

MODULAR FORMS AND k -COLORED GENERALIZED FROBENIUS PARTITIONS

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Dedicated to Professor George E. Andrews on the occasion of his 80th birthday

ABSTRACT. Let k and n be positive integers. Let $c\phi_k(n)$ denote the number of k -colored generalized Frobenius partitions of n and let $C\Phi_k(q)$ be the generating function of $c\phi_k(n)$. In this article, we study $C\Phi_k(q)$ using the theory of modular forms and discover new surprising properties of $C\Phi_k(q)$.

1. INTRODUCTION

A partition π of an integer n is a sequence of non-increasing positive integers which add up to n . We denote the number of partitions of n by $p(n)$. It is known that a partition π of n can be visualized using a Ferrers diagram by representing the positive integer m of the s -th part by m dots on the s -th row. An example showing the pictorial representation of the partition $4 + 4 + 4 + 2$ of the integer 14 is given in Figure 1.

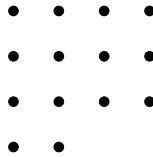


FIGURE 1

From the Ferrers diagram of a partition, we can construct a 2 by d matrix by carrying out the following steps:

Step 1. Remove all the dots lying on the diagonal of the diagram.

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Step 2. Fill the first row of the matrix with entries $r_{1,j}$, where $r_{1,j}$ is the number of dots on the j -th row that are to the right of the diagonal.

Step 3. Fill the second row of the matrix with entries $r_{2,j}$, where $r_{2,j}$ is the number of dots on the j -th column that are below the diagonal.

For example, after Step 1, we obtained Figure 2 from Figure 1. Carrying out Steps 2

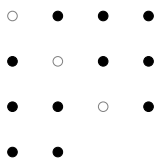


FIGURE 2

and 3, we arrive at the matrix

$$\begin{pmatrix} 3 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}.$$

It is clear that we can always construct a 2 by d matrix from any partition π with d dots along the diagonal of its Ferrers diagram, and the matrix obtained from a partition π using the above procedures is called a Frobenius symbol for the partition π . A Frobenius symbol, by construction, has strictly decreasing entries on each row.

One way to find new functions that are similar to the partition function $p(n)$ is to start with a modified version of the Frobenius symbol. In his 1984 AMS Memoir, G.E. Andrews [2, Section 4] introduced a generalized Frobenius symbol with at most k repetitions for each integer by relaxing the “strictly decreasing” property and allowing at most k -repetitions of each positive integer in each row. Andrews then used the generalized Frobenius symbol to define the generalized Frobenius partition of n . For a generalized Frobenius symbol with entries $r_{i,j}$, $i = 1, 2, 1 \leq j \leq d$, the generalized Frobenius partition of n is given by

$$n = d + \sum_{j=1}^d (r_{1,j} + r_{2,j}).$$

Andrews used the symbol $\phi_k(n)$ to denote the number of such partitions of n . As an example, we observe that $\phi_2(3) = 5$ and these are given by the following generalized Frobenius symbols with at most 2 repetitions on each row:

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that with this definition,

$$\phi_1(n) = p(n).$$

There are at most k -repetitions in each row of a generalized Frobenius symbol. In order to restore the “strictly decreasing” property of a Frobenius symbol from a generalized Frobenius symbol, Andrews colored the repeated parts using “colors” denoted by $1, 2, \dots, k$ and imposed an ordering on these parts as follows:

$$(1.1) \quad 0_1 < 0_2 < \dots < 0_k < 1_1 < 1_2 < \dots < 1_k < 2_1 < 2_2 < \dots < 2_k < \dots$$

Here, we use “ \prec ” to differentiate the inequality from the usual inequality “ $<$ ”. Andrews referred to a symbol obtained using *k*-colors in this way as a *k*-colored generalized Frobenius symbol.

Given a *k*-colored generalized Frobenius symbol with entries

$$r_{i,j} \in \{\ell_c \mid \ell \text{ and } c \text{ are non-negative integers with } 1 \leq c \leq k\}$$

and

$$r_{i,j+1} \prec r_{i,j}, \quad i = 1, 2 \quad \text{and} \quad 1 \leq j \leq d - 1.$$

Andrews associated a *k*-colored generalized Frobenius partition of *n* to a *k*-colored generalized Frobenius symbol $(r_{i,j})_{2 \times d}$ by setting

$$n = d + \sum_{j=1}^d (r_{1,j} + r_{2,j}),$$

where only the non-negative integer ℓ is added if $r_{i,j} = \ell_c$. He used the symbol $c\phi_k(n)$ to denote the number of such partitions of *n*. Observe that when *k* = 1, the 1-colored generalized Frobenius symbols coincide with the Frobenius symbols and $c\phi_1(n) = p(n)$. To help the reader understand *k*-colored generalized Frobenius symbols, we list the following 2-colored generalized Frobenius symbols which give rise to 2-colored generalized Frobenius partitions of 2:

$$(1.2) \quad \begin{pmatrix} 1_1 \\ 0_1 \end{pmatrix}, \quad \begin{pmatrix} 1_1 \\ 0_2 \end{pmatrix}, \quad \begin{pmatrix} 1_2 \\ 0_1 \end{pmatrix}, \quad \begin{pmatrix} 1_2 \\ 0_2 \end{pmatrix}, \\ \begin{pmatrix} 0_1 \\ 1_1 \end{pmatrix}, \quad \begin{pmatrix} 0_2 \\ 1_1 \end{pmatrix}, \quad \begin{pmatrix} 0_1 \\ 1_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 \\ 1_2 \end{pmatrix}, \quad \begin{pmatrix} 0_2 & 0_1 \\ 0_2 & 0_1 \end{pmatrix}.$$

Note that there are altogether nine 2-colored generalized Frobenius partitions of 2 and hence

$$c\phi_2(2) = 9.$$

The best way to study a new function such as the *k*-colored generalized Frobenius partition function $c\phi_k(n)$ is to study its generating function

$$C\Phi_k(q) := \sum_{n=0}^{\infty} c\phi_k(n)q^n.$$

In [2, Theorem 5.2], Andrews showed that

$$(1.3) \quad C\Phi_k(q) = \frac{1}{(q; q)_{\infty}^k} \sum_{m_1, \dots, m_{k-1} \in \mathbf{Z}} q^{Q(m_1, \dots, m_{k-1})},$$

where

$$(1.4) \quad Q(m_1, m_2, \dots, m_{k-1}) = \sum_{i=1}^{k-1} m_i^2 + \sum_{1 \leq i < j \leq k-1} m_i m_j$$

and

$$(a; q)_{\infty} = \prod_{j=1}^{\infty} (1 - aq^{j-1}), \quad |q| < 1.$$

Using (1.3), Andrews [2, Corollary 5.2] discovered alternative expressions for $C\Phi_k(q)$ when $k = 2, 3$, and 5 . To describe Andrews' identities, let $q = e^{2\pi i\tau}$ throughout this paper, so

$$\Theta_3(q) = \vartheta_3(0|2\tau) = \sum_{j=-\infty}^{\infty} q^{j^2} \quad \text{and} \quad \Theta_2(q) = \vartheta_2(0|2\tau) = \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2},$$

where

$$\vartheta_2(u|\tau) = \sum_{j=-\infty}^{\infty} e^{\pi i\tau(j+1/2)^2} e^{(2j+1)iu}$$

and

$$\vartheta_3(u|\tau) = \sum_{j=-\infty}^{\infty} e^{\pi i\tau j^2} e^{2jiu}.$$

Andrews showed that

$$(1.5) \quad C\Phi_2(q) = \frac{(q^2; q^4)_{\infty}}{(q; q^2)_{\infty}^4 (q^4; q^4)_{\infty}},$$

$$(1.6) \quad C\Phi_3(q) = \frac{1}{(q; q)_{\infty}^3} (\Theta_3(q)\Theta_3(q^3) + \Theta_2(q)\Theta_2(q^3))$$

$$(1.7) \quad = \frac{1}{(q; q)_{\infty}^3} \left(1 + 6 \sum_{j=0}^{\infty} \binom{j}{3} \frac{q^j}{1 - q^j} \right),$$

