of multiplicity m is $-m$. Shifting the contour gives

$$-\frac{L'}{L}(u,\chi) - P_{\chi,X}(u) = \frac{1}{\log x} \sum_\rho \frac{X^{\rho-u} - x^{\rho-u}}{(\rho-u)^2}. \tag{6.12}$$

The sum includes, with multiplicity, the primitive nontrivial and trivial zeros and the zeros of the additional Euler factors. For primes $p \mid N$ with $p \nmid d_\chi$, these zeros have real part zero and ordinates $(\arg\chi^*(p) + 2\pi k)/\log p$, $k \in \mathbb{Z}$. Since χ is odd, there is no pole at $u = 1$. For the contour shift, choose left vertical lines on which $\Re(u+v)$ is a negative even integer. The local zero count permits horizontal ordinates of size T at distance $\gg 1/\log(N(T+2))$ from the zero ordinates, including those of the Euler factors. On each fixed horizontal segment, the logarithmic derivative is then $O(\log^2(N(T+2)))$, so its integral against the Perron kernel tends to zero. First let the height tend to infinity, then move the left line to minus infinity. The functional equation bounds the logarithmic derivative on these lines; $x > 1$ supplies exponential decay on the left. The quadratic

denominator and the local zero count give absolute convergence of the nontrivial and Euler zero sums. The trivial-zero sum has additional geometric decay.

The nontrivial-zero contribution is bounded by $\epsilon_\sigma Q_\chi(\sigma,t)$, where

$$\epsilon_\sigma = \frac{x^{-\sigma}+X^{-\sigma}}{\sigma\log x}.$$

The remaining zeros contribute $O((\log N/\log x)x^{-1/2-\sigma})$, uniformly in t. Here the Euler zeros are the zeros of the additional factors $1-\chi^*(p)p^{-u}$, where p ranges over the prime divisors of N that do not divide d_χ. Since $|\chi^*(p)|=1$, the zeros of each such factor are

$$\rho_{p,k} = i\left(t_p + \frac{2\pi k}{\log p}\right), \qquad t_p = \frac{\arg\chi^*(p)}{\log p}, \quad k\in\mathbb{Z}.$$

Thus, for each fixed prime p, their ordinates have spacing $2\pi/\log p$, which gives

$$\sum_{k\in\mathbb{Z}} \frac{1}{(1/2+\sigma)^2+(t-t_p-2\pi k/\log p)^2} \ll 1+\log p.$$

Since $\sum_{p\mid N}(1+\log p)\ll \log N$, summing over these primes gives the stated error bound. The primitive trivial zeros give a smaller bound. At the reference line, put

$$a_N = \sigma_{\mathrm{ref}}\log x = 1-\frac{\log 2}{\log N} > \frac{9}{10}.$$

Thus

$$\epsilon_{\sigma_{\mathrm{ref}}} = \frac{e^{-a_N}+e^{-2a_N}}{a_N} \leqslant \frac{10}{9}\left(e^{-9/10}+e^{-9/5}\right) < \frac{2}{3}.$$

The comparison (6.10) therefore allows the logarithmic derivative on the right of (6.12) to be absorbed into the left side. Explicitly, the comparison $Q_\chi(\sigma_{\mathrm{ref}},t)\leqslant |g_\chi(\sigma_{\mathrm{ref}}+it)|+C\log(N(|t|+2))$ gives

$$(1-\epsilon_{\sigma_{\mathrm{ref}}})|g_\chi(\sigma_{\mathrm{ref}}+it)| \leqslant |P_{\chi,X}(1/2+\sigma_{\mathrm{ref}}+it)|+C\log(N(|t|+2)).$$

The bound $1-\epsilon_{\sigma_{\mathrm{ref}}}>1/3$ is uniform in N, the character, and the height. Hence

$$|g_\chi(\sigma_{\mathrm{ref}}+it)| \ll |P_{\chi,X}(1/2+\sigma_{\mathrm{ref}}+it)|+\log\big(N(|t|+2)\big).$$

Odd-character orthogonality leaves exactly the diagonal for this polynomial: $n,m\leqslant N/2$, and $w_X(N/2)=0$ removes the possible endpoint term. Thus averaging removes every oscillatory cross term before the integration in height. The resulting estimate does not require a bound for an individual character polynomial. Hence

$$\frac{2}{\varphi(N)}\sum_{\chi\in X_N^-}|P_{\chi,X}(1/2+\sigma+it)|^2 = \sum_{(n,N)=1}\frac{\Lambda^2(n)w_X^2(n)}{n^{1+2\sigma}} \ll \sigma^{-2}.$$

For the last step, Chebyshev's bound gives $B(u):=\sum_{n\leqslant u}\Lambda^2(n)\ll u\log(2u)$, so partial summation yields

$$\sum_{n\geqslant 1}\frac{\Lambda^2(n)}{n^{1+2\sigma}} = (1+2\sigma)\int_1^\infty \frac{B(u)}{u^{2+2\sigma}}\,\mathrm{d}u \ll \int_1^\infty \frac{\log(2u)}{u^{1+2\sigma}}\,\mathrm{d}u \ll \sigma^{-2}.$$

Average the square of the preceding pointwise bound and integrate against $(1+t^2)^{-1}$. The diagonal estimate then shows that the left side of (6.11) is $O((\log N)^2)$.

Transfer to the Hardy boundaries. For each character, write

$$\mathscr{E}_\chi(\sigma) = \int_{\mathbb{R}} \frac{|g_\chi(\sigma+it)|^2}{1+t^2}\,\mathrm{d}t. \tag{6.13}$$

For $0 < \sigma \leqslant \beta$ and any $0 < \sigma_{\mathrm{ref}} \leqslant 1/5$, we claim that

$$\mathscr{E}_\chi(\sigma) \ll \begin{cases} (\sigma_{\mathrm{ref}}/\sigma)^2\big(\mathscr{E}_\chi(\sigma_{\mathrm{ref}}) + (\log N)^2\big), & \sigma \leqslant \sigma_{\mathrm{ref}}, \\ \mathscr{E}_\chi(\sigma_{\mathrm{ref}}), & \sigma_{\mathrm{ref}} \leqslant \sigma. \end{cases} \tag{6.14}$$

For any $0 < \sigma \leqslant \beta$, we first relate the weighted norm of the full logarithmic derivative to that of its real part:

$$\int_{\mathbb{R}} \frac{|g_\chi(\sigma+it)|^2}{1+t^2}\,\mathrm{d}t \leqslant 2\int_{\mathbb{R}} \frac{|\Re g_\chi(\sigma+it)|^2}{1+t^2}\,\mathrm{d}t + \pi|g_\chi(\sigma+1)|^2. \tag{6.15}$$

To apply Parseval with the present weight, use the disk function

$$G(\zeta) = g_\chi\!\left(\sigma + \frac{1+\zeta}{1-\zeta}\right).$$

Here the disk Hardy space H^2 uses one circular boundary: bounded radial square means are equivalent to square summability of the Taylor coefficients. The Cayley map converts its normalized boundary measure into the weight used in the displayed inequality. The GRH logarithmic-derivative bound and the absolutely convergent Dirichlet series farther to the right give

$$|G(\zeta)| \ll_{N,\sigma} \log^2(2 + |1-\zeta|^{-1}).$$

This bound gives uniformly bounded radial square means, so $G \in H^2$. With normalized circle measure, if $G(\zeta) = \sum a_n \zeta^n$, then

$$\|G\|_2^2 = 2\|\Re G\|_2^2 - \Re(a_0^2).$$

The Cayley map sends that measure to $\mathrm{d}t/(\pi(1+t^2))$. This change of variables and $-\Re(a_0^2) \leqslant |a_0|^2$ give (6.15); its last term is uniformly $O(1)$ by the Dirichlet series at $3/2 + \sigma$.

For $0 < \sigma \leqslant \min(\sigma_{\mathrm{ref}}, \beta)$, positivity gives $Q_\chi(\sigma,t) \leqslant (\sigma_{\mathrm{ref}}/\sigma)Q_\chi(\sigma_{\mathrm{ref}},t)$. Writing $K = \sigma_{\mathrm{ref}}/\sigma \geqslant 1$, (6.10) implies

$$|\Re g_\chi(\sigma+it)| \leqslant K|g_\chi(\sigma_{\mathrm{ref}}+it)| + C(K+1)\log\big(N(|t|+2)\big).$$

Square this inequality and integrate against $(1+t^2)^{-1}$. Since $\int_{\mathbb{R}} \log^2(|t|+2)(1+t^2)^{-1}\,\mathrm{d}t < \infty$, the integrated logarithmic error is $O((\log N)^2)$. Applying (6.15) therefore gives

$$\mathscr{E}_\chi(\sigma) \ll K^2\big(\mathscr{E}_\chi(\sigma_{\mathrm{ref}}) + (\log N)^2\big).$$

The bounded center term is absorbed since $K \geqslant 1$ and $N \geqslant 4$. This proves the first case of (6.14). For $\sigma \geqslant \sigma_{\mathrm{ref}}$, the disk H^2 estimate bounds the squared modulus by the Poisson integral of its boundary squared modulus. This is the harmonic majorant used here. Under the same Cayley change of variable it becomes

$$|g_\chi(\sigma+it)|^2 \leqslant (P_\ell * |g_\chi(\sigma_{\mathrm{ref}}+i\,\cdot)|^2)(t), \qquad P_\ell(t) = \frac{\ell}{\pi(\ell^2+t^2)}, \quad \ell = \sigma - \sigma_{\mathrm{ref}}.$$

Here $*$ denotes convolution in t; for $\ell = 0$ the formula is understood as the identity. The disk H^2 boundary representation has no additional point mass at the point corresponding

to infinity. Multiply the resulting majorant by $\pi P_1(t)$ and integrate. By nonnegativity, Tonelli's theorem permits interchanging the two integrals. Then $P_1 * P_\ell = P_{1+\ell}$ and $P_{1+\ell} \leqslant (1+\ell)P_1$ give $\mathscr{E}_\chi(\sigma) \leqslant (1+\ell)\mathscr{E}_\chi(\sigma_{\mathrm{ref}})$. The factor is bounded since $0 \leqslant \ell \leqslant \beta$. This proves the second case of (6.14).

The character average. Set $\sigma_{\mathrm{ref}} = 2/\log N$. Equation (6.11) bounds the weighted square integrals of the logarithmic derivatives on this line, averaged over all odd characters modulo N:

$$\frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} \mathscr{E}_\chi(\sigma_{\mathrm{ref}}) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} \int_{\mathbb{R}} \frac{|g_\chi(\sigma_{\mathrm{ref}} + it)|^2}{1+t^2}\, \mathrm{d}t \ll (\log N)^2.$$

For $0 < \sigma \leqslant \min(\sigma_{\mathrm{ref}}, \beta)$, average the first case of (6.14) over the odd characters. With $K = \sigma_{\mathrm{ref}}/\sigma$, this gives $O(K^2(\log N)^2) = O(\sigma^{-2})$ on the left of the reference line. Here $K^2(\log N)^2 = 4\sigma^{-2}$ and $2|X_N^-|/\varphi(N) = 1$, so the additional $(\log N)^2$ term has the same bound after averaging. For $\sigma_{\mathrm{ref}} \leqslant \sigma \leqslant \beta$, average the second case of (6.14) to obtain $O((\log N)^2)$. These two ranges prove (6.9) with the asserted uniform constant. This proves the lemma. □

6.4.2. *The three components of the residual operator.* We apply the averaged estimate to the correlation operator and bound the diagonal and Beta-model operators on the same Hardy space. Positivity gives trace bounds for the first two components; the continued Beta model is controlled through its boundary kernel.

