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Ramanujan and the Modular j-Invariant

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Abstract. A new infinite product t_n was introduced by S. Ramanujan on the last page of his third notebook. In this paper, we prove Ramanujan's assertions about t_n by establishing new connections between the modular j-invariant and Ramanujan's cubic theory of elliptic functions to alternative bases. We also show that for certain integers n, t_n generates the Hilbert class field of $\mathbb{Q}(\sqrt{-n})$. This shows that t_n is a new class invariant according to H. Weber's definition of class invariants.

1 Introduction

Except for four entries, the last two pages in Ramanujan's third notebook, pages 392 and 393 in the pagination of [21, vol. 2], are devoted to values of the modular j-invariant. Recall [14, p. 81], [15, p. 224] that the invariants $J(\tau)$ and $j(\tau)$, for $\tau \in \mathbb{H} := \{\tau : \operatorname{Im}\tau > 0\}$, are defined by

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} \quad \text{and} \quad j(\tau) = 1728\,J(\tau), \tag{1.1}$$

where

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau), \tag{1.2}$$

$$g_2(\tau) = 60 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m\tau + n)^{-4},$$

and

$$g_3(\tau) = 140 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m\tau + n)^{-6}.$$

Furthermore, the function $\gamma_2(\tau)$ is defined by [15, p. 249]

$$\gamma_2(\tau) = \sqrt[3]{j(\tau)}, \tag{1.3}$$

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where that branch which is real when τ is purely imaginary is chosen.

At the top of page 392 in [21, vol. 2], which inexplicably is printed upside down, Ramanujan defines $J := J_n$ and $u := u_n$ by

$$J_n = \frac{1 - 16\alpha_n(1-\alpha_n)}{8\big(4\alpha_n(1-\alpha_n)\big)^{1/3}} \quad \text{and} \quad J_n = \frac{\sqrt[3]{4u_n}}{2^3}, \tag{1.4}$$

where n is a natural number. Here, if, as usual in the theory of elliptic functions, k, $0 < k < 1$, denotes the modulus, then $\sqrt{\alpha_n} := k_n$ is the *singular modulus*. To help identify J_n, first recall Ramanujan's definition of the *class invariant* G_n,

$$G_n := 2^{-1/4} q_n^{-1/24} \prod_{j=1}^{\infty} (1 + q_n^{2j-1}), \tag{1.5}$$

where n is a positive rational number and $q_n = \exp(-\pi\sqrt{n})$. In the notation of Weber [26], $G_n =: 2^{-1/4}\mathfrak{f}(\sqrt{-n})$. Furthermore [4],

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24}. \tag{1.6}$$

Now, from (1.4) and (1.6), we easily find that

$$J_n = \frac{1}{8} G_n^8 (1 - 4G_n^{-24}). \tag{1.7}$$

We now identify J_n with γ_2. First, following Ramanujan, set

$$f(-q) := \prod_{n=1}^{\infty} (1 - q^n) = q^{-1/24}\eta(\tau), \tag{1.8}$$

where $q = \exp(2\pi i \tau)$, $\tau \in \mathbb{H}$, and $\eta(\tau)$ denotes the Dedekind eta-function. From Cox's text [15, p. 257, Theorem 12.17],

$$\gamma_2(\tau) = \frac{\mathfrak{f}_2(\tau)^{24} + 16}{\mathfrak{f}_2(\tau)^8}, \tag{1.9}$$

where $\mathfrak{f}_2(\tau) = \sqrt{2} q^{1/24} f(-q^2)/f(-q)$ and $q = e^{2\pi i \tau}$. Hence, by (1.9),

$$\gamma_2(\tau) = 2^8 \frac{q^{2/3} f^{16}(-q^2)}{f^{16}(-q)} + \frac{f^8(-q)}{q^{1/3} f^8(-q^2)}. \tag{1.10}$$

Setting $\tau = (3 + \sqrt{-n})/2$, we deduce from (1.5) and (1.10) that

$$\gamma_2\left(\frac{3 + \sqrt{-n}}{2}\right) = \frac{2^8 - 2^6 G_n^{24}}{2^4 G_n^{16}} = -4G_n^8 (1 - 4G_n^{-24}). \tag{1.11}$$