and

$$(1.8) \quad C\Phi_5(q) = \frac{1}{(q; q)_{\infty}^5} \left(1 + 25 \sum_{j=1}^{\infty} \binom{j}{5} \frac{q^j}{(1 - q^j)^2} - 5 \sum_{j=1}^{\infty} \binom{j}{5} \frac{j q^j}{1 - q^j} \right),$$

where $\binom{j}{\cdot}$ is the Kronecker symbol. For (1.8), we have recorded the equivalent version of Andrews' identity found in the work of L.W. Kolitsch [23, Lemma 1]. Andrews [2, pp. 13–15] used Jacobi triple product identity (see for example [2, (3.1)]) and properties of theta series to prove (1.5) and (1.6). The proofs of (1.7) and (1.8) [2, pp. 26–27] are dependent on the work of H.D. Kloosterman [20, pp. 362, 358]. In a paragraph before the proofs of (1.7) and (1.8), Andrews [2, p. 26] mentioned that similar identity exists for $k = 7$, but this identity was not given in [2]. This missing identity, namely,

$$(1.9) \quad C\Phi_7(q) = \frac{1}{(q; q)_{\infty}^7} \left(1 + \frac{343}{8} \sum_{j=1}^{\infty} \binom{j}{7} \frac{q^j + q^{2j}}{(1 - q^j)^3} - \frac{7}{8} \sum_{j=1}^{\infty} \binom{j}{7} \frac{j^2 q^j}{1 - q^j} \right),$$

was later published by Kolitsch [23, Lemma 2].

Recently, N.D. Baruah and B.K. Sarmah [6, 7] used the method illustrated in Z. Cao's work [11] and found representations of $C\Phi_k(q)$ for $k = 4, 5$, and 6 . They showed that

$$(1.10) \quad C\Phi_4(q) = \frac{1}{(q; q)_{\infty}^4} (\Theta_3^3(q^2) + 3\Theta_3(q^2)\Theta_2^2(q^2)),$$

$$(1.11) \quad C\Phi_5(q) = \frac{1}{(q; q)_\infty^5} \left(\Theta_3(q^{10})\Theta_3^3(q^2) + 3\Theta_3(q^{10})\Theta_3(q^2)\Theta_2^2(q^2) + \frac{1}{2}\Theta_2(q^{5/2})\Theta_2^3(q^{1/2}) \right. \\ \left. + 3\Theta_2(q^{10})\Theta_2(q^2)\Theta_3^2(q) + \Theta_2(q^{10})\Theta_2^3(q^2) \right),$$

and

$$(1.12) \quad C\Phi_6(q) = \frac{1}{(q; q)_\infty^6} \left(\Theta_3^3(q)\Theta_3(q^2)\Theta_3(q^6) + \frac{3}{4}\Theta_2^3(q^{1/2})\Theta_2(q)\Theta_2(q^{3/2}) \right. \\ \left. + \Theta_3^2(q)\Theta_2(q^2)\Theta_2(q^6) \right).$$

Identities (1.10) and (1.11) can be found in [6, (2.2)] and [6, (2.13)] respectively, while (1.12) can be found in [7, (2.1)].

For $k > 7$, it is not clear if new identities associated with $C\Phi_k(q)$ could be derived using the methods of Andrews and Baruah-Sarmah. In fact, Andrews [2, p. 15] commented that as k increases, “the expressions quickly become long and messy”. The main goal of this paper is to discuss ways of finding new representations of $C\Phi_k(q)$. Using the theory of modular forms, we will derive all the identities mentioned above. In addition to providing new proofs to known identities, we will also construct new representations for $C\Phi_k(q)$ for the first time for $8 \leq k \leq 17$. In Section 2, we discuss the behavior of $C\Phi_k(q)$ as the modular form for each integer $k > 2$. In Section 3, we derive alternative representations of $C\Phi_k(q)$ for primes $k = 3, 5, 7, 11, 13$, and 17 and prove Kolitsch’s identities [23, p. 223]

$$(1.13) \quad c\phi_5(n) = p(n/5) + 5p(5n - 1)$$

and

$$(1.14) \quad c\phi_7(n) = p(n/7) + 7p(7n - 2).$$

We also discover and prove the identities

$$(1.15) \quad c\phi_{11}(n) = p(n/11) + 11p(11n - 5)$$

and

$$(1.16) \quad c\phi_{13}(n) = p(n/13) + 13p(13n - 7) + 26a(n),$$

where $p(x) = 0$ when x is not an integer and

$$q \prod_{j=1}^{\infty} \frac{(1 - q^{13j})}{(1 - q^j)^2} = \sum_{j=0}^{\infty} a(j)q^j.$$

It turns out that (1.15) is equivalent to Kolitsch’s identity for an 11-colored generalized Frobenius partition with order 11 [26, Theorem 3], which was first established using the results of F.G. Garvan, D. Kim, and D. Stanton [16]. Identity (1.16), on the other hand, is new. The proof of (1.16) motivates the discovery of a uniform method of treating identities such as (1.16). We discuss this method in Section 4 and derive analogues of (1.16) for $\ell = 17, 19$, and 23. This method also leads to the discovery of interesting modular functions that satisfy mysterious congruences. For example, if

$$h_\ell(\tau) = (q^\ell; q^\ell)_\infty C\Phi_\ell(q) - 1 - \ell(q^\ell; q^\ell)_\infty \sum_{j=1}^{\infty} p\left(\ell j - \frac{\ell^2 - 1}{24}\right) q^{j - 2\ell(\ell - 11)/2} \frac{\eta^{\ell - 11}(\ell\tau)}{\eta^{\ell - 11}(\tau)},$$

where $\eta(\tau)$ is the Dedekind eta function given by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

then for $\ell = 17, 19,$ and $23,$

$$h_{\ell}(\tau) \equiv 0 \pmod{\nu_{\ell}}$$

where

$$\nu_{\ell} = \ell^2 - \ell p \left(\ell n - \frac{\ell^2 - 1}{24} \right).$$

In Section 5, we discuss the cases for $k = 9$ and $15,$ the two composite odd integers less than $17.$ We derive the following congruence satisfied by $c\phi_k(n):$

$$(1.17) \quad c\phi_{p^{\alpha}N}(n) \equiv c\phi_{p^{\alpha-1}N}(n/p) \pmod{p^{2\alpha}},$$

where $c\phi_k(m) = 0$ if m is not an integer, p is a prime, and N and α are positive integers with $(N, p) = 1.$ The discovery of congruence (1.17) is motivated by congruences found in the study of $C\Phi_{10}(q)$ and $C\Phi_{14}(q)$ in Section 6, where identities associated with $k = 4, 6, 8, 10, 12,$ and 16 are given. More precisely, we discovered that

$$(1.18) \quad C\Phi_{2p}(q) \equiv \frac{\Theta_3(q^p)}{(q^p; q^p)_{\infty}} = C\Phi_2(q^p) \pmod{p^2},$$

which holds for any odd prime $p.$ The second equality follows from Andrews' identity for $C\Phi_2(q)$ (see also (3.1)). Congruence (1.18) can be viewed as an extension of Andrews' congruence [2, Corollary 10.2]

$$(1.19) \quad C\Phi_p(q) \equiv \frac{1}{(q^p; q^p)_{\infty}} \pmod{p^2}$$

if we rewrite (1.19) as

$$(1.20) \quad C\Phi_p(q) \equiv C\Phi_1(q^p) \pmod{p^2}$$

using the fact that

$$c\phi_1(n) = p(n).$$

The discovery of (1.18) leads to the congruence

$$(1.21) \quad C\Phi_{\ell p}(q) \equiv C\Phi_{\ell}(q^p) \pmod{p^2},$$

which holds for any distinct primes ℓ and $p.$ Congruence (1.21) eventually leads to (1.17).

There may be more surprising properties to be discovered for $c\phi_k(n),$ and we hope that this article will be helpful to future researchers who are interested in knowing more about these functions.