The three components act by boundary integration against the respective kernels

$$\frac{\mathscr{P}_N(s,\bar{z})}{(1+s)(1+\bar{z})}, \qquad \frac{\mathscr{D}_N(s+\bar{z})}{(1+s)(1+\bar{z})}, \qquad \frac{\mathscr{K}_N(s,\bar{z})}{(1+s)(1+\bar{z})}.$$

We denote these operators by $\mathcal{P}_N$, $\mathcal{D}_N$, and $\mathcal{K}_N$, so that $T_N = \mathcal{P}_N - \mathcal{D}_N - \mathcal{K}_N$.

Proposition 6.5. *The operators $\mathcal{P}_N$ and $\mathcal{D}_N$ are positive and trace class, and*

$$\|\mathcal{P}_N\|_{\mathrm{HS}} \leqslant \operatorname{tr} \mathcal{P}_N \ll (\log N)^2 + \alpha_N^{-2},$$
$$\|\mathcal{D}_N\|_{\mathrm{HS}} \leqslant \operatorname{tr} \mathcal{D}_N \ll \alpha_N^{-2},$$
$$\|\mathcal{K}_N\|_{\mathrm{HS}} \ll \alpha_N^{-2}.$$

The implied constants are independent of N and α_N.

Proof. For the correlation component, the Gram representation is

$$\mathscr{P}_N(s,\bar{z}) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} g_\chi(s)\overline{g_\chi(z)},$$

as in (2.13). It makes $\mathcal{P}_N$ positive and finite rank. The boundary norm is the sum over $\Re s = \alpha_N$ and $\Re s = \beta$, with $|1+s|^{-2} \asymp (1+|\Im s|^2)^{-1}$. Thus the averaged bound for weighted square integrals in (6.9) applies to each boundary and gives

$$\|\mathcal{P}_N\|_{\mathrm{HS}} \leqslant \operatorname{tr} \mathcal{P}_N = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} \left\| \frac{g_\chi(s)}{1+s} \right\|_{\mathcal{H}_N}^2 \ll (\log N)^2 + \alpha_N^{-2}.$$

For the diagonal component, recall from (2.11) that $\mathscr{D}_N(w) = \sum_{(n,N)=1} \Lambda^2(n) n^{-1-w}$, and set $d_n(s) = n^{-s}/(1+s)$. Each d_n belongs to $\mathcal{H}_N$, and

$$\|d_n\|^2 = \pi \left(\frac{n^{-2\alpha_N}}{1+\alpha_N} + \frac{n^{-2\beta}}{1+\beta} \right).$$

Because the inner product is linear in its first variable, $d_n \otimes d_n$ denotes the operator $h \mapsto \langle h, d_n \rangle d_n$. The positive rank-one series $\sum_{(n,N)=1} \Lambda^2(n) n^{-1} d_n \otimes d_n$ therefore has total trace

$$\operatorname{tr} \mathcal{D}_N = \pi \left(\frac{\mathscr{D}_N(2\alpha_N)}{1+\alpha_N} + \frac{\mathscr{D}_N(2\beta)}{1+\beta} \right) \ll \alpha_N^{-2}.$$

The bound $\sum_{n \leqslant u} \Lambda^2(n) \ll u \log(2u)$ and partial summation give $\mathscr{D}_N(2\sigma) \ll \sigma^{-2}$ for $0 < \sigma \leqslant \beta$. Hence the rank-one series converges in trace norm and defines $\mathcal{D}_N$ with the required diagonal kernel. For this positive series, convergence in trace norm means that the sum of the traces of the omitted terms tends to zero; it implies convergence in operator norm as well. The operators $\mathcal{P}_N$ and $\mathcal{D}_N$ act on $\mathcal{H}_N$, as does T_N by Proposition 5.3. The identity $\mathscr{E}_N = \mathscr{P}_N - \mathscr{D}_N - \mathscr{K}_N$ shows that $\mathcal{K}_N = \mathcal{P}_N - \mathcal{D}_N - T_N$ has the boundary kernel corresponding to $\mathscr{K}_N$.

For the model component, the arithmetic and analytic factors are separated as follows:

$$\mathscr{K}_N(s, \bar{z}) = \mathfrak{S}(N) \left(\frac{\pi}{N} \right)^{s+\bar{z}} P_N(s+\bar{z}) \zeta(1-s-\bar{z}) \mathcal{X}_1(s) \mathcal{X}_1(\bar{z}), \tag{6.16}$$

as in (2.12). Here

$$P_N(w) = \prod_{\substack{p>2 \\ p \nmid N}} \left(1 + \frac{p^{-w}}{p-2} \right), \qquad \mathcal{X}_1(s) = \frac{\Gamma((3/2-s)/2)}{\Gamma((3/2+s)/2)}.$$

We estimate these factors uniformly as the left boundary approaches zero. Let $0 < \sigma \leqslant 2\beta$ and put $P_*(\sigma) = \prod_{p>2}(1 + p^{-\sigma}/(p-2))$. The convergent product comparison

$$\left(1 + \frac{p^{-\sigma}}{p-2} \right) (1 - p^{-1-\sigma}) \leqslant 1 + \frac{2}{p(p-2)}$$

gives $P_*(\sigma) \ll \zeta(1+\sigma) \ll \sigma^{-1}$. Moreover,

$$\mathfrak{S}(N) P_N(\sigma) = 2C_2 P_*(\sigma) \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2+p^{-\sigma}}.$$

Each finite factor is at most p^σ, and $\prod_{p \mid N} p \leqslant N$. Hence, for $\Re w = \sigma$ in this range, $|N^{-w} \mathfrak{S}(N) P_N(w)| \ll \sigma^{-1}$. To bound the zeta factor, apply Euler summation at $R = 2 + |\tau|$. The truncated sum and the pole term are each $O(R^\sigma/\sigma)$, and the remainder is $O(R^\sigma)$. Consequently $|\zeta(1-\sigma-i\tau)| \ll \sigma^{-1}(2+|\tau|)^{2\beta}$ for $0 < \sigma \leqslant 2\beta$. Stirling's formula gives $\mathcal{X}_1(s) \ll (1+|\Im s|)^{-\Re s}$ uniformly for $0 \leqslant \Re s \leqslant \beta$, in particular a uniform bound there. The functional form (6.16) therefore bounds the squared model kernel on each boundary product by

$$C \alpha_N^{-4} \frac{(2+|t-y|)^{4\beta}}{(1+t^2)(1+y^2)}.$$

The displayed majorant is integrable because $4\beta < 1$; explicitly,

$$\int_{\mathbb{R}} \frac{\mathrm{d}t}{(1+t^2)(1+(t-u)^2)} = \frac{2\pi}{4+u^2}.$$

Thus the boundary-kernel operator has Hilbert–Schmidt norm $O(\alpha_N^{-2})$. The preceding identity identifies the restriction of this boundary-kernel operator to the Hardy trace space with $\mathcal{K}_N$. Restricting an operator to a closed subspace cannot increase its Hilbert–Schmidt norm, which proves the model bound.

This proves the proposition. □

6.4.3. *Proof of the baseline spectral zero-free theorem.* The triangle inequality gives the operator bound; the zero–norm conversion then gives the stated width.

Proof of Theorem 6.2. Applying the triangle inequality to the three bounds in Proposition 6.5 gives

$$\begin{aligned}\|T_N\|_{\mathrm{HS}} &\leqslant \|\mathcal{P}_N\|_{\mathrm{HS}} + \|\mathcal{D}_N\|_{\mathrm{HS}} + \|\mathcal{K}_N\|_{\mathrm{HS}} \\ &\ll (\log N)^2 + \alpha_N^{-2} \ll (\log N)^2.\end{aligned}$$

Combining this with $\|T_N\| \leqslant \|T_N\|_{\mathrm{HS}}$ proves (6.7).

Choose η_N^{GRH} to be a sufficiently small absolute multiple of $\mathfrak{S}(N)/(N\log N)$. Since $\mathfrak{S}(N) \ll \log\log N$, this width tends to zero and lies in $(0, 1/2)$ for sufficiently large N. Using $\delta_N^2 = \mathfrak{S}(N)(\log N)^3/N$, the corresponding norm bound is

$$\frac{\delta_N}{2\sqrt{\eta_N^{\mathrm{GRH}}(1-\eta_N^{\mathrm{GRH}})}} \geqslant \frac{\delta_N}{2\sqrt{\eta_N^{\mathrm{GRH}}}}.$$

The right-hand side is a fixed multiple of $(\log N)^2$. Choosing the width's constant sufficiently small makes this multiple large enough to dominate the operator bound. The choice is made once, using the absolute constant in the operator estimate, and hence remains uniform as the modulus varies. Theorem 6.1 then gives the stated zero-free half-space, including its quaternionic extension. The chosen width has the order asserted in (6.8). This proves the theorem. □

6.5. **The scale of the left Hardy boundary.** The normalization used above also balances the available analytic estimates. Keep $\beta = 1/8$ fixed, let $0 < \alpha < 1/10$, and write $\widehat{T}_{N,\alpha}$ for the operator on this strip and $\mathcal{Q}_\alpha$ for the squared norm of its Mellin functional. The proof of Proposition 6.5 applies with α in place of α_N. The Fourier calculation in Proposition 7.4, performed with the same endpoints, gives

$$\mathcal{Q}_\alpha = \frac{1}{2\pi}\int_0^\infty \frac{(Nu)^{2\alpha}|\Phi(u) - u\Phi'(u)|^2}{1+(Nu)^{-2(\beta-\alpha)}}\,\frac{\mathrm{d}u}{u}.$$

On the fixed support, $u^{2\alpha}$ is bounded above and below uniformly for $0 < \alpha < 1/10$, and the denominator lies between one and two for sufficiently large N. Consequently,

$$\mathcal{Q}_\alpha \asymp_\phi N^{2\alpha}, \qquad \|\widehat{T}_{N,\alpha}\| \leqslant M_\alpha := C\big((\log N)^2 + \alpha^{-2}\big).$$

The constants are uniform in α in this range, for sufficiently large N. For left boundary α, the Cauchy–Schwarz argument in (2.18) gives an error bound $\mathcal{Q}_\alpha\|\widehat{T}_{N,\alpha}\|$. Setting $x = \alpha\log N$, we have

$$\mathcal{Q}_\alpha M_\alpha \asymp_\phi (\log N)^2 e^{2x}(1+x^{-2}).$$

The last factor tends to infinity both as $x \downarrow 0$ and as $x \to \infty$, and has a positive finite minimum in between. Thus $\alpha \asymp 1/\log N$ gives the best order obtainable from these separate bounds; it does not assert optimality of the actual residual norm. The chosen $\alpha_N = 1/\log N$ also gives $N^{2\alpha_N} = e^2$. Appendix A develops the parameter dependence.