Hence, from (1.7), (1.11), and (1.3),

$$J_n = -\frac{1}{32}\gamma_2\left(\frac{3+\sqrt{-n}}{2}\right) = -\frac{1}{32}\sqrt[3]{j\left(\frac{3+\sqrt{-n}}{2}\right)}. \tag{1.12}$$

Ramanujan records fifteen values of J_n, $n \equiv 3 \pmod 4$, although some are not given very explicitly. He also records factors of certain polynomials in J_n. A complete discussion of this material can be found in Berndt's book [2, Chap. 34, Sect. 11].

Up to this point, all of Ramanujan's claims about the j-invariant are well known, but on page 393, Ramanujan records the following interesting result.

Theorem 1.1 *For $q = \exp(-\pi\sqrt{n})$, define*

$$t := t_n := \sqrt{3}q^{1/18}\frac{f(q^{1/3})f(q^3)}{f^2(q)}. \tag{1.13}$$

Then

$$t_n = \left(2\sqrt{64J_n^2 - 24J_n + 9} - (16J_n - 3)\right)^{1/6}. \tag{1.14}$$

Ramanujan then gives a table of polynomials satisfied by t_n, for five values of n.

Theorem 1.2 *For the values of n given below, we have the following table of polynomials $p_n(t)$ satisfied by t_n.*

n	$p_n(t)$
11	$t-1$
35	t^2+t-1
59	t^3+2t-1
83	t^3+2t^2+2t-1
107	t^3-2t^2+4t-1

The simplicity of these polynomials is remarkable, since the corresponding well-known polynomials (of the same degree) satisfied by J_n are considerably more complicated, especially in the latter three instances.

The form of Theorem 1.1 suggests hitherto unknown connections between the j-invariant and Ramanujan's cubic theory of elliptic functions to alternative bases developed by Berndt, S. Bhargava, and F. G. Garvan [3]. In Section 2, we offer some of these connections and show how they can be exploited to calculate particular values of $\gamma_2(\tau)$. We also use this connection to give a proof of Theorem 1.1 in Section 3.

In Section 4, we derive the polynomials in Theorem 1.2 from Theorem 1.1 and results in papers by A. G. Greenhill [17], [18]. In view of (1.14), there is clearly a relation between t_n and the Hilbert class field of $\mathbb{Q}(\sqrt{-n})$. We will show that in fact t_n generates the Hilbert class field of $\mathbb{Q}(\sqrt{-n})$ for certain integers n. In this final section we also provide some historical remarks on what Ramanujan likely knew about the j–invariant from the literature.

2 Connections with Ramanujan's Alternative Cubic Theory

Recall that the ordinary hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$$ {}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, $$

where $(a)_k = (a)(a+1)\cdots(a+k-1)$, and $|z| < 1$. In the alternative cubic theory, the base q is defined by

$$ q := q_3 := \exp\left(-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; 1-\alpha)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; \alpha)} \right), \quad 0 < \alpha < 1. \tag{2.1} $$

From the fundamental inversion formula for the cubic theta functions [3, Lemma 2.9], we find that

$$ \alpha := \alpha(q) = \frac{c^3(q)}{a^3(q)}, \tag{2.2} $$

where

$$ a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{2.3} $$

and

$$ c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}. \tag{2.4} $$

Following Ramanujan, define $M(q)$ and $N(q)$ by

$$ g_2(\tau) = \frac{4\pi^4}{3} M(q) \quad \text{and} \quad g_3(\tau) = \frac{8\pi^6}{27} N(q), $$

where $q = \exp(2\pi i\tau)$, $\tau \in \mathbb{H}$. Then, it was shown by Berndt, Bhargava, and Garvan [3, Theorems 4.2–4.5] (see also [11]) that

$$ M(q) = z^4(1 + 8\alpha), \quad N(q) = z^6(1 - 20\alpha - 8\alpha^2), \tag{2.5} $$