2. MODULAR PROPERTIES OF $C\Phi_k(q)$

In this section, we determine the modular properties of the function

$$\mathfrak{A}_k(q) := (q; q)_{\infty}^k C\Phi_k(q) = \sum_{m_1, \dots, m_{k-1} \in \mathbf{Z}} q^{Q(m_1, \dots, m_{k-1})}, \quad k > 1.$$

Let χ be a Dirichlet character and let $M_k(\Gamma_0(N), \chi)$ be the space of modular forms on $\Gamma_0(N)$ with weight k and multiplier $\chi.$ When χ is the trivial Dirichlet character, we write $M_k(\Gamma_0(N))$ for $M_k(\Gamma_0(N), \chi).$

Let

$$(2.1) \quad A_n = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 2 \end{pmatrix}_{n \times n}.$$

Then $\det(A_n) = n + 1$ and

$$(2.2) \quad A_n^{-1} = \begin{pmatrix} \frac{n}{n+1} & -\frac{1}{n+1} & -\frac{1}{n+1} & \cdots & -\frac{1}{n+1} \\ -\frac{1}{n+1} & \frac{n}{n+1} & -\frac{1}{n+1} & \cdots & -\frac{1}{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{n+1} & -\frac{1}{n+1} & -\frac{1}{n+1} & \cdots & \frac{n}{n+1} \end{pmatrix}.$$

Let n be a positive even integer and let

$$\chi(\cdot) = \left(\frac{(-1)^{n/2} \det(A_n)}{\cdot} \right) = \left(\frac{(-1)^{n/2} (n + 1)}{\cdot} \right).$$

Since all the diagonal components of A_n and $(n + 1)A_n^{-1}$ are even, we deduce from [30, Corollary 4.9.5(3)] that if

$$\theta(\tau; A_n) = \sum_{m \in \mathbf{Z}^n} e^{\pi i \tau \cdot m^t A_n m} = \sum_{m \in \mathbf{Z}^n} q^{\frac{1}{2} m^t A_n m} = \sum_{m \in \mathbf{Z}^n} q^{Q(m_1, \dots, m_n)},$$

then

$$(2.3) \quad \theta(\tau; A_n) = \sum_{m \in \mathbf{Z}^n} q^{Q(m_1, \dots, m_n)} = \mathfrak{A}_{n+1} \in M_{n/2}(\Gamma_0(n + 1), \chi).$$

Next, let $n > 1$ be an odd positive integer and let

$$B_n = \begin{pmatrix} A_n & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $\det B_n = 2(n + 1)$. We have

$$\begin{aligned} \theta(\tau; B_n) &= \sum_{m \in \mathbf{Z}^{n+1}} e^{\pi i \tau \cdot m^t B_n m} = \sum_{m \in \mathbf{Z}^n} q^{\frac{1}{2} m^t A_n m} \sum_{m_{n+1} \in \mathbf{Z}} q^{m_{n+1}^2} \\ &= \sum_{m \in \mathbf{Z}^n} q^{Q(m_1, \dots, m_n)} \sum_{m_{n+1} \in \mathbf{Z}} q^{m_{n+1}^2} \\ &= \mathfrak{A}_{n+1}(q) \Theta_3(q). \end{aligned}$$

Note that

$$B_n^{-1} = \begin{pmatrix} A_n^{-1} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Let

$$\chi(\cdot) = \left(\frac{(-1)^{(n+1)/2} \det(B_n)}{\cdot} \right) = \left(\frac{2(-1)^{(n+1)/2} (n + 1)}{\cdot} \right).$$

Since all the diagonal components of B_n and $2(n+1)B_n^{-1}$ are even, we deduce from [30, Corollary 4.9.5(3)] that

$$(2.4) \quad \theta(\tau; B_n) = \mathfrak{A}_{n+1}(q)\Theta_3(q) \in M_{(n+1)/2}(\Gamma_0(2(n+1)), \chi).$$

Similarly, let

$$C_n = \begin{pmatrix} A_n & 0 \\ 0 & 4 \end{pmatrix}.$$

Then $\det C_n = 4(n+1)$. Note that

$$C_n^{-1} = \begin{pmatrix} A_n^{-1} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

Let

$$\chi(\cdot) = \left(\frac{(-1)^{(n+1)/2} \det(C_n)}{\cdot} \right) = \left(\frac{(-1)^{(n+1)/2} (n+1)}{\cdot} \right).$$

Since all the diagonal components of C_n and $4(n+1)C_n^{-1}$ are even, we find from [30, Corollary 4.9.5(3)] that

$$(2.5) \quad \theta(\tau; C_n) = \mathfrak{A}_{n+1}(q)\Theta_3(q^2) \in M_{(n+1)/2}(\Gamma_0(4(n+1)), \chi).$$

From (2.3), (2.4), and (2.5), we deduce the following theorem.

Theorem 2.1. *If $k = 2r + 1$ is odd, then*

$$\mathfrak{A}_k(q) \in M_{(k-1)/2}(\Gamma_0(k), \left(\frac{(-1)^r \cdot k}{\cdot} \right)).$$

If $k = 2r$ is even, then

$$\Theta_3(q)\mathfrak{A}_k(q) \in M_{k/2}(\Gamma_0(2k), \left(\frac{2(-1)^r \cdot k}{\cdot} \right))$$

and

$$\Theta_3(q^2)\mathfrak{A}_k(q) \in M_{k/2}(\Gamma_0(4k), \left(\frac{(-1)^r \cdot k}{\cdot} \right)).$$

3. GENERATING FUNCTION OF $c\phi_k(n)$ WHEN k IS A PRIME

In this section, we will derive expressions for $C\Phi_k(q)$ when k is a prime number less than 18.

3.1. Case $k = 2$.

Our proof for $k = 2$ is exactly the same as that of Andrews' proof of (1.5), and we include it for the sake of completeness. From (1.3), we find that

$$(3.1) \quad C\Phi_2(q) = \frac{\Theta_3(q)}{(q; q)_\infty^2}.$$

Using Jacobi triple product identity (see [2, (3.1)]), we deduce that

$$(3.2) \quad \Theta_3(q) = (-q; q^2)_\infty^2 (q^2; q^2)_\infty.$$

Substituting (3.2) into (3.1) and simplifying, we complete the proof of (1.5).

3.2. Case $k = 3$.

From Theorem 2.1, we deduce that $\mathfrak{A}_3(q)$ is a modular form of weight 1 on $\Gamma_0(3)$ with multiplier $\left(\frac{-3}{\cdot}\right)$. Comparing the coefficients of $\mathfrak{A}_3(q)$ with the known Eisenstein series of weight 1 [15, Theorem 4.8.1] on $\Gamma_0(3)$ with multiplier $\left(\frac{-3}{\cdot}\right)$, we deduce that

$$\mathfrak{A}_3(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} = 1 + 6 \sum_{j=1}^{\infty} \binom{3}{j} \frac{q^j}{1-q^j}.$$

This is equivalent to (1.7). Another proof of (1.7) can also be found, for example, in the article by J.M. Borwein, P.B. Borwein, and F.G. Garvan [10, p. 43].

We next show that (1.6) follows from a general identity. Let $\omega = (1 + \sqrt{-d})/2$, with $d \equiv 3 \pmod{4}$. Observe that the set

$$S = \{m + n\omega | m, n \in \mathbf{Z}\}$$

is a disjoint union of

$$S_0 = \{m + n\omega | m, n \in \mathbf{Z}, n \equiv 0 \pmod{2}\}$$

and

$$S_1 = \{m + n\omega | m, n \in \mathbf{Z}, n \equiv 1 \pmod{2}\}.$$

Let

$$N(m + n\omega) = m^2 + mn + \left(\frac{d+1}{4}\right)n^2.$$

Then

$$\sum_{v \in S} q^{N(v)} = \sum_{v \in S_0} q^{N(v)} + \sum_{v \in S_1} q^{N(v)}.$$

Simplifying the above, we deduce that

$$(3.3) \quad \sum_{m,n \in \mathbf{Z}} q^{m^2+mn+(\frac{d+1}{4})n^2} = \Theta_3(q)\Theta_3(q^d) + \Theta_2(q)\Theta_2(q^d).$$

Identity (1.6) follows from (3.3) with $d = 3$.

3.3. Case $k = 5$.

We first establish three representations of $C\Phi_5(q)$.