The next section proves the exact quadratic-form identity and the uniform functional bound, then combines them with the spectral estimates to bound the Goldbach error and prove Theorems 1.1 and 1.2.

7. Hardy: the arithmetic error as an exact quadratic form

We express the signed error in the weighted Goldbach correlation as an exact quadratic form of the residual operator. The construction has two parts: Mellin inversion identifies the arithmetic error, and a bounded functional on the Hardy space selects the vector at which the operator gives that error. We then combine the Hardy and spectral estimates to bound the relative Goldbach error in terms of a zero-free width and prove the two main theorems.

Throughout, $N \geqslant 4$ is even. The weight ϕ is a fixed nonzero function in $C_c^\infty(\mathbb{R})$ supported in $[1/3, 2/3]$, with $0 \leqslant \phi \leqslant 1$ and $\phi(u) = \phi(1-u)$. Put

$$\Phi(u) = \sqrt{u}\,\phi(u), \qquad c_\phi = \int_0^1 \phi(u)^2 \,\mathrm{d}u.$$

The arithmetic quantities to be compared are

$$\mathcal{G}(N) = \frac{1}{N} \sum_{\substack{n+m=N \\ (nm,N)=1}} \Lambda(n)\Lambda(m)\phi(n/N)\phi(m/N),$$

$$\mathfrak{S}(N) = 2C_2 \prod_{\substack{p \mid N \\ p>2}} \frac{p-1}{p-2}, \qquad C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right),$$

as in (1.3) and (1.2). The sum is over ordered pairs of positive integers. Whenever GRH(N) is assumed, it means that every nontrivial zero of every primitive Dirichlet L-function whose conductor divides N has real part $1/2$. The Hardy construction will additionally require $N > e^{10}$. Our target is $c_\phi \mathfrak{S}(N) - \mathcal{G}(N)$; retaining its sign gives an exact identity before any norm estimate is applied.

7.1. Mellin inversion and the arithmetic residual.

We evaluate the character, diagonal, and model integrals on a common Mellin contour. Their combination will represent the same arithmetic discrepancy on every permitted contour. The residual is formed by subtracting the exact diagonal and the continuous model from the character kernel:

$$\mathscr{E}_N(s,z) = \mathscr{P}_N(s,z) - \mathscr{D}_N(s+z) - \mathscr{K}_N(s,z).$$

We evaluate these three terms in order, keeping the same Mellin normalization in each.

7.1.1. *The character integral.* The odd-character family is $X_N^- = \{\chi \bmod N : \chi(-1) = -1\}$, with all characters extended by zero at nonunits. The character moment is

$$\mathcal{P}(N) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} |P_\chi|^2.$$

Here each character contributes the polynomial

$$P_\chi = \sum_{n \geqslant 1} \frac{\Lambda(n)\chi(n)}{\sqrt{n}} \Phi(n/N).$$

Writing $g_\chi(s) = L'/L(1/2+s,\chi)$, the analytic kernel is

$$\mathscr{P}_N(s,z) = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} g_\chi(s) g_{\bar\chi}(z).$$

Here s and z are independent holomorphic variables. We use the Mellin convention

$$\widetilde{\Phi}(s) = \int_0^\infty \Phi(u) u^{s-1} \,\mathrm{d}u, \qquad \Phi(u) = \frac{1}{2\pi i} \int_{(\sigma)} \widetilde{\Phi}(s) u^{-s} \,\mathrm{d}s.$$

Here $\int_{(\sigma)}$ runs upward along $\Re s = \sigma$; the same tilde notation and Mellin convention apply to the other weights below. The decay needed for the contour shifts is

$$\widetilde{\Phi}(\sigma + it) \ll_{\phi, I} (1+|t|)^{-6} \quad (\sigma \in I), \tag{7.1}$$

for every fixed compact real interval I. We prove this estimate together with the character inversion below.

Lemma 7.1. *The Mellin inversion formula above and the decay estimate* (7.1) *hold, with decay uniform on compact real intervals for the contour. Under* GRH(N)*, for every* $0 < \sigma \leqslant 1/9$ *one has the absolutely convergent identities*

$$P_\chi = -\frac{1}{2\pi i} \int_{(\sigma)} N^s \widetilde{\Phi}(s) g_\chi(s) \,\mathrm{d}s, \tag{7.2}$$

$$\mathcal{P}(N) = \frac{1}{(2\pi i)^2} \int_{(\sigma)} \int_{(\sigma)} N^{s+z} \widetilde{\Phi}(s) \widetilde{\Phi}(z) \mathscr{P}_N(s,z) \,\mathrm{d}s \,\mathrm{d}z. \tag{7.3}$$

Proof. Write the Mellin transform as the Fourier integral of $e^{\sigma x}\Phi(e^x)$. This function has fixed compact support, and six integrations by parts give the asserted decay uniformly for σ in a compact real interval. Fourier inversion then gives the absolutely convergent Mellin inversion formula.

On $\Re s = 1$, absolute convergence follows from $\sum_n \Lambda(n) n^{-3/2} < \infty$. Mellin decay therefore permits termwise inversion and proves (7.2) on that line. For an odd character, the inducing primitive L-function is entire. Under GRH its nontrivial zeros lie on $\Re u = 1/2$; its trivial zeros and the zeros of the removed Euler factors lie in $\Re u \leqslant 0$. Thus $L'/L(1/2+s,\chi)$ has no poles between σ and 1.

On the horizontal sides of the contour rectangle, the bound in Lemma 3.4 gives

$$\left| N^s \widetilde{\Phi}(s) \frac{L'}{L}(1/2+s,\chi) \right| \ll_{\phi,\sigma} N(1+T)^{-6} \log^2\big(N(T+2)\big), \qquad s = x \pm iT, \quad \sigma \leqslant x \leqslant 1.$$

For fixed N and σ, these contributions vanish as $T \to \infty$. The contour is shifted for each fixed modulus and positive left line. Uniformity in the moving arithmetic boundary is supplied later by the Hardy estimates, rather than by this shift. Cauchy's theorem proves the shifted identity. Multiplying the formulas for χ and $\bar{\chi}$, then averaging over X_N^-, proves (7.3): $\overline{P_\chi} = P_{\bar{\chi}}$, the two minus signs cancel, and absolute convergence permits Fubini. This proves the lemma. □

7.1.2. *The diagonal and Beta integrals.* The diagonal and continuous-model kernels are

$$\mathscr{D}_N(w) = \sum_{\substack{n\geqslant 1\\(n,N)=1}} \frac{\Lambda(n)^2}{n^{1+w}}, \qquad \mathscr{K}_N(s,z) = N^{-s-z}\mathscr{S}_N(s+z)K_-(s,z).$$

The corresponding finite diagonal is

$$\mathcal{D}(N) = N^{-1} \sum_{(n,N)=1} \Lambda(n)^2 \phi(n/N)^2.$$

The density series and signed Beta factor are

$$\mathscr{S}_N(w) = \sum_{k\geqslant 1} \mathfrak{S}(kN)k^{-w} \quad (\Re w > 1),$$
$$K_-(s,z) = B(1/2-s, s+z) + B(1/2-z, s+z) - B(1/2-s, 1/2-z).$$

Here $B(s,z) = \Gamma(s)\Gamma(z)/\Gamma(s+z)$ is Euler's Beta function. Lemma 4.3 continues $\mathscr{S}_N$ to $\Re w > 0$, with only a simple pole at $w = 1$. In the range of the contours below, $\mathscr{K}_N$ means the continued model constructed in Section 4.

To encode the two positive difference families and the negative sum family, define the following convolutions:

$$C_0(y) = \int_0^\infty \phi(u)\phi(u+y)\,\mathrm{d}u, \qquad J_0(y) = \int_0^y \phi(u)\phi(y-u)\,\mathrm{d}u, \tag{7.4}$$
$$H_0(y) = 2C_0(y) - J_0(y), \qquad J_0(1) = c_\phi.$$

The value $J_0(1) = c_\phi$ follows from $\phi(u) = \phi(1-u)$. Write $\widetilde{H}_0$ for the Mellin transform of H_0. The Beta–Mellin relation below reduces the model integral to one Mellin variable, and a contour shift then gives its value.

Lemma 7.2. *For $0 < \sigma \leqslant 1/9$, the diagonal and model integrals converge absolutely and satisfy*

$$\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} N^{s+z}\widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{D}_N(s+z)\,\mathrm{d}s\,\mathrm{d}z = \mathcal{D}(N), \tag{7.5}$$

$$I_N(\Phi) := \frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)} \widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{S}_N(s+z)K_-(s,z)\,\mathrm{d}s\,\mathrm{d}z = -c_\phi\mathfrak{S}(N). \tag{7.6}$$

In the model integral, the factor N^{s+z} has cancelled the N^{-s-z} in $\mathscr{K}_N$.

Proof. Since $|\mathscr{D}_N(s+z)| \leqslant \mathscr{D}_N(2\sigma)$ on the product contour, Mellin decay permits termwise integration of its Dirichlet series. Inversion in both variables gives

$$\sum_{(n,N)=1} \frac{\Lambda(n)^2}{n}\Phi(n/N)^2 = \frac{1}{N}\sum_{(n,N)=1}\Lambda(n)^2\phi(n/N)^2 = \mathcal{D}(N),$$

which proves (7.5).