$$ M(q^3) = z^4\left(1 - \frac{8}{9}\alpha\right), \quad \text{and} \quad N(q^3) = z^6\left(1 - \frac{4}{3}\alpha + \frac{8}{27}\alpha^2\right), \tag{2.6} $$

where

$$ z := {}_2F_1(\tfrac{1}{3}, \tfrac{2}{3}; 1; \alpha). $$

It follows from (1.1) and (1.2) that

$$j(\tau) = 1728\frac{M^3(q)}{M^3(q) - N^2(q)}. \tag{2.7}$$

Now, employing (2.5) and (2.6) in (2.7), we deduce that, respectively,

$$j(\tau) = 27\frac{(1+8\alpha)^3}{\alpha(1-\alpha)^3} \tag{2.8}$$

and

$$j(3\tau) = 27\frac{\big(1+8(1-\alpha)\big)^3}{(1-\alpha)\alpha^3}. \tag{2.9}$$

Alternatively, we show how to derive (2.9) from (2.1) and (2.8). In (2.1), set $q = \exp(2\pi i\tau)$. Then, replacing α by $1-\alpha$ in (2.1), we find that

$$\exp\left(-\frac{2\pi}{\sqrt{3}}\,\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\alpha)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\alpha)}\right) = \exp\big(-2\pi i/(3\tau)\big). \tag{2.10}$$

It is well known that $j(\tau)$ is a modular function on the full modular group [14, p. 84], [15, p. 221]. In particular,

$$j(-1/\tau) = j(\tau). \tag{2.11}$$

Hence, by (2.11), (2.10), and (2.8),

$$j(3\tau) = j\left(-\frac{1}{3\tau}\right) = 27\frac{\big(1+8(1-\alpha)\big)^3}{(1-\alpha)\alpha^3},$$

and thus (2.9) has been established again.

Identity (2.8) is truly a cubic analogue of the well-known identity [8, p. 115, Theorem 4.4]

$$j(\tau) = 256\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda)^2},$$

where

$$\lambda := \lambda(q) = q\frac{\psi^4(q^2)}{\varphi^4(q)},$$

with

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} \quad \text{and} \quad \varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q = e^{\pi i\tau}.$$

We remark here that the discovery of identities (2.8) and (2.9) has recently motivated a proof of Ramanujan's cubic transformation via Goursat's transformation formulas. For more details, see [11].

Next, we use (2.8) and (2.9) to calculate some values for $\gamma_2(\tau)$. We emphasize that we could utilize (1.11) in our procedures. However, our calculations avoid the need for the values of class invariants. Instead, we employ some simple modular equations from the alternative cubic theory.

Examples

$$\gamma_2(3\sqrt{-1}) = 2^2\sqrt{3}(2+3\sqrt{3})(2\sqrt{3}+1)(2+\sqrt{3})^{2/3}, \tag{2.12}$$

$$\gamma_2(3\sqrt{-2}) = 2^2\cdot 5\cdot(3\sqrt{6}-1)(3\sqrt{6}+5)(5+2\sqrt{6})^{2/3}, \tag{2.13}$$

$$\gamma_2(\sqrt{-5/3}) = 3\sqrt{5}(3\sqrt{5}-2)(7-2\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^{10/3}, \tag{2.14}$$

$$\gamma_2(\sqrt{-15}) = 3\sqrt{5}(3\sqrt{5}+2)(7+2\sqrt{5})\left(\frac{\sqrt{5}+1}{2}\right)^{10/3}, \tag{2.15}$$

$$\begin{aligned}\gamma_2(\sqrt{-7/3}) = {}&\frac{4}{9}(45-26\sqrt{3}+2\sqrt{21})(34+13\sqrt{7})\\ &\cdot(142-78\sqrt{3}+6\sqrt{21}-13\sqrt{7})^{1/3},\end{aligned} \tag{2.16}$$

$$\begin{aligned}\gamma_2(\sqrt{-21}) = {}&\frac{4}{9}(45+26\sqrt{3}-2\sqrt{21})(34+13\sqrt{7})\\ &\cdot(142+78\sqrt{3}-6\sqrt{21}-13\sqrt{7})^{1/3}.\end{aligned} \tag{2.17}$$

Remark Note that if we use (1.11) to evaluate $\gamma_2(\tau)$, we obtain a product of two rather "complicated" algebraic numbers. On the other hand, our new method produces much simpler expressions for $\gamma_2(\tau)$.