Theorem 3.1. *The following identities hold:*

$$(3.4) \quad C\Phi_5(q) = \frac{1}{(q; q)_\infty^5} \left(1 + 25 \sum_{j=1}^{\infty} \binom{j}{5} \frac{q^j}{(1-q^j)^2} - 5 \sum_{j=1}^{\infty} \binom{j}{5} \frac{j q^j}{1-q^j} \right)$$

$$(3.5) \quad = \frac{1}{(q^5; q^5)_\infty} + 25q \frac{(q^5; q^5)_\infty}{(q; q)_\infty^6}$$

$$(3.6) \quad = \frac{1}{(q^5; q^5)_\infty} + 5 \sum_{j=1}^{\infty} p(5j-1)q^j.$$

Proof. From Theorem 2.1, we deduce that

$$\mathfrak{A}_5(q) \in M_2 \left(\Gamma_0(5), \left(\frac{5}{\cdot} \right) \right).$$

Since [31, Theorem 1.34]

$$\dim M_2 \left(\Gamma_0(5), \left(\frac{5}{\cdot} \right) \right) = 2,$$

we deduce that the two modular forms

$$(3.7) \quad E_{5,1} = \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \left(\frac{d}{5} \right) m q^{md} = \sum_{j=1}^{\infty} \left(\frac{j}{5} \right) \frac{q^j}{(1-q^j)^2}$$

and

$$E_{5,2} = 1 - 5 \sum_{m=1}^{\infty} \sum_{d=1}^{\infty} \left(\frac{d}{5} \right) d q^{md} = 1 - 5 \sum_{j=1}^{\infty} \left(\frac{j}{5} \right) \frac{j q^j}{1 - q^j},$$

which are in $M_2(\Gamma_0(5), (\frac{5}{\cdot}))$ (see [15, Section 4.6]), form a basis for this space of modular forms. By comparing Fourier coefficients of $\mathfrak{A}_5(q), E_{5,1}$, and $E_{5,2}$, we deduce that

$$\mathfrak{A}_5(q) = 25E_{5,1} + E_{5,2},$$

and the proof of (3.4) is complete.

Before we begin with our proof of (3.5), we observe that if $p > 3$, then by Theorem 2.1,

$$C\Phi_p(q)(q^p; q^p)_{\infty}$$

is a modular function on $\Gamma_0(p)$. This implies that the function can be expressed in terms of combinations of infinite products. For more details, see for example the paper by H.H. Chan, H. Hahn, R.P. Lewis, and S.L. Tan [12]. In [2, Corollary 10.2], Andrews showed that if p is a prime, then

$$C\Phi_p(q) = \frac{1}{(q^p; q^p)_{\infty}} + p^2 G_p(q)$$

for some $G_p(q)$ analytic inside $|q| < 1$ with integral power series coefficients. He then asked [2, Problem 6] for explicit closed forms for $G_p(q)$. Since

$$G_p(q)(q^p; q^p)_{\infty} = \frac{1}{p^2} (C\Phi_p(q)(q^p; q^p)_{\infty} - 1),$$

we conclude that $G_p(q)$ is a modular function on $\Gamma_0(p)$ for $p > 3$. This provides an answer to Andrews' question. The above discussion also gives us a way to derive alternative expressions for $C\Phi_p(q)$ whenever the functions invariant under $\Gamma_0(p)$ can be expressed as a rational function of a single modular function. This happens for $p = 5, 7$, and 13 . We now use this fact to derive an expression for $C\Phi_5(q)$. It is known from T. Kondo's work [28] that every modular function on $\Gamma_0(5)$ is a rational function of $\eta^6(5\tau)/\eta^6(\tau)$, where

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{j=1}^{\infty} (1 - e^{2\pi i j \tau}).$$

Since $C\Phi_5(q)(q^5; q^5)_{\infty}$ is a modular function on $\Gamma_0(5)$, we deduce that

$$C\Phi_5(q)(q^5; q^5)_{\infty} = 1 + 25 \frac{\eta^6(5\tau)}{\eta^6(\tau)}.$$

This completes the proof of (3.5).

Using the fact that

$$\frac{1}{(q; q)_\infty} = \sum_{j=0}^\infty p(j)q^j$$

and Ramanujan’s identity [9, Theorem 2.3.4],

$$(3.8) \quad \sum_{j=1}^\infty p(5j - 1)q^j = 5q \frac{(q^5; q^5)_\infty^5}{(q; q)_\infty^6},$$

we deduce (3.6) from (3.5). □

Remark 3.1. Identity (3.4) is Andrews’ (1.8), which was first proved using results found in Kloosterman’s work [20]. Identity (3.6) immediately implies (1.13). We emphasize here that our proof of (1.13) is different from Kolitsch’s proof as we have used (3.5) instead of (1.8).

As shown in (1.11), there is a fourth representation of $C\Phi_5(q)$ due to Baruah and Sarmah. This identity can be proved by realizing that

$$\mathfrak{A}_5(q) \in M_2 \left(\Gamma_0(40), \left(\frac{5}{\cdot} \right) \right),$$

together with the fact that the space $M_2 \left(\Gamma_0(40), \left(\frac{5}{\cdot} \right) \right)$ is spanned by the modular forms

$$\begin{aligned} &\Theta_3(q)\Theta_3(q^5), && \Theta_3^3(q)\Theta_3(q^5), && \Theta_3(q^2)\Theta_3^3(q^{10}), && \Theta_3^3(q^2)\Theta_3(q^{10}), \\ &\Theta_3(q)\Theta_3(q^5)\Theta_2^2(q^2), && \Theta_3(q^2)\Theta_3(q^{10})\Theta_2^2(q^2), && \Theta_2^3(q^{1/2})\Theta_2(q^{5/2}), \\ &\Theta_3^2(q)\Theta_2(q^2)\Theta_2(q^{10}), && \Theta_3^2(q^2)\Theta_2(q^{10}), && \text{and } \Theta_3^2(q^5)\Theta_2(q^2)\Theta_2(q^{10}). \end{aligned}$$

3.4. Case $k = 7$.

Theorem 3.2. *The following identities are true:*

$$(3.9) \quad C\Phi_7(q) = \frac{1}{(q; q)_\infty^7} \left(1 - \frac{7}{8} \sum_{k=1}^\infty \binom{k}{7} \frac{k^2 q^k}{1 - q^k} + \frac{343}{8} \sum_{k=1}^\infty \binom{k}{7} \frac{q^k + q^{2k}}{(1 - q^k)^3} \right)$$

$$(3.10) \quad = \frac{1}{(q^7; q^7)_\infty} + 49q \frac{(q^7; q^7)_\infty^3}{(q; q)_\infty^4} + 343q^2 \frac{(q^7; q^7)_\infty^7}{(q; q)_\infty^8}$$

$$(3.11) \quad = \frac{1}{(q^7; q^7)_\infty} + 7 \sum_{j=1}^\infty p(7j - 2)q^j.$$

Proof. Before giving the proof of (3.9), we observe that (3.9) is the same as (1.9). We will prove (3.9) using the theory of modular forms. Note that by Theorem 2.1, we have $\mathfrak{A}_7(q) \in M_3 \left(\Gamma_0(7), \left(\frac{-7}{\cdot} \right) \right)$. The space $M_3 \left(\Gamma_0(7), \left(\frac{-7}{\cdot} \right) \right)$ is spanned by

$$\begin{aligned} E_{7,1} &= \sum_{m=1}^\infty \sum_{d=1}^\infty \binom{d}{7} m^2 q^{md} = \sum_{j=1}^\infty \binom{j}{7} \frac{q^j + q^{2j}}{(1 - q^j)^3}, \\ E_{7,2} &= 1 - \frac{7}{8} \sum_{m=1}^\infty \sum_{d=1}^\infty \binom{d}{7} d^2 q^{md} = 1 - \frac{7}{8} \sum_{j=1}^\infty \binom{j}{7} \frac{j^2 q^j}{1 - q^j}, \end{aligned}$$

and

$$S_7 = \eta^3(\tau)\eta^3(7\tau).$$

By comparing Fourier coefficients of these modular forms, we deduce that

$$\mathfrak{A}_7(q) = \frac{343}{8}E_{7,1} + E_{7,2}.$$

This completes the proof of (3.9).

The proof of (3.10) is similar to the proof of (3.5). We recall that modular functions invariant under $\Gamma_0(7)$ is a rational function of

$$\frac{\eta^4(7\tau)}{\eta^4(\tau)}.$$

Since $(q^7; q^7)_\infty C\Phi_7(q)$ is such a function, we conclude that

$$(q^7; q^7)_\infty C\Phi_7(q) = 1 + 49\frac{\eta^4(7\tau)}{\eta^4(\tau)} + 343\frac{\eta^8(7\tau)}{\eta^8(\tau)},$$

and the proof of (3.10) is complete.