For the model, Euler's integral gives

$$\begin{aligned}\int_0^r u^{-1/2-s}(r-u)^{w-1}\,\mathrm{d}u &= r^{w-1/2-s}B(1/2-s,w),\\ \int_0^\infty u^{-1/2-s}(u+r)^{w-1}\,\mathrm{d}u &= r^{w-1/2-s}B(1/2-s,1/2+s-w).\end{aligned} \tag{7.7}$$

Both identities are absolutely convergent when $\Re s = \sigma$ and $0 < \Re w < 1/2+\sigma$. Insert Mellin inversion for $\phi(u) = u^{-1/2}\Phi(u)$ into the convolution transforms. The substitutions $r = u+y$ for C_0 and $y = u+r$ for J_0, followed by (7.7), give

$$\begin{aligned}\widetilde{C}_0(w) &= \frac{1}{2\pi i}\int_{(\sigma)}\widetilde{\Phi}(s)\widetilde{\Phi}(w-s)B(1/2-s,w)\,\mathrm{d}s,\\ \widetilde{J}_0(w) &= \frac{1}{2\pi i}\int_{(\sigma)}\widetilde{\Phi}(s)\widetilde{\Phi}(w-s)B(1/2-s,1/2+s-w)\,\mathrm{d}s.\end{aligned}$$

Here r remains in the fixed support of ϕ, while the real-part versions of (7.7) and the integrability of $\widetilde{\Phi}$ justify the interchanges. On $\Re w = 2\sigma$, replacing s by $w-s$ gives the second Beta term for C_0. Adding these two expressions and subtracting the expression for J_0 gives, for $0 < \eta \leqslant 1/9$ and $\Re w = 2\eta$,

$$\widetilde{H}_0(w) = \frac{1}{2\pi i}\int_{(\eta)}\widetilde{\Phi}(s)\widetilde{\Phi}(w-s)K_-(s,w-s)\,\mathrm{d}s. \tag{7.8}$$

The support of ϕ makes C_0 vanish for $y > 1/3$ and J_0 vanish outside $[2/3, 4/3]$. Their derivatives through order six are bounded, so $\widetilde{H}_0$ is holomorphic for $\Re w > 0$. For every compact interval $I \subset (0,\infty)$, the derivatives through order six of $e^{\sigma x}H_0(e^x)$ are integrable uniformly for $\sigma \in I$: they decay exponentially as $x \to -\infty$ and vanish for large positive x. All boundary terms vanish, so six integrations by parts give

$$\widetilde{H}_0(\sigma+it) \ll_{\phi,I} (1+|t|)^{-6} \qquad (\sigma \in I,\ t \in \mathbb{R}). \tag{7.9}$$

Put $a_0 = \int_0^\infty \phi(u)\,\mathrm{d}u$. Integration over the positive quadrant assigns half the product integral to C_0 and all of it to J_0, giving

$$\int_0^\infty C_0(y)\,\mathrm{d}y = \frac{a_0^2}{2}, \quad \int_0^\infty J_0(y)\,\mathrm{d}y = a_0^2, \quad \widetilde{H}_0(1) = 0. \tag{7.10}$$

The support restrictions separately give the integer values

$$C_0(k) = 0\ (k \geqslant 1), \quad J_0(k) = 0\ (k \geqslant 2), \quad H_0(1) = -c_\phi. \tag{7.11}$$

On the original product contour, Euler's integral bounds the three Beta factors uniformly, while $\mathscr{S}_N(2\sigma+it) \ll_{N,\sigma} 1+|t|$. The model integrand is therefore dominated by

a constant times

$$(1+|v|)^{-6}(1+|y|)^{-6}(1+|v+y|),$$

an integrable function on $\mathbb{R}^2$. The change of variables $(v,y)\mapsto(v,v+y)$ has Jacobian 1, so Fubini and (7.8) give the single-contour expression

$$I_N(\Phi)=\frac{1}{2\pi i}\int_{(2\sigma)}\mathscr{S}_N(w)\widetilde{H}_0(w)\,\mathrm{d}w,\qquad w=s+z.$$

This absolute majorant permits the change of variables before either integration is evaluated. It also shows that the orientation and the normalization are unchanged when the two heights are replaced by their sum.

The possible pole at $w=1$ is removable in the product $\mathscr{S}_N(w)\widetilde{H}_0(w)$, because $\mathscr{S}_N$ has a simple pole there and $\widetilde{H}_0(1)=0$. Applying (4.6) and (7.9) on $I=[2\sigma,3/2]\subset(0,\infty)$ gives horizontal integrals $O_{N,\phi,\sigma}(T^{-5})$. We may therefore move the w-contour to $\Re w=3/2$. On that line the density series $\mathscr{S}_N(w)=\sum_{k\geqslant1}\mathfrak{S}(kN)k^{-w}$ converges absolutely. Termwise Mellin inversion therefore completes the evaluation:

$$\begin{aligned}I_N(\Phi)&=\frac{1}{2\pi i}\int_{(2\sigma)}\mathscr{S}_N(w)\widetilde{H}_0(w)\;\mathrm{d}w\\&=\sum_{k\geqslant1}\mathfrak{S}(kN)H_0(k)=-c_\phi\mathfrak{S}(N).\end{aligned}\tag{7.12}$$

Only $k=1$ survives by (7.11), proving (7.6). The pole cancellation concerns $\widetilde{H}_0(1)$; the surviving sampled value is $H_0(1)$. This proves the lemma. □

7.1.3. *The arithmetic discrepancy.* We can now combine the three evaluations without any further contour argument. Write the residual integral as

$$r_\sigma(N)=\frac{1}{(2\pi i)^2}\int_{(\sigma)}\int_{(\sigma)}N^{s+z}\widetilde{\Phi}(s)\widetilde{\Phi}(z)\mathscr{E}_N(s,z)\,\mathrm{d}s\,\mathrm{d}z.\tag{7.13}$$

The signs of the three kernels are essential: the model is subtracted in $\mathscr{E}_N$, but its own weighted integral is negative.

Proposition 7.3. *Under* GRH(N)*, the residual integral $r_\sigma(N)$ is absolutely convergent for $0<\sigma\leqslant1/9$ and satisfies*

$$\mathcal{G}(N)=c_\phi\mathfrak{S}(N)-r_\sigma(N).\tag{7.14}$$

In particular, $r_\sigma(N)$ is independent of σ; we write it as $r(N)$.

Proof. Lemmas 7.1 and 7.2 evaluate the three terms of the residual as

$$r_\sigma(N)=\mathcal{P}(N)-\mathcal{D}(N)+c_\phi\mathfrak{S}(N).$$

The identity $\mathcal{G}(N)=\mathcal{D}(N)-\mathcal{P}(N)$ from Lemma 3.1 gives (7.14). Its arithmetic side has no contour parameter. The model enters with the opposite sign to its own Mellin evaluation because it is subtracted in the residual kernel. The convergence constants may depend on N and σ; the uniform estimate follows from the Hardy norm below. This proves the proposition. □

7.2. **The residual as a quadratic form.** For even $N > e^{10}$ satisfying GRH(N), we construct a vector v_N of uniformly bounded norm such that $r(N) = \langle T_N v_N, v_N\rangle$. This transfers the spectral estimates to the arithmetic discrepancy without a factor growing with N. We use the strip and boundary

$$\alpha_N = \frac{1}{\log N}, \qquad \beta = \frac{1}{8}, \qquad \sigma_* = \frac{1}{9}, \qquad \Delta_N = \beta - \alpha_N,$$
$$S_N = \{s : \alpha_N < \Re s < \beta\}, \qquad \Gamma_N = (\alpha_N + i\mathbb{R}) \sqcup (\beta + i\mathbb{R}).$$

The Hardy space $\mathcal{H}_N = H^2(S_N)$ consists of holomorphic functions with bounded paired vertical L^2 means as the two lines approach the boundary. As proved in Lemma 5.2, its boundary traces h_{α_N}, h_β have norm

$$\|h\|^2_{\mathcal{H}_N} = \int_{\mathbb{R}} |h_{\alpha_N}(t)|^2 \,\mathrm{d}t + \int_{\mathbb{R}} |h_\beta(t)|^2 \,\mathrm{d}t.$$

Thus both boundary lines contribute to the norm, and the two traces must come from the same holomorphic function in the strip. The inner product is $\langle h, g\rangle = \int_{\Gamma_N} h(s)\overline{g(s)}\,|\mathrm{d}s|$, linear in its first variable. Both boundaries carry positive arc length, with no factor $1/(2\pi)$. On this space the compact selfadjoint operator constructed in Proposition 5.3 is

$$(T_N h)(s) = \int_{\Gamma_N} \frac{\mathscr{E}_N(s, \bar{z})}{(1+s)(1+\bar{z})} h(z)\,|\mathrm{d}z|.$$

The next argument restores the weight to this normalized kernel.

7.2.1. *The functional norm.* Define the linear functional

$$\ell_N(h) = \frac{1}{2\pi i} \int_{(\sigma)} N^s \widetilde{\Phi}(s)(1+s)h(s)\,\mathrm{d}s.$$

Here $\alpha_N < \sigma < \beta$. The bound proved below gives a unique Riesz vector $v_N \in \mathcal{H}_N$ characterized by $\ell_N(h) = \langle h, v_N\rangle$. Write $\mathcal{Q}_N = \|\ell_N\|^2 = \|v_N\|^2_{\mathcal{H}_N}$. For the calculation of this norm, introduce the companion weight

$$V(u) = \Phi(u) - u\Phi'(u).$$

We use the Fourier–Laplace representation of Lemma 5.2:

$$(\mathcal{U}_H \psi)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(\tau) e^{s\tau}\,\mathrm{d}\tau,$$
$$\|\mathcal{U}_H \psi\|^2_{\mathcal{H}_N} = \int_{\mathbb{R}} |\psi(\tau)|^2 w_{H,N}(\tau)\,\mathrm{d}\tau.$$

Here ψ ranges over $L^2(\mathbb{R}, w_{H,N}(\tau)\,\mathrm{d}\tau)$, with weight

$$w_{H,N}(\tau) = e^{2\alpha_N \tau} + e^{2\beta\tau}.$$

Its two terms correspond to the two boundary lines in the Hardy norm. This representation identifies both the norm and the contour independence of ℓ_N; the factor $1+s$ in the functional is encoded by V.

Proposition 7.4 (The functional norm)**.** *The functional ℓ_N is continuous and independent of the interior contour. Its squared norm and uniform bound are*

$$\mathcal{Q}_N = \frac{1}{2\pi}\int_0^\infty \frac{(Nu)^{2\alpha_N}|V(u)|^2}{1+(Nu)^{-2\Delta_N}}\,\frac{\mathrm{d}u}{u} \leqslant \frac{e^2}{2\pi}\int_0^\infty |V(u)|^2\,\frac{\mathrm{d}u}{u}. \tag{7.15}$$

Moreover,

$$\mathcal{Q}_N = \frac{e^2}{2\pi}\int_0^\infty |V(u)|^2\,\frac{\mathrm{d}u}{u} + O_\phi\left(\frac{1}{\log N}\right). \tag{7.16}$$

Proof. Integration by parts gives $\widetilde{V}(s) = (1+s)\widetilde{\Phi}(s)$; the compact support of Φ removes the boundary term. Apply the Fourier–Laplace representation (5.2)–(5.3), first to a density ψ of compact support. Fubini and Mellin inversion give

$$\begin{aligned}\ell_N(\mathcal{U}_H\psi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\psi(\tau)\left\{\frac{1}{2\pi i}\int_{(\sigma)}\widetilde{V}(s)(Ne^{\tau})^s\,\mathrm{d}s\right\}\mathrm{d}\tau\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\psi(\tau)V(e^{-\tau}/N)\,\mathrm{d}\tau.\end{aligned}$$

The first interchange is absolute because $\widetilde{V}$ is integrable on the contour and ψ has compact support. In $L^2(w_{H,N})$, the last integral has squared norm

$$\frac{1}{2\pi}\int_{\mathbb{R}}\frac{|V(e^{-\tau}/N)|^2}{w_{H,N}(\tau)}\,\mathrm{d}\tau.$$

Weighted Cauchy–Schwarz gives the upper bound, with equality for a density proportional to $\overline{V(e^{-\tau}/N)}/w_{H,N}(\tau)$. This density has compact support for each fixed N, so the norm is exact. The weight is strictly positive, so this equality case is admissible in the weighted space. Its norm is finite because its support is compact.