We will provide proofs of (2.12) and (2.13) and say a few words about the proofs of identities (2.14)–(2.17).

Proof of (2.12) We first recall the definition of a modular equation of degree n in the cubic theory [3]. Suppose that, for some positive integer n,

$$\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\beta)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\beta)} = n\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\alpha)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\alpha)}. \tag{2.18}$$

Then a *modular equation of degree n* in Ramanujan's alternative cubic theory is a relation between α and β induced by (2.18). As in the classical theory, we say that β *has degree n over* α.

If n is a positive rational number, then the singular modulus $\sqrt[3]{\alpha_n}$ in the alternative cubic theory is that unique positive number such that

$$\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\alpha_n)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\alpha_n)}=\sqrt{n}. \tag{2.19}$$

Now suppose that $\beta = 1-\alpha$. Then, from (2.18),

$$\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\beta)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\beta)}=\sqrt{n},$$

i.e., by (2.19), $\beta=\alpha_n$, the cube of the singular modulus.

We now demonstrate how this principle can be used to calculate values of $\gamma_2(\tau)$.

Suppose that β has degree 3 over α in the cubic theory. Then [3, Lemma 7.4]

$$\beta^{1/3}=\frac{1-\sqrt[3]{1-\alpha}}{1+2\sqrt[3]{1-\alpha}}. \tag{2.20}$$

Assuming that $\beta=1-\alpha$, we conclude, as above, that $\beta=\alpha_3$. Hence, from (2.20),

$$\alpha_3^{1/3}=\frac{1-\alpha_3^{1/3}}{1+2\alpha_3^{1/3}},$$

which implies that

$$\alpha_3^{1/3}=\frac{\sqrt{3}-1}{2}. \tag{2.21}$$

From (2.1), note that $q=e^{-2\pi}$. Thus, by (2.8) and a straight-forward calculation, we find that

$$\gamma_2(\sqrt{-1})=12,$$

which is well known [15, p. 261]. From (2.9) and (2.21), we also find that

$$\gamma_2(3i)=\frac{24(19-6\sqrt{3})}{\sqrt[3]{18-6\sqrt{3}}(3\sqrt{3}-5)}.$$

Upon simplifications, we arrive at (2.12).

Next, assume that β has degree 2 over α and degree 3 over $1-\alpha$. Then, from (2.18) we can deduce that

$$\frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1;1-\beta)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1;\beta)}=\sqrt{6}.$$

Thus, $\beta=\alpha_6$. Since β has degree 2 over α, from [3, Theorem 7.1],

$$(\alpha\beta)^{1/3}+\{(1-\alpha)(1-\beta)\}^{1/3}=1. \tag{2.22}$$

Since β has degree 3 over $1-\alpha$, by (2.20),

$$\beta^{1/3} = \frac{1-\alpha^{1/3}}{1+2\alpha^{1/3}},$$

from which it follows that

$$\alpha^{1/3} = \frac{1-\beta^{1/3}}{1+2\beta^{1/3}}. \tag{2.23}$$

If $x = \alpha_6^{1/3}$, substitution of (2.23) into (2.22) yields, after a lengthy calculation,

$$(1+x+x^2)(1-8x+10x^2) = 0. \tag{2.24}$$

Checking numerically the two real roots, we find that the appropriate root of (2.24) is

$$\alpha_6^{1/3} = \frac{4-\sqrt{6}}{10}. \tag{2.25}$$

From (2.1), $q = e^{-2\pi\sqrt{2}}$, and so from (2.8), we easily find that $\gamma_2(\sqrt{2}i) = 20$, which is well-known [15, p. 261]. Next, substituting (2.25) into (2.9) and simplifying, we deduce (2.13).