Ramanujan discovered that [9, Theorem 2.4.2]

$$(3.12) \quad \sum_{j=1}^{\infty} p(7j - 2)q^j = 7q\frac{(q^7; q^7)_\infty^3}{(q; q)_\infty^4} + 49q^2\frac{(q^7; q^7)_\infty^7}{(q; q)_\infty^8}.$$

Using (3.12) and (3.10), we deduce (3.11). □

Identity (3.11) immediately implies Kolitsch’s identity (1.14). We emphasize here that our proof of (1.14) uses (3.10) instead of (3.9).

As in the case for $k = 5$, we are able to find a representation of $C\Phi_7(q)$ in terms of theta functions. This new identity is an analogue of (1.11). We first observe that $\mathfrak{A}_7(q) \in M_3(\Gamma_0(28), (\frac{-28}{\cdot}))$. Furthermore the modular forms

$$\begin{aligned} &\Theta_3^5(q)\Theta_3(q^7), \quad \Theta_3^3(q)\Theta_3^3(q^7), \quad \Theta_3(q)\Theta_3^5(q^7), \quad \Theta_3^4(q)\Theta_2(q^{1/2})\Theta_2(q^{7/2}), \\ &\Theta_3^4(q^7)\Theta_2(q^{1/2})\Theta_2(q^{7/2}), \quad \Theta_3^4(q)\Theta_2(q)\Theta_2(q^7), \quad \Theta_3^4(q^7)\Theta_2(q)\Theta_2(q^7), \\ &\Theta_2^3(q^{1/2})\Theta_2^3(q^{7/2}), \quad \Theta_2^3(q)\Theta_2^3(q^7), \quad \Theta_2^5(q)\Theta_2(q^7), \quad \text{and} \quad \Theta_2(q)\Theta_2^5(q^7) \end{aligned}$$

form a basis for $M_3(\Gamma_0(28), (\frac{-28}{\cdot}))$. Hence, we deduce that

$$(3.13) \quad \begin{aligned} C\Phi_7(q) = \frac{1}{(q; q)_\infty^7} &\left(-\frac{15}{32}\Theta_3^5(q)\Theta_3(q^7) + \frac{55}{16}\Theta_3^3(q)\Theta_3^3(q^7) - \frac{63}{32}\Theta_3(q)\Theta_3^5(q^7) \right. \\ &+ \frac{15}{16}\Theta_3^4(q)\Theta_2(q^{1/2})\Theta_2(q^{7/2}) + \frac{105}{16}\Theta_3^4(q^7)\Theta_2(q^{1/2})\Theta_2(q^{7/2}) \\ &- \frac{15}{16}\Theta_3^4(q)\Theta_2(q)\Theta_2(q^7) + \frac{525}{16}\Theta_3^4(q^7)\Theta_2(q)\Theta_2(q^7) \\ &+ \frac{105}{32}\Theta_2^3(q^{1/2})\Theta_2^3(q^{7/2}) + \frac{95}{8}\Theta_2^3(q)\Theta_2^3(q^7) \\ &\left. + \frac{15}{16}\Theta_2^5(q)\Theta_2(q^7) - \frac{189}{16}\Theta_2(q)\Theta_2^5(q^7) \right). \end{aligned}$$

We next prove some congruences satisfied by $c\phi_7(n)$ using (3.13) and (3.10).

Theorem 3.3. *For any integer $n \geq 0$,*

$$c\phi_7(5n + 3) \equiv 0 \pmod{5}.$$

Proof. From (3.13), we deduce that

$$\begin{aligned} \sum_{j=0}^{\infty} c\phi_7(j)q^j &\equiv \frac{1}{(q; q)_7^{\infty}} \left(\Theta_3(q)\Theta_3^5(q^7) + \Theta_2(q)\Theta_2^5(q^7) \right) \pmod{5} \\ &\equiv \frac{1}{(q^5; q^5)_2^{\infty}} \left((q; q)_{\infty}^3 \Theta_3(q)\Theta_3(q^{35}) + (q; q)_{\infty}^3 \Theta_2(q)\Theta_2(q^{35}) \right) \pmod{5}. \end{aligned} \tag{3.14}$$

Using Jacobi's identity for $(q; q)_{\infty}^3$ [9, Theorem 1.3.9], we find that

$$(q; q)_{\infty}^3 \Theta_3(q) = \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^i (2i + 1) q^{i(i+1)/2+j^2}. \tag{3.15}$$

Now, observe that

$$m = \frac{i(i + 1)}{2} + j^2$$

is equivalent to

$$8m + 1 = (2i + 1)^2 + 8j^2.$$

If $8m \equiv -1 \pmod{5}$, then $m \equiv 3 \pmod{5}$. Since

$$\left(\frac{-8}{5} \right) = -1,$$

we deduce that

$$(2i + 1)^2 + 8j^2 \equiv 0 \pmod{5}$$

holds if and only if

$$2i + 1 \equiv j \equiv 0 \pmod{5}.$$

Similarly, we have

$$(3.16) \quad q^{35/4} (q; q)_{\infty}^3 \Theta_2(q) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^i (2i + 1) q^{9+i(i+1)/2+j(j+1)}.$$

Observe that

$$m = 9 + \frac{i(i + 1)}{2} + j(j + 1)$$

is equivalent to

$$8m - 69 = (2i + 1)^2 + 2(2j + 1)^2.$$

Note that if $8m - 69 \equiv 0 \pmod{5}$, then $m \equiv 3 \pmod{5}$. Since

$$\left(\frac{-2}{5} \right) = -1,$$

we deduce that

$$(2i + 1)^2 + 2(2j + 1)^2 \equiv 0 \pmod{5}$$

holds if and only if

$$2i + 1 \equiv 2j + 1 \equiv 0 \pmod{5}.$$

From (3.14), (3.15), and (3.16), we conclude that if $m \equiv 3 \pmod{5}$, then

$$c\phi_7(m) \equiv 0 \pmod{5}$$

or, equivalently,

$$c\phi_7(5n + 3) \equiv 0 \pmod{5}$$

for any integer $n \geq 0$. □

Remark 3.2. It is possible to deduce Theorem 3.3 without using (3.13). We first recall a recent result of F.G. Garvan and J.A. Sellers [17] which states that if p is a prime number and $0 < r < p$, then the congruence

$$c\phi_k(pn + r) \equiv 0 \pmod{p}, \quad \text{for all } n \in \mathbf{N},$$

implies that

$$c\phi_{pN+k}(pn + r) \equiv 0 \pmod{p}, \quad \text{for all } n \in \mathbf{N}.$$

In [2, (10.3)], Andrews showed that for all integers $n \geq 0$,

$$(3.17) \quad c\phi_2(5n + 3) \equiv 0 \pmod{5}.$$

Applying the result of Garvan and Sellers with $p = 5$, $r = 3$, $N = 1$, and $k = 2$, we complete the proof of Theorem 3.3.

Our next set of congruences are consequences of (3.10).