Compactly supported densities are dense in the weighted space. The original contour integral is also continuous there: $e^{2\sigma\tau} \leqslant w_{H,N}(\tau)$ bounds restriction to the interior line in L^2, and $(1+s)\widetilde{\Phi}(s)$ is square integrable. The two expressions therefore agree for every Hardy vector. The density expression proves contour independence.

The substitution $\tau = -\log(Nu)$ gives the u-integral in (7.15). On the fixed support of V,

$$(Nu)^{2\alpha_N} = e^2\{1+O_\phi((\log N)^{-1})\}, \qquad (Nu)^{-2\Delta_N} = O_\phi(N^{-1/4}),$$

which proves (7.16). Finally, $(Nu)^{2\alpha_N} = e^2u^{2/\log N} \leqslant e^2$ and the denominator in (7.15) is at least 1 on this support. This proves the uniform bound in (7.15), hence $\mathcal{Q}_N \ll_\phi 1$. This proves the proposition. □

7.2.2. *Recovering the quadratic form.* Proposition 5.4 gives the evaluation vectors

$$k_z(s) = \frac{1}{4\Delta_N}\csc\left(\frac{\pi(s+\bar{z}-2\alpha_N)}{2\Delta_N}\right), \qquad h(s) = \langle h, k_s\rangle.$$

It also identifies the residual with the matrix elements of T_N:

$$\langle T_N k_z, k_s\rangle = \frac{\mathscr{E}_N(s,\bar{z})}{(1+s)(1+\bar{z})}.$$

Thus applying the weight functional to the evaluation vectors will restore exactly the two Mellin factors. The conjugations are fixed by the convention for the inner product.

Theorem 7.1 (The exact residual identity)**.** *For every even $N > e^{10}$ satisfying* GRH(N)*, the residual satisfies*

$$r(N) = \langle T_N v_N, v_N\rangle, \qquad |r(N)| \leqslant \mathcal{Q}_N\|T_N\| \ll_\phi \|T_N\|. \tag{7.17}$$

Consequently, $\mathcal{G}(N) = c_\phi\mathfrak{S}(N) - \langle T_N v_N, v_N\rangle$, and

$$|\mathcal{G}(N) - c_\phi\mathfrak{S}(N)| \ll_\phi \|T_N\|. \tag{7.18}$$

Proof. Put $\Psi_N(s) = N^s\widetilde{\Phi}(s)(1+s)$ and fix $\sigma = \sigma_* = 1/9$. The reproducing property and uniqueness of the Riesz vector give

$$v_N = \frac{1}{2\pi}\int_{\mathbb{R}} \overline{\Psi_N(\sigma+iy)}k_{\sigma+iy}\,\mathrm{d}y.$$

This is a Bochner integral, an integral of Hilbert-space vectors defined by convergence in the Hilbert norm. In the present setting it is the norm limit of the integrals over bounded intervals; integrability of the vector norms ensures that this limit exists. Indeed, (5.7) gives $\|k_s\|^2 \leqslant [8\min(\Re s - \alpha_N, \beta - \Re s)]^{-1}$. At $\Re s = \sigma_*$ both distances are at least $1/90$, so $\sup_y \|k_{\sigma_*+iy}\|^2 \leqslant 45/4$, while Ψ_N is integrable by Mellin decay. The coefficient is conjugated because the inner product is conjugate-linear in its second entry. When expanding the quadratic form, one may first perform both integrations over bounded intervals. The evaluation bound then permits passage to the full lines by dominated convergence, preserving the order of the two inner-product entries.

The kernel identity established in the proof of Proposition 5.4 yields

$$\langle T_N v_N, v_N\rangle = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} \Psi_N(\sigma+ix)\overline{\Psi_N(\sigma+iy)}\frac{\mathscr{E}_N(\sigma+ix,\sigma-iy)}{(1+\sigma+ix)(1+\sigma-iy)}\,\mathrm{d}x\,\mathrm{d}y.$$

The same evaluation bound gives absolute convergence. Since Φ is real, $\overline{\Psi_N(\sigma+iy)} = \Psi_N(\sigma-iy)$. Replace y by $-y$; reversing the limits cancels the sign from $\mathrm{d}y$. Both Mellin variables now run upward, and their factors $1+s$ and $1+z$ cancel the kernel denominators. The remaining integral is exactly the residual integral in (7.13). This proves the identity in (7.17).

Cauchy–Schwarz and Proposition 7.4 give $|r(N)| \leqslant \|T_N\|\,\|v_N\|^2 = \mathcal{Q}_N\|T_N\|$. Combining this with (7.14) gives (7.18), the estimate used for Theorems 1.1 and 1.2. This proves the theorem. □

7.3. **Proofs of the main theorems.** We now combine the spectral bounds of Theorem 6.1 with the Hardy estimate of Theorem 7.1. The following lemma expresses the relative Goldbach error in terms of a zero-free width and gives a criterion for the asymptotic formula.

Lemma 7.5 (An arithmetic estimate from a zero-free width)**.** *Assume* GRH(N) *for an even $N > e^{10}$. If $0 < \eta_N \leqslant 1/4$ and $\mathfrak{X}_N(q) \neq 0$ for $\Re q > 1 - \eta_N$, then*

$$\frac{|\mathcal{G}(N) - c_\phi\mathfrak{S}(N)|}{c_\phi\mathfrak{S}(N)} \ll_\phi \sqrt{\frac{(\log N)^3}{N\eta_N\mathfrak{S}(N)}}. \tag{7.19}$$

As $N \to \infty$ through even integers satisfying these hypotheses, the relative Goldbach asymptotic follows from the condition

$$\frac{N\eta_N \mathfrak{S}(N)}{(\log N)^3} \longrightarrow \infty. \tag{7.20}$$

Proof. Insert the bound $\|T_N\| \leqslant \delta_N/(2\sqrt{\eta_N(1-\eta_N)})$ from Theorem 6.1 into the Hardy estimate in Theorem 7.1 and divide by $c_\phi \mathfrak{S}(N)$. Using $\delta_N^2 = \mathfrak{S}(N)(\log N)^3/N$ and $1 - \eta_N \geqslant 3/4$ gives (7.19). Under (7.20), the right-hand side of (7.19) tends to zero. This proves the lemma. □

We now prove Theorems 1.1 and 1.2 under Hypotheses S and R, respectively.

Proof of Theorem 1.1. Hypothesis S excludes zeros in $\Re q > 1 - \eta_N$, with the prescribed width

$$\eta_N = \frac{h(N)^2(\log N)^3}{N\mathfrak{S}(N)} \leqslant \frac{1}{4}, \qquad h(N) \longrightarrow \infty,$$

as in (1.7). Here h is fixed in advance. Substitution in Theorem 6.1 gives

$$\|T_N\| \leqslant \frac{\mathfrak{S}(N)}{2h(N)\sqrt{1-\eta_N}} \leqslant \frac{\mathfrak{S}(N)}{\sqrt{3}\,h(N)}.$$

The estimate in Theorem 7.1 now gives

$$\mathcal{G}(N) = c_\phi \mathfrak{S}(N) + O_\phi\left(\frac{\mathfrak{S}(N)}{h(N)}\right),$$

as in (1.8). Its implied constant depends only on ϕ; the required lower bound on N may depend on the admissibility and growth of h. Since $h(N) \to \infty$, the error is $o(\mathfrak{S}(N))$; if $h(N)/\mathfrak{S}(N) \to \infty$, it is also $o(1)$. This proves the theorem. □

Proof of Theorem 1.2. Hypothesis R puts every spectral zero on $\Re q = 1/2$. By Theorem 6.1, this is equivalent to $\|T_N\| \leqslant \delta_N$. The estimate in Theorem 7.1 yields

$$\mathcal{G}(N) = c_\phi \mathfrak{S}(N) + O_\phi\left(\sqrt{\frac{\mathfrak{S}(N)}{N}}(\log N)^{3/2}\right),$$

as in (1.9). The zero classification also shows that R implies S for every prescribed admissible width: all exceptional real pairs have been excluded. The functional symmetries alone provide no such norm bound. This proves the theorem. □

The next section places these results alongside the twin-prime problem and discusses the possible scope of classical methods.

8. Goldbach, twin primes, and classical methods

Goldbach's conjecture and the twin-prime conjecture both concern the simultaneous primality of two linear expressions, but their arithmetic questions are different. The twin-prime conjecture asks for infinitely many prime pairs $(p, p+2)$, with the difference fixed. Goldbach's conjecture asks for a prime pair $(p, N-p)$ for every even $N \geqslant 4$; its asymptotic version requires uniform control as the target N varies.

The twin-prime problem fits the general orbit-and-polynomial formulation of the affine sieve of Bourgain, Gamburd, and Sarnak [1]. For example, let

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad F(x, y) = x(x + 2y).$$

Then $\langle U \rangle v = \{(n, 1)^{\mathsf{T}} : n \in \mathbb{Z}\}$ and $F(U^n v) = n(n + 2)$. The desired prime pairs are therefore the orbit points with $n > 1$ at which both linear factors are prime. This is a unipotent orbit; the expansion theorems for semisimple groups do not apply to it directly.

For Goldbach, the corresponding polynomial is $F_N(n) = n(N-n)$, and the relevant positive range is finite for each fixed N. The polynomial, the range, and the local congruence data all vary with N. An orbit description is still possible, but obtaining representations for every sufficiently large even target requires estimates uniform in this varying family. This distinction matters when comparing the two conjectures through a common sieve framework.

We do not know whether sufficiently ingenious refinements of sieve methods or the circle method can yield the Goldbach asymptotic formula. The present difficulties do not warrant excluding this possibility.

The central advance of this paper is the exact identification of the signed Goldbach error with a quadratic form of the residual operator. For every even $N > e^{10}$ satisfying GRH(N), Theorem 7.1 gives

$$c_\phi \mathfrak{S}(N) - \mathcal{G}(N) = \langle T_N v_N, v_N \rangle.$$

The operator T_N is compact and selfadjoint, and the vector v_N determined by the weight satisfies $\|v_N\|^2 \ll_\phi 1$. The identity thus retains the full signed arithmetic discrepancy and transfers operator norm bounds to error bounds without a factor growing with N. Hypotheses S and R enter only after this identity has been established: they control $\|T_N\|$ and hence the size of the inner product. They yield, respectively, the relative asymptotic formula of Theorem 1.1 and the sharper remainder of Theorem 1.2.

Nor can we rule out that sieve methods might produce a positive lower bound sufficient to establish Goldbach representations, even without the full asymptotic formula. In pursuing such lower bounds, we have found some remarkable phenomena, whose analysis will be presented elsewhere.