Unlike (2.13), the proofs of identities (2.14)–(2.17) require only a single modular equation in each case. For (2.14) and (2.15), we use Ramanujan's modular equation of degree 5 [3, Theorem 5.6], namely,

$$(\alpha\beta)^{1/3} + \{(1-\alpha)(1-\beta)\}^{1/3} + 3\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6} = 1,$$

and show that

$$\alpha_5 = \frac{1}{2}\left(1 - \frac{11\sqrt{5}}{25}\right).$$

This corresponds to $q = e^{-2\pi\sqrt{5/3}}$ and gives us the value for $\gamma_2(\sqrt{-5/3})$. Next, observe that

$$1-\alpha_5 = \frac{1}{2}\left(1 + \frac{11\sqrt{5}}{25}\right)$$

differs from the value α_5 by only a sign change in $\sqrt{5}$. Since (2.9) relates $\gamma_2(\sqrt{-15})$ and $1-\alpha_5$ in exactly the same way as $\gamma_2(\sqrt{-5/3})$ and α_5, we can simply replace $\sqrt{5}$ by $-\sqrt{5}$ in the expression for $\gamma_2(\sqrt{-5/3})$ to obtain $\gamma_2(\sqrt{-15})$.

For the proofs of identities (2.16) and (2.17), we use a new modular equation of degree 7 recently discovered by Chan and W.-C. Liaw [13]. Suppose β has degree 7 over α. Let $w = (x+y-1)(x+y+1)$ and $s = xy$, where $x = \sqrt{\alpha\beta}$ and $y = \sqrt{(1-\alpha)(1-\beta)}$. Then

$$w^4 - 2^4\cdot 3\cdot 7sw^3 + (2^8\cdot 3\cdot 7^2s^2 - 2^3\cdot 3^5\cdot 5s)w^2 - (2^{12}\cdot 7^3s^3 + 2^7\cdot 3^5\cdot 11s^2 + 3^7\cdot 13s)w - 3^5s(2^6s+3^2)^2 = 0.$$

From the modular equation above, we deduce that

$$\alpha_7 = \frac{1}{2} - \frac{13}{36}\sqrt{3} + \frac{1}{36}\sqrt{21}.$$

This value corresponds to $q = e^{-2\pi\sqrt{7/3}}$ and gives us the value for $\gamma_2(\sqrt{-7/3})$. The value for $\gamma_2(\sqrt{-21})$ can be obtained by replacing $\sqrt{3}$ by $-\sqrt{3}$ in the expression for $\gamma_2(\sqrt{-7/3})$.

3 Theorem 1.1 and the Alternative Cubic Theory

In this section, we will derive the following q-identity from the alternative cubic theory and deduce Theorem 1.1 as a corollary of this identity.

Theorem 3.1 *Set*

$$h(q) = \frac{f^{12}(-q^3)}{q f^6(-q) f^6(-q^9)},$$

where $f(-q)$ is given as in (1.8). Then

$$h(q) - 27h(q)^{-1} = \gamma_2(3\tau) + 6, \tag{3.1}$$

where $q = \exp(2\pi i \tau)$.

Proof of Theorem 3.1 From [3, Lemma 5.1] or [10],

$$c(q^3) = 3q \frac{f^3(-q^9)}{f(-q^3)}. \tag{3.2}$$

By combining Entry 1(v) in Chapter 20 of Ramanujan's second notebook [1, p. 346] with a Lambert series representation for $a(q)$ found by both Ramanujan and the Borweins [9], we find that

$$a(q^3) = \frac{f^3(-q) + 3q f^3(-q^9)}{f(-q^3)}. \tag{3.3}$$

(For several historical remarks on this Lambert series, see [3, p. 4167].) Hence, by (3.2) and (3.3), we deduce that

$$\frac{1}{\alpha^{1/3}(q^3)} = \frac{f^3(-q)}{3q f^3(-q^9)} + 1 = \frac{s}{3} + 1, \tag{3.4}$$