Theorem 3.4. *For any integer $n \geq 0$, we have*

$$(3.18) \quad c\phi_7(7n + 3) \equiv c\phi_7(7n + 5) \equiv c\phi_7(7n + 6) \equiv 0 \pmod{7^3}.$$

Proof. From (3.10), we find that

$$(3.19) \quad \sum_{k=0}^{\infty} c\phi_7(k)q^k \equiv \frac{1}{(q^7; q^7)_{\infty}} + 49q \frac{(q^7; q^7)_{\infty}^3}{(q; q)_{\infty}^4} \pmod{7^3}.$$

Let

$$q \frac{(q^7; q^7)_{\infty}^3}{(q; q)_{\infty}^4} = \sum_{j=0}^{\infty} a(j)q^j.$$

Then

$$(3.20) \quad \sum_{n=0}^{\infty} c\phi_7(7n + r)q^n \equiv 49 \sum_{n=0}^{\infty} a(7n + r)q^n \pmod{7^3}, \quad 1 \leq r \leq 6.$$

By the binomial theorem, we find that

$$(3.21) \quad \sum_{j=0}^{\infty} a(j)q^j \equiv q(q^7; q^7)_{\infty}^2 (q; q)_{\infty}^3 \equiv (q^7; q^7)_{\infty}^2 \left(\sum_{i=0}^{\infty} (-1)^i (2i + 1) q^{i(i+1)/2+1} \right) \pmod{7}.$$

Since

$$1 + \frac{i(i+1)}{2} \equiv 0, 1, 2, \text{ or } 4 \pmod{7},$$

we deduce that

$$(3.22) \quad a(7n + 3) \equiv a(7n + 5) \equiv a(7n + 6) \equiv 0 \pmod{7}.$$

Combining (3.20) with (3.22) we complete the proof of (3.18). \square

3.5. Case $k = 11$.

Theorem 3.5. *We have*

$$(3.23) \quad C\Phi_{11}(q) = \frac{1}{(q^{11}; q^{11})_\infty} + 11 \sum_{j=1}^\infty p(11j - 5)q^j.$$

Proof. By Theorem 2.1, we know that $\mathfrak{A}_{11}(q) \in M_5(\Gamma_0(11), (\frac{-11}{11}))$. The dimension of $M_5(\Gamma_0(11), (\frac{-11}{11}))$ is 5 [31, Theorem 1.34], and this space is spanned by the modular forms

$$\begin{aligned} & \frac{\eta^{11}(11\tau)}{\eta(\tau)}, \quad \frac{\eta^{11}(\tau)}{\eta(11\tau)}, \quad (q; q)_\infty^{11} \sum_{j=1}^\infty p(11j - 5)q^j, \\ & \sum_{m=1}^\infty \sum_{d=1}^\infty \binom{d}{11} m^4 q^{md} \quad \text{and} \quad \frac{1275}{11} + \sum_{m=1}^\infty \sum_{d=1}^\infty \binom{d}{11} d^4 q^{md}. \end{aligned}$$

By comparing the coefficients of $\mathfrak{A}_{11}(q)$ with those of the five modular forms above, we deduce that

$$\mathfrak{A}_{11}(q) = \frac{\eta^{11}(\tau)}{\eta(11\tau)} + 11(q; q)_\infty^{11} \sum_{j=1}^\infty p(11j - 5)q^j.$$

This proves (3.23). □

It is immediate that (3.23) implies (1.15). There is no analogue of (3.4) and (3.9) for $k = 11$, but an analogue for (1.11) and (3.13) exists. This expression is complicated and we will give such identities if we do not have other representations for $(q^k; q^k)_\infty C\Phi_k(q)$ when k is composite (see Section 6).

3.6. Case $k = 13$.

Theorem 3.6. *We have*

$$(3.24) \quad \begin{aligned} C\Phi_{13}(q) = & \frac{1}{(q^{13}; q^{13})_\infty} + 169 \left(q \frac{(q^{13}; q^{13})_\infty}{(q; q)_\infty^2} + 36q^2 \frac{(q^{13}; q^{13})_\infty^3}{(q; q)_\infty^4} + 494q^3 \frac{(q^{13}; q^{13})_\infty^5}{(q; q)_\infty^6} \right. \\ & + 3380q^4 \frac{(q^{13}; q^{13})_\infty^7}{(q; q)_\infty^8} + 13182q^5 \frac{(q^{13}; q^{13})_\infty^9}{(q; q)_\infty^{10}} \\ & \left. + 28561q^6 \frac{(q^{13}; q^{13})_\infty^{11}}{(q; q)_\infty^{12}} + 28561q^7 \frac{(q^{13}; q^{13})_\infty^{13}}{(q; q)_\infty^{14}} \right) \end{aligned}$$

$$(3.25) \quad = \frac{1}{(q^{13}; q^{13})_\infty} + 13 \sum_{j=1}^\infty p(13j - 7)q^j + 26q \frac{(q^{13}; q^{13})_\infty}{(q; q)_\infty^2}.$$

Proof. From the discussion at the end of Section 3.3, we know that

$$(q^{13}; q^{13})_\infty C\Phi_{13}(q)$$

is a modular function invariant under $\Gamma_0(13)$, and since modular functions invariant under $\Gamma_0(13)$ are rational functions of $H = \eta^2(13\tau)/\eta^2(\tau)$ [28], we deduce that

$$(q^{13}; q^{13})_\infty C\Phi_{13}(q) = 1 + 169 \left(H + 36H^2 + 494H^3 + 3380H^4 + 13182H^5 + 28561H^6 + 28561H^7 \right),$$

and (3.24) follows.

Around 1939, motivated by Ramanujan’s identities (3.8) and (3.12), H. Zuckerman [44, eq. (1.15)] discovered that

$$(3.26) \quad \sum_{j=1}^\infty p(13j - 7)q^j = 11q \frac{(q^{13}; q^{13})_\infty}{(q; q)_\infty^2} + 468q^2 \frac{(q^{13}; q^{13})_\infty^3}{(q; q)_\infty^4} + 6422q^3 \frac{(q^{13}; q^{13})_\infty^5}{(q; q)_\infty^6} + 43940q^4 \frac{(q^{13}; q^{13})_\infty^7}{(q; q)_\infty^8} + 171366q^5 \frac{(q^{13}; q^{13})_\infty^9}{(q; q)_\infty^{10}} + 371293q^6 \frac{(q^{13}; q^{13})_\infty^{11}}{(q; q)_\infty^{12}} + 371293q^7 \frac{(q^{13}; q^{13})_\infty^{13}}{(q; q)_\infty^{14}}.$$

Using (3.26) to simplify (3.24), we deduce that

$$C\Phi_{13}(q) = \frac{1}{(q^{13}; q^{13})_\infty} + 13 \sum_{j=1}^\infty p(13j - 7)q^j + 26q \frac{(q^{13}; q^{13})_\infty}{(q; q)_\infty^2},$$

and this yields (3.25). □

Identity (3.25) immediately implies (1.16).

We observe that the appearance of

$$\sum_{j=1}^\infty p(13j - 7)q^j$$

simplifies (3.24), leading to (3.25) with only three terms on the right hand side. Identity (3.25) is clearly an analogue of Kolitsch’s identities (3.6) and (3.11).

In Section 4, we will prove identities involving both $C\Phi_k(q)$ and

$$\sum_{j=1}^\infty p \left(kj - \frac{k^2 - 1}{24} \right) q^j$$

when $k > 3$ is a prime. This method appears to yield the simplest (in terms of the number of modular forms involved) representation of $C\Phi_k(q)$ for any prime $k > 3$, and it does not involve the construction of basis for

$$M_{(k-1)/2} \left(\Gamma_0(k), \left(\frac{(-1)^{(k-1)/2} k}{\cdot} \right) \right).$$

Constructing such basis could get complicated for large k , as we shall see in the next subsection.

3.7. Case $k = 17$.

Let

$$(3.27) \quad \mathcal{E}_a(\tau) = q^{17B_2(a/17)/2} \prod_{m=1}^\infty (1 - q^{17(m-1)+a})(1 - q^{17m-a}),$$

where $B_2(x) = x^2 - x + 1/6$. Let

$$\begin{aligned} f_{17,1} &= \eta^7(\tau)\eta(17\tau), & f_{17,2}(\tau) &= \eta(\tau)\eta^7(17\tau), \\ g_{17,1}(\tau) &= \frac{1}{8}(17E_2(17\tau) - E_2(\tau)), \\ g_{17,2}(\tau) &= \eta^4(17\tau) \sum_{k=0}^7 \mathcal{E}_{2,3^k}(\tau)\mathcal{E}_{14,3^k}(\tau)\mathcal{E}_{4,3^k}(\tau)^2\mathcal{E}_{12,3^k}(\tau)\mathcal{E}_{6,3^k}(\tau)\mathcal{E}_{10,3^k}(\tau)^2\mathcal{E}_{8,3^k}(\tau), \\ h_{17,1}(\tau) &= g_{17,1}^2(\tau), & h_{17,2}(\tau) &= g_{17,1}(\tau)g_{17,2}(\tau), & h_{17,3}(\tau) &= g_{17,2}^2(\tau), \\ h_{17,4}(\tau) &= \eta^4(\tau)\eta^4(17\tau), & h_{17,5}(\tau) &= \frac{1}{24}(289E_4(17\tau) - E_4(\tau)). \end{aligned}$$