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Appendix A. Dependence on the strip parameter

The main text fixes $\alpha_N = 1/\log N$. To discuss that choice, we introduce the family $\alpha_{N,c} = c/\log N$, so the main normalization is $c = 1$. We track the dependence of the Hardy realization and spectral conditions on c, then distinguish fixed-c asymptotics from the limit $c \downarrow 0$ at fixed N.

A.1. **The Hardy realization with a variable boundary.** We construct the Hardy realization as the left boundary varies and identify its unchanged arithmetic pairing. The arithmetic weights and residual kernel remain those of the main text, while the Hardy space, operator and representing vector acquire c.

Fix $c > 0$ and an even $N \geqslant 4$ with $N > e^{10c}$. Put

$$\alpha_{N,c} = \frac{c}{\log N}, \qquad \Delta_{N,c} = \beta - \alpha_{N,c}. \tag{A.1}$$

Use (2.14), (2.15), (2.17) and (2.36) with this left boundary to obtain $S_{N,c}$, $\mathcal{H}_{N,c}$, $\Gamma_{N,c}$, $T_{N,c}$ and $\ell_{N,c}$; write $v_{N,c}$ for the Riesz vector and $\mathcal{Q}_{N,c} = \|v_{N,c}\|^2$.

Proposition A.1 (Hardy realization with a variable boundary)**.** *Assume* GRH(N). *Then* $\alpha_{N,c} < 1/10$, $\Delta_{N,c} > 1/40$, *and* $T_{N,c}$ *is a compact selfadjoint Hilbert–Schmidt operator on* $\mathcal{H}_{N,c}$. *The common contour* $\sigma_* = 1/9$ *remains at distance greater than* $1/90$ *from the left boundary and exactly* $1/72$ *from the right. The point-evaluation estimate* (5.8) *holds with the distance* $d_{N,c}$ *to the new boundary. With* V *as in Subsubsection 2.5.2, the functional norm satisfies*

$$\begin{aligned} \mathcal{Q}_{N,c} &= \frac{e^{2c}}{2\pi} \int_0^\infty |V(u)|^2 \, \frac{\mathrm{d}u}{u} + O_{\phi,c}\!\left(\frac{1}{\log N}\right), \\ \mathcal{Q}_{N,c} &\leqslant \frac{e^{2c}}{2\pi} \int_0^\infty |V(u)|^2 \, \frac{\mathrm{d}u}{u}. \end{aligned} \tag{A.2}$$

The first line holds as $N \to \infty$ *with* c *fixed; the second holds throughout the permitted range. Moreover,*

$$r(N) = \langle T_{N,c} v_{N,c}, v_{N,c} \rangle, \qquad |r(N)| \leqslant \mathcal{Q}_{N,c} \|T_{N,c}\|. \tag{A.3}$$

The left side is independent of c*: varying the realization changes the operator and vector, but not their arithmetic pairing. At* $c = 1$ *all these objects reduce to those of the main text.*

Proof. The inequalities for the strip and contour follow from $\alpha_{N,c} = c/\log N < 1/10$, $\beta = 1/8$ and $\sigma_* = 1/9$. The proof of Proposition 5.3 applies on every such closed strip, and the Fourier–Laplace and point-evaluation arguments apply to arbitrary strip endpoints. This proves the operator properties and the stated pointwise estimate.

In (2.37), $N^{2\alpha_{N,c}} = e^{2c}$. On the fixed support of V in $(0,1)$, the denominator is at least one and $u^{2\alpha_{N,c}} \leqslant 1$, giving the upper bound in (A.2). For fixed c, $u^{2\alpha_{N,c}} = 1 + O_{\phi,c}(1/\log N)$ uniformly on this support, while the remaining denominator tends uniformly to one with a power saving in N. This gives its asymptotic formula. The proof of Theorem 7.1, on the same fixed interior contour, gives (A.3). Its arithmetic side depends only on the residual and the weight, so it is independent of c. The specialization at $c = 1$ follows from the definitions. This proves the proposition. □

A.2. **Spectral conditions and their quantifiers.** We establish the zero–norm correspondence for each realization and state its asymptotic consequences with c fixed before N grows. This order of quantifiers is essential for the estimates.

Let $\mathfrak{X}_{N,c}$ be the product (1.6) formed from $T_{N,c}$, retaining the scale (1.5), which is independent of c. Use $\mathrm{R}(N,c)$ and $\mathrm{S}(N,c;h_c)$ for the corresponding conditions of Section 1. For each fixed $c>0$, preassign a positive h_c on the even integers, tending to infinity and satisfying

$$\eta_{N,c}=\frac{h_c^2(N)(\log N)^3}{N\mathfrak{S}(N)}\leqslant\frac{1}{4} \tag{A.4}$$

eventually.

Proposition A.2 (Spectral conditions for a fixed boundary parameter)**.** *Assume* $\mathrm{GRH}(N)$ *and* $N>e^{10c}$*. Then*

$$\begin{aligned}&\mathrm{R}(N,c)\quad\Longleftrightarrow\quad\|T_{N,c}\|\leqslant\delta_N,\\ &\mathfrak{X}_{N,c}(q)\neq0\quad(\Re q>1-\eta)\quad\Longleftrightarrow\quad\|T_{N,c}\|\leqslant\frac{\delta_N}{2\sqrt{\eta(1-\eta)}},\qquad0<\eta<\tfrac12.\end{aligned} \tag{A.5}$$

The half-space is open, so zeros on its boundary are allowed. For every admissible $\eta_{N,c}$*,*

$$\mathrm{S}(N,c;h_c)\quad\Longleftrightarrow\quad\|T_{N,c}\|\leqslant\frac{\mathfrak{S}(N)}{2h_c(N)\sqrt{1-\eta_{N,c}}}. \tag{A.6}$$

For fixed $c>0$ *and sufficiently large even* N *satisfying GRH,*

$$\|T_{N,c}\|\ll_c(\log N)^2,\qquad\eta_{N,c}^{\mathrm{GRH}}\asymp_c\frac{\mathfrak{S}(N)}{N\log N},$$

and

$$\mathcal{G}(N)=c_\phi\mathfrak{S}(N)+\begin{cases}O_{\phi,c}(\mathfrak{S}(N)/h_c(N)), & \mathrm{S}(N,c;h_c),\\ O_{\phi,c}(\delta_N), & \mathrm{R}(N,c).\end{cases}$$

GRH is assumed in both cases. In the S estimate the implied constant is independent of h_c*, although the required lower bound on* N *depends on the growth and admissibility of* h_c*.*

Proof. The proof of Theorem 6.1 uses only compactness, selfadjointness and the Hilbert–Schmidt property, supplied here by Proposition A.1. It therefore proves (A.5) for $\mathfrak{X}_{N,c}$. Substituting (A.4) and the scale (1.5) gives (A.6).

In the component estimates of Proposition 6.5, replacing $1/\log N$ by $c/\log N$ changes the constants to ones depending on the fixed c and gives $\|T_{N,c}\|\ll_c(\log N)^2$. The zero–norm correspondence then supplies the stated GRH width. Combining (A.3) with the upper bound in (A.2) gives the two arithmetic estimates. For S, use $1-\eta_{N,c}\geqslant3/4$; the functional bound is independent of h_c. For R, use $\|T_{N,c}\|\leqslant\delta_N$. This proves the proposition. □

The eventual hypotheses assert these conditions for all sufficiently large even N satisfying $\mathrm{GRH}(N)$, with lower bounds $N_0(c)$ and $N_0(c,h_c)$ on N. Neither these lower bounds nor the admissibility and divergence of h_c need be uniform in c. The main hypotheses are their specializations at $c=1$. These conclusions concern sufficiently large N; covering every even $N\geqslant4$ requires a separate check of the finite initial range.

A.3. **Why the boundary has order** $1/\log N$**.** The comparison in Subsection 6.5 balances the functional norm against the available operator bound. Here we justify its uniformity and record the dependence on c.

Use $\widehat{T}_{N,\alpha}$, $\mathcal{Q}_\alpha$ and M_α from that comparison, with $0 < \alpha < 1/10$.

Lemma A.3 (Balance of the functional and operator bounds)**.** *For sufficiently large even N satisfying* GRH(N)*, uniformly for $0 < \alpha < 1/10$,*

$$\mathcal{Q}_\alpha \asymp_\phi N^{2\alpha}, \qquad \|\widehat{T}_{N,\alpha}\| \leqslant M_\alpha = C((\log N)^2 + \alpha^{-2}),$$

where C is absolute. Setting $\alpha = c/\log N$ gives

$$\mathcal{Q}_\alpha M_\alpha \asymp_\phi (\log N)^2 e^{2c}(1 + c^{-2}).$$

The factor $e^{2c}(1+c^{-2})$ tends to infinity as $c \downarrow 0$ and as $c \to \infty$. The ratio of the sufficient width for error $O(\mathfrak{S}(N)/h_c(N))$ to the width supplied by M_α is, when both widths are admissible,

$$\asymp_\phi \frac{h_c^2(N)(\log N)^4}{\mathfrak{S}^2(N)} e^{4c}(1 + c^{-2})^2.$$

Proof. The denominator of (2.37) lies between one and two on the fixed support for large N, and $u^{2\alpha}$ is bounded above and below there, uniformly in α. This proves $\mathcal{Q}_\alpha \asymp_\phi N^{2\alpha}$. The component estimates with left boundary α give $M_\alpha = C((\log N)^2 + \alpha^{-2})$ with an absolute C. Substitution of $\alpha = c/\log N$ gives the stated product. As $c \downarrow 0$, its factor $e^{2c}(1 + c^{-2})$ is at least c^{-2}; as $c \to \infty$, it is at least e^{2c}. This proves divergence at both ends.

For widths at most $1/4$, Theorem 6.1 makes a width comparable to δ_N^2/M^2 equivalent to a norm bound of size M. The stated arithmetic error is ensured by a norm bound of size $\mathfrak{S}(N)/(h_c(N)\mathcal{Q}_\alpha)$ by the Cauchy–Schwarz argument in (2.18) with left boundary α, retaining the factor $\mathcal{Q}_\alpha$. Dividing this sufficient width by the width supplied by M_α therefore gives

$$\asymp_\phi \frac{h_c^2(N)\mathcal{Q}_\alpha^2 M_\alpha^2}{\mathfrak{S}^2(N)},$$

which is the asserted ratio. This proves the lemma. □

Lemma A.3 shows that a fixed positive c gives the best order available from these bounds; they do not select a universal optimal constant. Changing c therefore changes the constants, not the logarithmic gap in the available estimates.

A.4. **The obstruction at a fixed modulus.** We examine the different limit in which N is fixed and $c \downarrow 0$. A pole of a logarithmic derivative supplies a lower bound for the residual operator norm and obstructs a zero-free width uniform in c.