where

$$s = \frac{f^3(-q)}{q f^3(-q^9)}. \tag{3.5}$$

Substituting (3.4) into (2.8) (with q replaced by q^3), we find that

$$\gamma_2(3\tau) = \frac{(s+3)(s+9)(s^2+27)}{s(s^2+9s+27)}. \tag{3.6}$$

On the other hand, if we cube Entry 1(iv) in Chapter 20 of Ramanujan's second notebook with q replaced by q^3 [1, p. 345], we find that

$$h(q) = s + 9 + \frac{27}{s}, \tag{3.7}$$

with s given by (3.5). Now, from (3.6) and (3.7),

$$\begin{aligned}\gamma_2(3\tau) &= \frac{(s^2+9s+27-3s)(s^2+27+9s-9s)}{s(s^2+9s+27)} \\ &= \frac{(sh(q)+3s)(sh(q)-9s)}{s^2h(q)} \\ &= h(q)-6-27h(q)^{-1},\end{aligned}$$

which is (3.1). This completes the proof of Theorem 3.1.

As a corollary, we deduce Theorem 1.1.

Proof of Theorem 1.1 In view of (1.12) and (1.13), we set $\tau = (3+\sqrt{-n})/6$ in (3.1) to deduce that

$$t_n^6 - 27t_n^{-6} = -32J_n + 6, \tag{3.8}$$

since $h(-e^{-\pi\sqrt{n}/3}) = -27t_n^{-6}$. Solving the quadratic equation (3.8) in t_n^6 yields (1.14) immediately. This completes the proof of Theorem 1.1.

Remark Theorem 3.1 can be proved using the theory of modular forms. From M. Newman's criterion ([20] or [12]), it is easy to show that $h(q)$ generates the function field associated with $\Gamma_0(9)$. On the other hand, $\gamma_2(3\tau)$ is a modular function for the group $\Gamma_0(9)$ [15, p. 250, Proposition 12.3]. Hence, $\gamma_2(3\tau)$ is a rational function of $h(q)$. For detailed computations of identities similar to (3.1), see [12].

4 Calculating the Polynomials in Theorem 1.2

Proof of Theorem 1.2 It is well known that $J_{11} = 1$ [15, p. 261]. Thus, we find that

$$t_{11} = (2\cdot 7 - 13)^{1/6} = 1,$$

as desired.

Secondly, from a paper of W. E. Berwick [6],

$$J_{35} = \sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^4.$$

Hence,

$$\begin{aligned}t_{35} &= \left(2\sqrt{64\cdot 5\left(\frac{\sqrt{5}+1}{2}\right)^8 - 24\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^4 + 9} - \left(16\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^4 - 3\right)\right)^{1/6} \\ &= \left(2\sqrt{7349+3276\sqrt{5}} - 117 - 56\sqrt{5}\right)^{1/6}.\end{aligned}$$

It is easy to verify that if $a^2 - db^2 = c^2$, then

$$\sqrt{a \pm b\sqrt{d}} = \sqrt{\frac{a+c}{2}} \pm \sqrt{\frac{a-c}{2}}. \tag{4.1}$$

Now, since

$$7349^2 - 5 \cdot 3276^2 = 589^2,$$

we find that

$$\sqrt{7349 + 3276\sqrt{5}} = \sqrt{\frac{7349+589}{2}} + \sqrt{\frac{7349-589}{2}} = \sqrt{3969} + \sqrt{3380} = 63 + 26\sqrt{5},$$

by (4.1). Hence,

$$t_{35} = \left(2\left(63 + 26\sqrt{5}\right) - 117 - 56\sqrt{5}\right)^{1/6} = (9 - 4\sqrt{5})^{1/6} = \frac{\sqrt{5}-1}{2}.$$

Hence, t_{35} is a root of $t^2 + t - 1$, and the second result is established.