From Theorem 2.1, we know that $\mathfrak{A}_{17}(q) \in M_8(\Gamma_0(17), (\frac{17}{\cdot}))$. By [31, Theorem 1.34], we find that

$$\dim M_8(\Gamma_0(17), (\frac{17}{\cdot})) = 12.$$

Let

$$\begin{aligned} B_{17,1} &= f_{17,1}h_{17,1}, & B_{17,2} &= f_{17,1}h_{17,2}, & B_{17,3} &= f_{17,1}h_{17,3}, & B_{17,4} &= f_{17,1}h_{17,4}, \\ B_{17,5} &= f_{17,1}h_{17,5}, & B_{17,6} &= f_{17,2}h_{17,1}, & B_{17,7} &= f_{17,2}h_{17,2}, & B_{17,8} &= f_{17,2}h_{17,3}, \\ B_{17,9} &= f_{17,2}h_{17,4}, & B_{17,10} &= f_{17,2}h_{17,5}, & B_{17,11} &= \frac{\eta^{17}(\tau)}{\eta(17\tau)}, & \text{and } B_{17,12} &= \frac{\eta^{17}(17\tau)}{\eta(\tau)}. \end{aligned}$$

One can verify that $\{B_{17,j} | 1 \leq j \leq 12\}$ forms a basis of $M_8(\Gamma_0(17), (\frac{17}{\cdot}))$. By comparing the Fourier coefficients of $\mathfrak{A}_{17}(q)$ and $B_{17,j}, 1 \leq j \leq 12$, we deduce the following identity.

Theorem 3.7. *We have*

$$\begin{aligned} (3.28) \quad C\Phi_{17}(q) &= \frac{1}{(q; q)_{\infty}^{17}} \left(\frac{1491529}{118} B_{17,1} - \frac{20931981}{236} B_{17,2} - \frac{117030839}{236} B_{17,3} \right. \\ &\quad + \frac{78308596}{59} B_{17,4} - \frac{988669}{236} B_{17,5} + \frac{424841849}{59} B_{17,6} \\ &\quad - \frac{10654955751}{236} B_{17,7} - \frac{17109438979}{236} B_{17,8} + \frac{7515406274}{59} B_{17,9} \\ &\quad \left. + \frac{91750275}{236} B_{17,10} + B_{17,11} + 6975757441 B_{17,12} \right). \end{aligned}$$

Note that all the coefficients of $B_{17,j}, j \neq 11$, are divisible by 17^2 . Therefore,

$$C\Phi_{17}(q) \equiv \frac{1}{(q^{17}; q^{17})_{\infty}} \pmod{17^2}$$

or, equivalently,

$$c\phi_{17}(n) \equiv p(n/17) \pmod{17^2}.$$

This is a special case of Andrews' congruence [2, Theorem 10.2 and Corollary 10.2]

$$(3.29) \quad c\phi_p(n) \equiv c\phi_1(n/p) \pmod{p^2},$$

which is true for all primes p .

In the next section, we will provide an analogue for Kolitsch's identities (3.6) and (3.11) for $C\Phi_{17}(q)$.

4. k -COLORED GENERALIZED FROBENIUS PARTITIONS AND ORDINARY PARTITIONS

Kolitsch’s identities (1.13), (1.14), and Andrews’ congruence (3.29) show a close relation between k -colored generalized Frobenius partitions and ordinary partitions. In this section, we will give a more precise description of the relation and prove (3.6), (3.11), (3.23), and (3.25) in a uniform way. We will also give an alternative representation for $C\Phi_{17}(q)$ and illustrate for any prime $\ell > 3$ a general procedure to express $C\Phi_\ell(q)$ in terms of other modular functions, one of which involves generating functions for $p(\ell n - (\ell^2 - 1)/24)$.

Let

$$F(\tau) \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} := F \left(\frac{a\tau + b}{c\tau + d} \right).$$

Let ℓ be a prime ≥ 5 and let $\mathcal{A}_\ell(\tau)$ denote the function $\mathfrak{A}_\ell(q)$ when the function $\mathfrak{A}_\ell(q)$ is viewed as a function of τ with $q = e^{2\pi i\tau}$. By Theorem 2.1,

$$\mathfrak{A}_\ell(q) = \mathcal{A}_\ell(\tau) = (q; q)_\infty^\ell C\Phi_\ell(q) = \sum_{m_1, \dots, m_{\ell-1} \in \mathbf{Z}} q^{Q(m_1, \dots, m_{\ell-1})}$$

is a modular form of weight $(\ell - 1)/2$ with character $\chi_{(-1)(\ell-1)/2\ell}$ on $\Gamma_0(\ell)$, where $Q(m_1, \dots, m_{\ell-1})$ is the quadratic form defined by (1.4) and χ_d is the character defined by $\chi_d(\cdot) = \left(\frac{d}{\cdot}\right)$. It follows that

$$f_\ell(\tau) = \frac{\eta(\ell\tau)}{\eta(\tau)^\ell} \mathcal{A}_\ell(\tau)$$

is a modular function on $\Gamma_0(\ell)$. On the other hand, $\eta(\ell^2\tau)/\eta(\tau)$ is a modular function on $\Gamma_0(\ell^2)$, and by a lemma of A.O.L. Atkin and J. Lehner [5, Lemma 7], we find that

$$\frac{\eta(\ell^2\tau)}{\eta(\tau)} \Big|_{U_\ell} := \frac{1}{\ell} \sum_{k=0}^{\ell-1} \frac{\eta(\ell^2\tau)}{\eta(\tau)} \Big| \begin{pmatrix} 1 & k \\ 0 & \ell \end{pmatrix} = (q^\ell; q^\ell)_\infty \sum_{j=1}^\infty p \left(\ell j - \frac{\ell^2 - 1}{24} \right) q^j$$

is also a modular function on $\Gamma_0(\ell)$.

Set

$$(4.1) \quad g_\ell(\tau) = 1 + \ell \frac{\eta(\ell^2\tau)}{\eta(\tau)} \Big|_{U_\ell} = 1 + \ell (q^\ell; q^\ell)_\infty \sum_{j=1}^\infty p \left(\ell j - \frac{\ell^2 - 1}{24} \right) q^j.$$

We now compare the analytic behaviors of $f_\ell(\tau)$ and $g_\ell(\tau)$ at cusps associated with $\Gamma_0(\ell)$.

Lemma 4.1. *Let $\ell \geq 5$ be a prime and let*

$$\delta_\ell = \frac{\ell^2 - 1}{24}.$$

At the cusp ∞ , we have

$$f_\ell(\tau) - g_\ell(\tau) = \begin{cases} \ell(\ell - p(\ell - \delta_\ell))q + O(q^2) & \text{if } \ell \leq 23, \\ \ell^2 q + \left(\frac{1}{4} \ell^2 (\ell^2 - 2\ell + 9) - \ell p(2\ell - \delta_\ell) \right) q^2 + O(q^3) & \text{if } 29 \leq \ell \leq 47, \\ \ell^2 q + \frac{1}{4} \ell^2 (\ell^2 - 2\ell + 9) q^2 + O(q^3) & \text{if } \ell \geq 53. \end{cases}$$

At the cusp 0, we have

$$(f_\ell(\tau) - g_\ell(\tau)) \Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = \begin{cases} O(q) & \text{if } \ell = 5, 7, 11, \\ 2q^{-1} - 4 + O(q) & \text{if } \ell = 13, \\ 2q^{-(\ell^2-1)/24+(\ell-1)/2}(1 - q + O(q^2)) & \text{if } \ell \geq 17. \end{cases}$$

Proof. It is clear from the definition of $g_\ell(\tau)$ that

$$(4.2) \quad \begin{aligned} g_\ell(\tau) &= 1 + \ell(q^\ell; q^\ell)_\infty \sum_{n \geq (\ell^2-1)/24\ell} p(\ell n - \delta_\ell) q^n \\ &= \begin{cases} 1 + \ell p(\ell - \delta_\ell) q + O(q^2) & \text{if } \ell \leq 23, \\ 1 + \ell p(2\ell - \delta_\ell) q^2 + O(q^3) & \text{if } 29 \leq \ell \leq 47, \\ 1 + O(q^3) & \text{if } \ell \geq 53. \end{cases} \end{aligned}$$