Proposition A.4 (The obstruction at a fixed modulus)**.** *Assume* GRH(N) *for a fixed even $N \geqslant 4$. Choose a nontrivial zero $\rho = 1/2 + i\gamma$ of an odd character modulo N, and let $m_\chi(\rho)$ be its multiplicity for $L(s,\chi)$, with value zero when it is not a zero. Put*

$$C_{N,\gamma} = \frac{2}{\varphi(N)} \sum_{\chi \in X_N^-} m_\chi^2(\rho) > 0.$$

Then

$$\liminf_{c\downarrow 0} c\|T_{N,c}\| \geqslant \frac{2C_{N,\gamma}\log N}{1+\gamma^2} > 0.$$

In particular, there are $z_N^{\mathrm{obs}} > 0$ *and* $0 < c_N^{\mathrm{obs}} \leqslant \log N/24$ *such that*

$$\|T_{N,c}\| \geqslant z_N^{\mathrm{obs}}/c \qquad (0 < c < c_N^{\mathrm{obs}}). \tag{A.7}$$

The condition $\mathrm{R}(N,c)$ *fails for all sufficiently small* c*. In the same range, every zero-free width* $0 < \eta \leqslant 1/4$ *satisfies*

$$\eta \leqslant \frac{\delta_N^2 c^2}{3(z_N^{\mathrm{obs}})^2}.$$

Proof. At $s = \sigma + i\gamma$, Laurent expansion gives

$$\mathscr{P}_N(s,\bar{s}) = \frac{C_{N,\gamma}}{\sigma^2} + O_{N,\gamma}(\sigma^{-1}). \tag{A.8}$$

The difference between $\mathscr{D}_N(w)$ and $(\zeta'/\zeta)'(1+w)$ consists of higher prime powers and corrections at primes dividing N, and is analytic near zero. Hence

$$\mathscr{D}_N(2\sigma) = \frac{1}{4\sigma^2} + O_N(1). \tag{A.9}$$

For the model, factor $P_N(w) = \zeta(1+w)R_N(w)$, where

$$R_N(w) = \prod_{p\mid N}(1-p^{-1-w})\prod_{\substack{p>2\\ p\nmid N}}\left(1+\frac{2p^{-w}-p^{-2w}}{p(p-2)}\right).$$

This product is analytic near zero, and its Euler factors give $\mathfrak{S}(N)R_N(0) = 1$. Consequently $\mathfrak{S}(N)P_N(2\sigma) = 1/(2\sigma) + O_N(1)$. Using $|\mathcal{X}_1(i\gamma)| = 1$ and $\zeta(1-2\sigma) = -1/(2\sigma) + O(1)$ in (2.12), we obtain

$$\mathscr{K}_N(s,\bar{s}) = -\frac{1}{4\sigma^2} + O_{N,\gamma}(\sigma^{-1}). \tag{A.10}$$

The quadratic singularities of the diagonal and model cancel, so

$$\mathscr{E}_N(s,\bar{s}) = \frac{C_{N,\gamma}}{\sigma^2} + O_{N,\gamma}(\sigma^{-1}).$$

Take $\sigma = 2c/\log N = 2\alpha_{N,c}$. For $c < \log N/24$, the distance to the strip boundary is $\sigma/2$. The general-strip version of (5.8) gives

$$\|T_{N,c}\| \geqslant \frac{4\sigma|\mathscr{E}_N(s,\bar{s})|}{|1+s|^2}.$$

Since $c/\sigma = \log N/2$, the residual expansion proves the stated lower limit and hence (A.7). Since δ_N is fixed, $\mathrm{R}(N,c)$ fails for sufficiently small c. Likewise, a zero-free width $0 < \eta \leqslant 1/4$ must satisfy

$$\frac{z_N^{\mathrm{obs}}}{c} \leqslant \frac{\delta_N}{2\sqrt{\eta(1-\eta)}}, \qquad \eta \leqslant \frac{\delta_N^2 c^2}{3(z_N^{\mathrm{obs}})^2}.$$

Here the last inequality uses $1-\eta \geqslant 3/4$. This proves the proposition. □

No positive width is uniform in this fixed-modulus limit. This does not decide the eventual validity of R or S with c fixed before N tends to infinity.

Appendix B. A numerical experiment at $N = 30030$

We recover the number of unordered Goldbach representations of 30030 from character integrals, exact finite corrections, and an error interval containing a unique integer, and compare the calculation with the classical Hardy–Littlewood integral prediction. This route uses finite prime and prime-power data for corrections and error checks, without requiring an enumeration of all Goldbach pairs. The count is unconditional; its Hardy interpretation and the finite spectral picture assume GRH(N).

Throughout, $N = 30030 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and $\varphi(N) = 5760$. The counting moments use all 5760 characters; the Hardy residual uses the 2880 odd characters.

B.1. **A wide weight and exact finite corrections.** A wider weight than ϕ is needed to count all representations. Reciprocal logarithms remove the von Mangoldt weights, with finite corrections for proper prime powers and the cutoff near the endpoints. Set $A = 16$ and $L = 1000$. We use the cutoff

$$\eta(x) = P\left(\frac{x - A}{L - A}\right) P\left(\frac{N - x - A}{L - A}\right),$$

where $P(t) = 0$ for $t \leqslant 0$, $P(t) = 1$ for $t \geqslant 1$, and

$$P(t) = \sum_{j=7}^{13} \binom{13}{j} t^j (1-t)^{13-j} \qquad (0 < t < 1).$$

With zero extension off the support, put

$$W(u) = \frac{\sqrt{1-u}\,\eta(Nu)}{\sqrt{\log(Nu)\log(N(1-u))}}, \qquad f(u) = \frac{W(u)}{\sqrt{u}},$$

and define

$$\mathcal{J} = \frac{1}{N} \sum_{\substack{n+m=N \\ (nm,N)=1}} \Lambda(n)\Lambda(m) f(n/N) f(m/N).$$

Let $r_G(N)$ count unordered prime pairs. Write $\mathcal{E} = \sum_{p \leqslant q,\ p+q=N,\ p<L} (1 - \eta(p)^2)$, with p, q prime, and let $\mathcal{C}_{\mathrm{pp}}$ be the ordered sum of $\eta(p^k)^2/(k\ell)$ over $p^k + q^\ell = N$, $(pq, N) = 1$, $k, \ell \geqslant 1$, $\max(k, \ell) \geqslant 2$.

Proposition B.1 (The wide-weight counting identity)**.** *The cutoff η is C^6, symmetric about $N/2$, and equal to 1 on $[L, N - L]$. On its support,*

$$f(u) f(1-u) = \frac{\eta(Nu)^2}{\log(Nu)\log(N(1-u))}.$$

At $N = 30030$, the exact counting identity is

$$r_G(N) = \frac{N}{2}\mathcal{J} - \frac{\mathcal{C}_{\mathrm{pp}}}{2} + \mathcal{E}. \tag{B.1}$$

Proof. Since $P'(t) = 12012t^6(1-t)^6$, its first six derivatives match the constant extensions. The cutoff properties and the formula for f follow. A prime-power pair contributes $\eta(p^k)^2/(k\ell)$ to $N\mathcal{J}$; subtracting $\mathcal{C}_{\mathrm{pp}}$ leaves the weighted ordered prime pairs. Every prime pair is coprime to N: otherwise a common prime divisor would force $N = 2p$. Also, $N/2 = 15015$ is not a prime power. Division by two therefore counts unordered pairs, and $\mathcal{E}$ restores the missing edge weights. This proves the proposition. □

Exact rational evaluation of the 75 edge pairs and 26 unordered proper-prime-power pairs gives

$$\mathcal{E} = 44.35776816334822525\ldots, \qquad \mathcal{C}_{\mathrm{pp}}/2 = 10.16758713116445818\ldots.$$

B.2. **Character integrals and their numerical evaluation.** Using the Mellin convention (2.7), we evaluate the character integrals

$$I_\chi = \frac{1}{2\pi}\int_{\mathbb{R}} N^{1+it}\widetilde{W}(1+it)\frac{L'}{L}(3/2+it,\chi)\,\mathrm{d}t.$$

We also use the diagonal moment

$$\mathcal{D} = \sum_{\substack{1\leqslant n<N\\(n,N)=1}} a_n^2,$$

where the coefficients are

$$a_n = \frac{\Lambda(n)}{\sqrt{n}}W(n/N).$$

Lemma B.2 (Character moments on the counting line)**.** *The integrals converge absolutely and satisfy*

$$I_\chi = -\sum_{1\leqslant n<N} a_n\chi(n),$$
$$\mathcal{J} = \frac{1}{5760}\sum_{\chi}\chi(-1)|I_\chi|^2, \quad \mathcal{D} = \frac{1}{5760}\sum_{\chi}|I_\chi|^2. \tag{B.2}$$

All characters, including the principal character, occur; no GRH or spectral hypothesis is required.

Proof. The line $\Re(1/2+s) = 3/2$ gives absolute convergence of L'/L. The C^6 Mellin decay permits termwise inversion, yielding the finite sum. Orthogonality selects $n+m = N$ in $\mathcal{J}$ and $n = m$ in $\mathcal{D}$, since $0 < n, m < N$ on the support. For $n+m = N$, $a_n a_m = N^{-1}\Lambda(n)\Lambda(m)f(n/N)f(m/N)$, proving the normalization. This proves the lemma. □

We integrate with mesh $h = 0.1$ and height $T = 1600$, using full weights at the two outer endpoints. A Fourier transform of the weight in the logarithmic variable, sampled on 2^{20} points over a period $2\pi/h$, supplies the Mellin values. The Hurwitz formula for L and L' uses 24 Euler–Maclaurin corrections and a height-dependent cutoff chosen from the remainder bound. Phase recurrences are reset every 32 heights. Character transforms on cyclic groups of orders $2, 4, 6, 10, 12$ give all 5760 values; conjugation supplies negative heights. The integrated moments are

$$\widehat{\mathcal{J}} \approx 0.05799599193959489, \qquad \widehat{\mathcal{D}} \approx 0.29331844084848136.$$

The finite identity in (B.2) will be used to check their error, without presupposing a representation count.

B.3. **The main integral, residual, and integer enclosure.** The classical Hardy–Littlewood prediction and the wide-weight main integral are, respectively,

$$\mathfrak{S}(N)\int_2^{N/2}\frac{\mathrm{d}x}{\log x\log(N-x)}\approx 908.1405628685,$$
$$\mathcal{A}:=\mathfrak{S}(N)\int_A^{N/2}\frac{\eta(x)^2\,\mathrm{d}x}{\log x\log(N-x)}\approx 851.831147718741.$$

These use 65-digit quadrature split at the cutoff transition; C_2 is evaluated by an accelerated zeta product, removing primes up to 97 and retaining logarithmic terms through order 60. The wide-weight main term with finite corrections is $\mathcal{A}-\mathcal{C}_{\mathrm{pp}}/2+\mathcal{E}\approx 886.021328750925$. Put $c_f=\int_0^1 f(u)f(1-u)\,\mathrm{d}u$ and $\mathcal{B}=\mathfrak{S}(N)c_f-\mathcal{J}$. Since $\mathcal{A}=N\mathfrak{S}(N)c_f/2$, the identity (B.1) also reads

$$r_G(N)=\mathcal{A}-\frac{N}{2}\mathcal{B}-\frac{\mathcal{C}_{\mathrm{pp}}}{2}+\mathcal{E}.$$

The integrated moments give the residual contribution $-N\widehat{\mathcal{B}}/2\approx 18.978671254276$, and hence

$$\begin{aligned}\widehat{r}_G&\approx 851.831147718741+18.978671254276\\&\qquad-10.167587131164+44.357768163348\\&\approx 905.0000000052.\end{aligned}\tag{B.3}$$

This displays the correction to the main integral. To turn the numerical value into an exact count, we bound the error in the character moment.