For $n = 59$, Greenhill [18] showed that u_{59}, defined by (1.4), is a root of the equation

$$u - 392 \cdot 2^{1/3} u^{2/3} + 1072 \cdot 4^{1/3} u^{1/3} - 2816 = 0.$$

Since, by (1.4) and (1.12), $u_{59} = -\gamma_2^3/256$, where $\gamma_2 = \gamma_2(\frac{3+\sqrt{-59}}{2})$, it follows that γ_2 satisfies

$$\gamma_2^3 + 3136\gamma_2^2 + 68608\gamma_2 + 720896 = 0. \tag{4.2}$$

Set $x := t^6 := t_n^6$. From (3.8), we find that

$$\gamma_2 = x - 27x^{-1} - 6. \tag{4.3}$$

Substituting (4.3) into (4.2) and turning to *Mathematica*, we find that

$$\begin{aligned} &x^6 + 3118x^5 + 31003x^4 + 25355x^3 - 837081x^2 + 2273022x - 19683 \\ &\quad = (x^3 + 13x^2 + 115x - 1)(x^3 + 3105x^2 - 9477x + 19683) \\ &\quad = 0. \end{aligned}$$

Numerically checking the roots, we see that x is a root of the first factor above, *i.e.*,

$$t^{18} + 13t^{12} + 115t^6 - 1 = 0.$$

We use *Mathematica* to factor this polynomial and find that

$$\begin{aligned} t^{18} + 13t^{12} + 115t^6 - 1 &= (t^3 + 2t - 1)(t^3 + 2t + 1)(t^6 - 2t^4 + 2t^3 + 4t^2 - 2t + 1) \\ &\quad \times (t^6 - 2t^4 - 2t^3 + 4t^2 + 2t + 1). \end{aligned}$$

Again, numerically calculating the roots, we find that

$$t^3 + 2t - 1 = 0,$$

as claimed by Ramanujan.

For $n = 83$, u_{83} is a root of [17]

$$u - 1740 \cdot 2^{1/3}u^{2/3} + 2000 \cdot 4^{1/3}u^{1/3} - 32000 = 0,$$

and for $n = 107$, u_{107} is a root of [18, p. 405]

$$u - 79 \cdot 80 \cdot 2^{1/3}u^{2/3} - 69 \cdot 800 \cdot 4^{1/3}u^{1/3} - 17 \cdot 16000 = 0.$$

The derivations of $p_{83}(t)$ and $p_{107}(t)$ from the relations above are similar to the case $n = 59$ and we omit the details here. This completes the proof of Theorem 1.2.

Our derivations of $p_n(t)$ for $n = 59, 83,$ and 107 are not satisfactory as they involve Greenhill's polynomials. Greenhill discovered these polynomials using Russell's modular equations and relations between u_n and Weber's class invariants. His computations are laborious. An effective and direct method for evaluating $p_n(t)$ is highly desirable.

Let n be squarefree and set $\omega = (3 + \sqrt{-n})/2$. It is well known that if $K_n = \mathbb{Q}(\sqrt{-n})$, $3 \nmid n$, and $n \equiv 3 \pmod 4$, then $K_n\big(\gamma_2(\omega)\big)$ is the Hilbert class field of K_n. This means that the degree $\Big(K_n\big(\gamma_2(\omega)\big) : K_n\Big) = h_{K_n}$, the class number of K_n. In view of (3.8), we conclude that $K_n\big(\gamma_2(\omega)\big)$ is a subfield of $K_n(t_n)$. Now, the class numbers of K_n for $n = 11, 35, 59, 83$ and 107 are respectively $1, 2, 3, 3,$ and 3. These class numbers coincide with the degrees of $p_n(t)$ and hence, $K_n(t_n) = K_n\big(\gamma_2(\omega)\big)$ for these integers. It is therefore natural to conjecture that for certain integers n, t_n generates the Hilbert class field of K_n. Indeed, we have the following result:

Theorem 4.1 *Let n be squarefree, $n \equiv 11 \pmod{24}$, and suppose that the class number of K_n is odd. Then t_n is a real unit generating the Hilbert class field of K_n.*

In [7, p. 291, Section 8], B. J. Birch relates a function g suggested by A. O. L. Atkin satisfying

$$g^6(\tau) - 27g^{-6}(\tau) = \gamma_2(\tau) + 6, \tag{4.4}$$

where $g(\tau)$ is real and positive when z is purely imaginary. He then established the following [7, p. 292]:

Lemma 4.2 *If $n \equiv 11 \pmod{24}$, then $g^2\big((3 + \sqrt{-n})/2\big)$ generates the Hilbert class field of K_n.*

Birch did not indicate the explicit form of $g(\tau)$, and so no examples were given to verify his results. In [25], H. M. Stark successfully determined an expression for $g(\tau)$. Using Kronecker's limit formula, he computed the first few numerical examples for (4.4).