Lemma B.3 (From moment error to a counting interval)**.** *Form $\widehat{\mathcal{J}}$ and $\widehat{\mathcal{D}}$ from approximations $\widehat{I}_\chi$ by* (B.2)*. If $\varepsilon_I^2\geqslant 5760^{-1}\sum_\chi|I_\chi-\widehat{I}_\chi|^2$ with $\varepsilon_I\geqslant 0$, then*

$$|\mathcal{J}-\widehat{\mathcal{J}}|\leqslant 2\sqrt{\widehat{\mathcal{D}}}\,\varepsilon_I+\varepsilon_I^2.$$

With exact finite corrections, set

$$\widehat{r}_G=\frac{N}{2}\widehat{\mathcal{J}}-\frac{\mathcal{C}_{\mathrm{pp}}}{2}+\mathcal{E},\qquad \varepsilon_r=\frac{N}{2}(2\sqrt{\widehat{\mathcal{D}}}\,\varepsilon_I+\varepsilon_I^2).$$

Then $r_G(N)\in[\widehat{r}_G-\varepsilon_r,\widehat{r}_G+\varepsilon_r]\cap\mathbb{Z}$.

Proof. Writing $e_\chi=I_\chi-\widehat{I}_\chi$ and using $|\chi(-1)|=1$, the difference of the moments is bounded by the average of $2|\widehat{I}_\chi||e_\chi|+|e_\chi|^2$. Cauchy–Schwarz gives the stated bound; (B.1) and integrality give the enclosure. The term $\mathcal{A}$ cancels, so its quadrature error does not enter this interval. This proves the lemma. □

For the error check, the 3280 nonzero coefficients a_n are enclosed using rational bounds for logarithms and square roots. Discrete logarithms to the roots $(2,2,3,2,2)$ modulo $(3,5,7,11,13)$ place them in the character-transform array. Rational enclosures of the

roots of unity and each transform stage, compared with the stored integral approximations, give

$$\left(\frac{1}{5760}\sum_{\chi}|I_\chi-\widehat{I}_\chi|^2\right)^{1/2}<\varepsilon_I=1.68935\times10^{-5}.$$

This finite-transform check bounds the total integral error, including quadrature, tails, L-evaluation and roundoff. It yields $\varepsilon_r\approx0.274758682716$; outward rounding gives

$$[904.725241,\ 905.274759]\cap\mathbb{Z}=\{905\}. \tag{B.4}$$

Thus the integral value 905.0000000052 determines the exact integer 905, compared with the Hardy–Littlewood prediction 908.1405628685. As a final check, direct rational summation in (B.1) and a separate prime enumeration both give 905 unordered pairs (1810 ordered pairs). This enumeration is a check on the count already certified from the integrals and finite corrections.

B.4. **The Hardy interpretation and the spectral picture.** We now take $c=1$ in Appendix A, as in the main text. Here $\alpha_{N,1}=1/\log N\approx0.09699366$, so the Hardy line $\sigma_*=1/9$ lies between $\alpha_{N,1}$ and $\beta=1/8$. Under GRH the same operator $T_{N,1}$ realizes the wide-weight residual; changing the weight changes its testing vector.

Proposition B.4 (The Hardy residual for the wide weight)**.** *Assume* GRH(N) *at* $N=30030$. *The wide-weight residual has the representation*

$$\mathcal{B}=\langle T_{N,1}v_W,v_W\rangle,$$

where v_W *is the Riesz vector of the bounded functional on* $\mathcal{H}_{N,1}$ *defined by*

$$\ell_W(h)=\frac{1}{2\pi i}\int_{(1/9)}N^s\widetilde{W}(s)(1+s)h(s)\,\mathrm{d}s.$$

Proof. The real weight f is C^6 with compact support in $(0,1)$. Define C_f, J_f, and H_f by (7.4), with f in place of ϕ. At integer arguments, they satisfy $C_f(k)=0$ for $k\geqslant1$, $J_f(k)=0$ for $k\geqslant2$, and $J_f(1)=c_f$; integration also gives $\widetilde{H}_f(1)=0$. These identities and C^6 Mellin decay justify model inversion and boundedness of ℓ_W. Applying the odd-character argument of Section 7 gives the quadratic identity with the unchanged residual kernel. This proves the proposition. □

The principal-character integral remains on its absolute-convergence line; this identification does not invoke Hypothesis R. For the finite spectral model, use 2562 nodes on the two boundary lines at height 40 with mesh 1/16. The positive trapezoidal weights w_i have half weight at the endpoints and no $1/(2\pi)$ factor. The finite spectral function is

$$X_{30030}(q):=\det\left(I-\frac{4q(1-q)}{\delta_N^2}C^2\right),$$

where $C=(C_{ij})$ is the matrix with entries

$$C_{ij}=\sqrt{w_i}\,\frac{\mathscr{E}_N(s_i,\overline{s_j})}{(1+s_i)(1+\overline{s_j})}\sqrt{w_j}.$$

Proposition B.5 (Zeros of the finite spectral model)**.** *Let $\lambda_1, \dots, \lambda_{2562}$ be the real eigenvalues of C. Then*

$$X_{30030}(q) = \prod_{j=1}^{2562} \left(1 - \frac{4\lambda_j^2}{\delta_N^2} q(1-q)\right), \qquad X_{30030}(1-q) = X_{30030}(q).$$

If $\|C\| \leqslant \delta_N$, every complex zero has real part $1/2$.

Proof. Unitary diagonalization gives the product and symmetry. For $\lambda_j \neq 0$, the zeros satisfy $q(1-q) = \delta_N^2/(4\lambda_j^2)$, whose roots have real part $1/2$ when $|\lambda_j| \leqslant \delta_N$. A zero eigenvalue contributes the constant factor one. This proves the proposition. □

Boundary L'/L values use the Hurwitz formula and character FFT with 64 direct terms and 24 Euler–Maclaurin corrections. Extracting zeta factors accelerates the prime corrections in $\mathscr{D}_N$ and P_N. At prime cutoff 2×10^5, the extreme eigenvalues are approximately -0.353421 and 0.357462, with $\|C\|/\delta_N \approx 0.826864 < 1$ and $\delta_N = 0.432310239587548\ldots$. Doubling the cutoff from 10^5 changes either endpoint by less than 3×10^{-8}. The computed finite-model zeros therefore lie on the critical line.

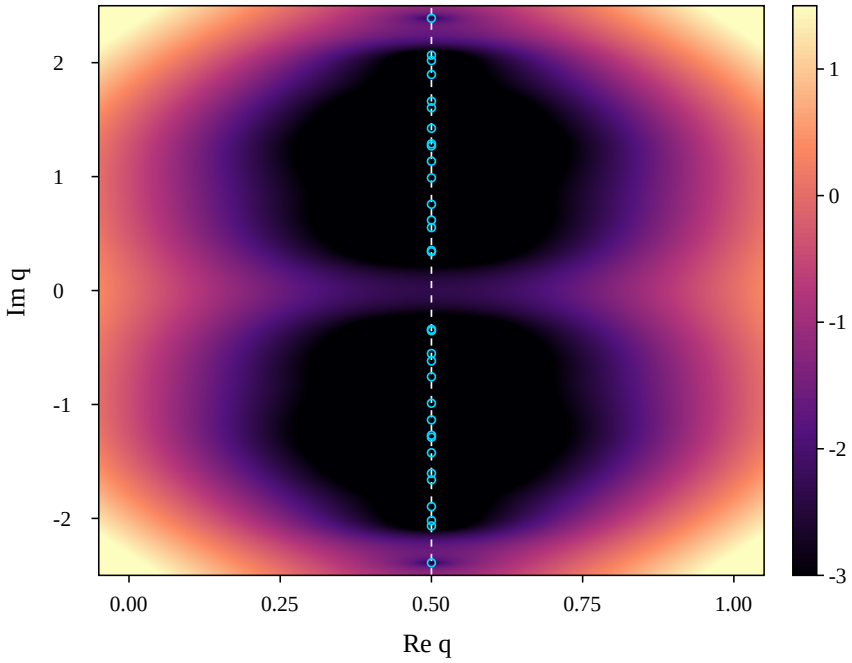


Figure B.1. The finite spectral product $\log_{10}|X_{30030}(q)|$; the circles mark its displayed zeros.

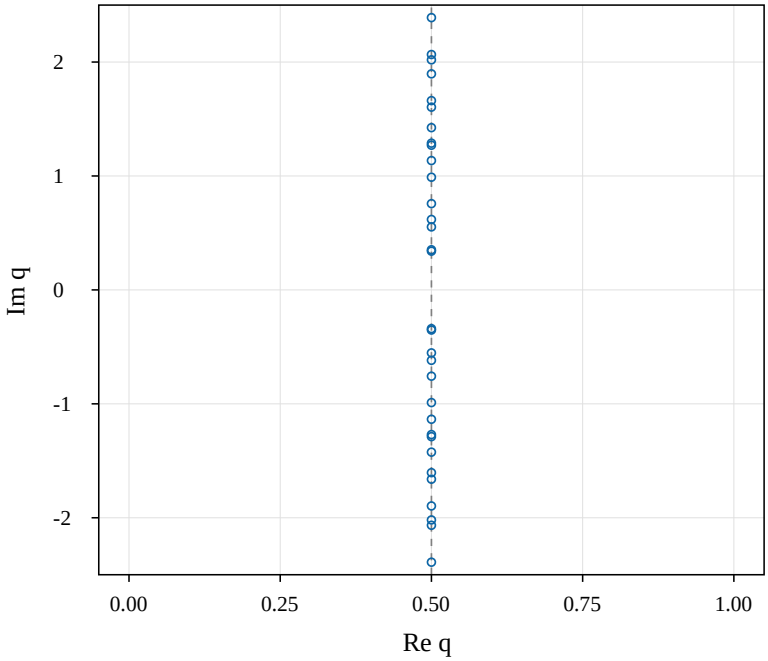


Figure B.2. The displayed zeros of $X_{30030}(q)$. Both figures use the full computed eigenvalue list.

Cutoff stability and pointwise high-precision checks do not bound the full operator's truncation or discretization error. The figures illustrate the finite spectral model; they do not certify Hypothesis R for $T_{30030,1}$.

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