In view of (3.1) and (4.4), we conclude, by solving $x - 27/x = y - 27/y$, that

$$g^6(\tau) = h(e^{2\pi i\tau/3}) \quad \text{or} \quad \frac{27}{h(e^{2\pi i\tau/3})}.$$

By considering the q-expansion of $h(\tau)$ and $\gamma_2(\tau)$, we conclude that

$$g^6(\tau) = h(e^{2\pi i\tau/3}). \tag{4.5}$$

Combining the identification (4.5) and Lemma 4.2, we conclude that $t_n^2 = 3\Big(g\big((3+\sqrt{-n})/2\big)\Big)^{-2}$ generates the Hilbert class field of K_n. We emphasize here that this result is independent of the condition on the class number of K_n.

To obtain Theorem 4.1, we use the idea given in Birch's paper. We quote the following simple but useful lemma (see [7, Lemma 5]):

Lemma 4.3 *If $K(\theta)$ is a quadratic extension of K with $\theta^2 \in K$, and if $\alpha \in K(\theta)$ with $\alpha^2 \in K$, then either $\alpha \in K$ or $\theta\alpha \in K$.*

Proof of Theorem 4.1 By Söhngen's Theorem [24], t_n is abelian over K_n. Hence, t_n lies in a quadratic extension $K_n(\theta)$, where $\theta^2 \in K_n$. This implies that $t_n \in H_n(\theta)$, where H_n denotes the Hilbert class field of K_n. Now, the hypotheses of Lemma 4.3 are satisfied with $K = H_n$ and $\alpha = t_n$ (note that $\theta^2 \in K_n \subset H_n$). Thus, $t_n \in H_n$ or $\theta t_n \in H_n$.

If $t_n \in H_n$, we are done. Otherwise, suppose $\theta t_n \in H_n$. Let

$$\theta t_n = \zeta, \quad \text{with} \quad \zeta \in H_n.$$

Squaring both sides, we obtain

$$\theta^2 t_n^2 = \zeta^2.$$

Next, since all the quantities above are in H_n, we deduce that

$$N(\theta^2)N(t_n^2) = N(\zeta)^2, \tag{4.6}$$

where N denotes the $H_n|K_n$ norm. Now, $N(t_n^2) = \pm 1$, since t_n is a real unit by Deuring's Theorem [16], and $N(\theta^2) = \theta^{2h}$ ($\theta \in K_n$), where h is the class number of K_n. Since h is odd, we may rearrange (4.6) and deduce that

$$\theta = \pm\frac{N(\zeta)}{\theta^{h-1}} \quad \text{or} \quad \pm\frac{N(\zeta)}{i\theta^{h-1}}.$$

If $\theta = \pm N(\zeta)\theta^{1-h}$, then $\theta \in K_n$, and so $t_n \in H_n$. If $\theta = \pm N(\zeta)(i\theta^{h-1})^{-1}$, then $\theta \in K_n(i)$. This means that we can replace θ by i. But now, if $t_n \in H_n(i)$, then t_n is fixed by complex conjugation since t_n is real. Hence, $t_n \in H_n$. This completes the proof of Theorem 4.1.

Historical Remarks It has often been conjectured that Ramanujan had seen Greenhill's book [19] on elliptic functions. For example, see a letter by K. Ananda Rau and the following commentary in the book by R. A. Rankin and one of the authors [5, pp. 289, 290]. On pages 327–329, Greenhill [19] briefly summarizes much of R. Russell's work on modular equations. There are several modular equations of the type found by Russell in Ramanujan's notebooks, and this is further evidence that Ramanujan had seen both of Russell's papers [22], [23], and therefore also both of Greenhill's papers [17], [18], since all four of these papers appear in volumes 19 and 21 of the *Proceedings of the London Mathematical Society*.

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