On the other hand, we have

$$\begin{aligned} Q(m_1, \dots, m_{\ell-1}) &= (m_1^2 + \dots + m_{\ell-1}^2) + \frac{1}{2} \sum_{i \neq j} m_i m_j \\ &= \frac{1}{2} (m_1^2 + \dots + m_{\ell-1}^2) + \frac{1}{2} (m_1 + \dots + m_{\ell-1})^2. \end{aligned}$$

From this, we see that $Q(m_1, \dots, m_{\ell-1}) = 1$ if and only if exactly one of m_j is ± 1 and the other are all 0, or $m_i = 1$ and $m_j = -1$ for some i, j with $i \neq j$ and all others are 0. Likewise, we can check that $Q(m_1, \dots, m_{\ell-1}) = 2$ if and only if there are two 1's and two -1 's among m_j , or there are two 1's and one -1 among m_j , or there are two -1 's and one 1 among m_j . Thus, the number of integer solutions of $Q(m_1, \dots, m_{\ell-1}) = 2$ is

$$\frac{1}{4}(\ell - 1)(\ell - 2)(\ell - 3)(\ell - 4) + 2 \cdot \frac{1}{2}(\ell - 1)(\ell - 2)(\ell - 3) = \frac{1}{4}\ell(\ell - 1)(\ell - 2)(\ell - 3).$$

Consequently, we have

$$\mathcal{A}_\ell(\tau) = 1 + \ell(\ell - 1)q + \frac{1}{4}\ell(\ell - 1)(\ell - 2)(\ell - 3)q^2 + \dots$$

and

$$\begin{aligned} f_\ell(\tau) &= \frac{(q^\ell; q^\ell)_\infty}{(1 - q - q^2 + \dots)^\ell} \left(1 + \ell(\ell - 1)q + \frac{1}{4}\ell(\ell - 1)(\ell - 2)(\ell - 3)q^2 + \dots \right) \\ &= 1 + \ell^2 q + \frac{1}{4}\ell^2(\ell^2 - 2\ell + 9)q^2 + \dots \end{aligned}$$

Together with (4.2), this yields the first half of the lemma. We next consider the analytic behavior of $f_\ell(\tau) - g_\ell(\tau)$ at 0.

Recall that if Λ is an even integral lattice of rank n and Λ' is its dual lattice, then their theta series $\theta_\Lambda(\tau)$ and $\theta_{\Lambda'}(\tau)$ are related by the transformation formula (see [38, Chapter VII, Proposition 16])

$$(4.3) \quad \theta_{\Lambda'}(-1/\tau) = \left(\frac{\tau}{i}\right)^{n/2} \nu(\Lambda) \theta_\Lambda(\tau),$$

where $\nu(\Lambda)$ is the volume of the lattice Λ . Here, we let Λ be the lattice whose Gram matrix is $\ell A_{\ell-1}^{-1}$, where A_n and A_n^{-1} are given by (2.1) and (2.2), respectively. The determinant of $\ell A_{\ell-1}^{-1}$ is $\ell^{\ell-1} / \det(A_{\ell-1}) = \ell^{\ell-2}$. Hence

$$\nu(\Lambda) = \ell^{\ell/2-1}.$$

Let $\mathcal{B}_\ell(\tau)$ be the theta series of Λ and observe that the theta series of Λ' is $\mathcal{A}_\ell(\tau/\ell)$. Thus, by (4.3), we have

$$\mathcal{A}_\ell\left(-\frac{1}{\ell\tau}\right) = \ell^{\ell/2-1} \left(\frac{\tau}{i}\right)^{(\ell-1)/2} \mathcal{B}_\ell(\tau).$$

Together with

$$\eta\left(-\frac{1}{\ell\tau}\right) = \sqrt{\frac{\ell\tau}{i}} \eta(\ell\tau) \quad \text{and} \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}} \eta(\tau),$$

we deduce that

$$(4.4) \quad f_\ell(\tau) \Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = \frac{1}{\ell} \frac{\eta(\tau)}{\eta(\ell\tau)^\ell} \mathcal{B}_\ell(\tau).$$

We now consider $g_\ell(-1/\ell\tau)$.
We have

$$\ell \left(\frac{\eta(\ell^2\tau)}{\eta(\tau)} \Big| U_\ell \right) \Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = \sum_{k=0}^{\ell-1} \frac{\eta(\ell^2\tau)}{\eta(\tau)} \Big| \begin{pmatrix} k\ell & -1 \\ \ell^2 & 0 \end{pmatrix}.$$

For $k = 0$, the transformation formula for $\eta(\tau)$ yields

$$(4.5) \quad \frac{\eta(\ell^2\tau)}{\eta(\tau)} \Big| \begin{pmatrix} 0 & -1 \\ \ell^2 & 0 \end{pmatrix} = \frac{1}{\ell} \frac{\eta(\tau)}{\eta(\ell^2\tau)}.$$

For $1 \leq k \leq \ell - 1$, we find that

$$(4.6) \quad \eta\left(\ell^2 \frac{k\ell\tau - 1}{\ell^2\tau}\right) = \eta\left(k\ell - \frac{1}{\tau}\right) = e^{2\pi i k\ell/24} \eta(-1/\tau) = e^{2\pi i k\ell/24} \sqrt{\frac{\tau}{i}} \eta(\tau).$$

Next, since $(k, \ell) = 1$, there exist integers a and k' such that $kk' - a\ell = 1$. This implies that

$$(4.7) \quad \begin{aligned} \eta\left(\frac{k\ell\tau - 1}{\ell^2\tau}\right) &= \eta\left(\tau - \frac{k'}{\ell}\right) \Big| \begin{pmatrix} k & a \\ \ell & k' \end{pmatrix} \\ &= \left(\frac{k'}{\ell}\right) i^{(1-\ell)/2} e^{2\pi i \ell(k+k')/24} \sqrt{\frac{\ell\tau}{i}} \eta\left(\tau - \frac{k'}{\ell}\right). \end{aligned}$$

It follows from (4.6) and (4.7) that

$$\begin{aligned} \frac{\eta(\ell^2\tau)}{\eta(\tau)} \Big| \begin{pmatrix} k\ell & -1 \\ \ell^2 & 0 \end{pmatrix} &= \frac{1}{\sqrt{\ell}} \left(\frac{k'}{\ell}\right) i^{(\ell-1)/2} e^{-2\pi i \ell k'/24} \frac{\eta(\tau)}{\eta(\tau - k'/\ell)} \\ &= \frac{1}{\sqrt{\ell}} \left(\frac{k'}{\ell}\right) i^{(\ell-1)/2} e^{-2\pi i m k'/\ell} + O(q), \end{aligned}$$

where $m = (\ell^2 - 1)/24$. Hence,

$$\begin{aligned}
 \sum_{k=1}^{\ell-1} \frac{\eta(\ell^2\tau)}{\eta(\tau)} \left| \begin{pmatrix} k\ell & -1 \\ \ell^2 & 0 \end{pmatrix} \right| &= \frac{i^{(\ell-1)/2}}{\sqrt{\ell}} \sum_{k=1}^{\ell-1} \left(\frac{k'}{\ell} \right) e^{-2\pi i m k' / \ell} + O(q) \\
 &= \frac{i^{(\ell-1)/2}}{\sqrt{\ell}} \left(\frac{-m}{\ell} \right) \sum_{n=1}^{\ell-1} \left(\frac{n}{\ell} \right) e^{2\pi i n / \ell} + O(q) \\
 (4.8) \qquad &= i^{(\ell-1)/2} \left(\frac{-m}{\ell} \right) \begin{cases} 1 + O(q) & \text{if } \ell \equiv 1 \pmod{4}, \\ i + O(q) & \text{if } \ell \equiv 3 \pmod{4} \end{cases} \\
 &= \left(\frac{8}{\ell} \right) \left(\frac{-m}{\ell} \right) + O(q) \\
 &= \left(\frac{12}{\ell} \right) + O(q),
 \end{aligned}$$

where we have used Gauss' result [4, Section 9.10] in our third equality. Combining (4.4), (4.5), and (4.8), we find that

$$\begin{aligned}
 (f_\ell(\tau) - g_\ell(\tau)) \left| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} \right| &= \frac{1}{\ell} q^{-(\ell^2-1)/24} (q; q)_\infty \left(\frac{\mathcal{B}_\ell(\tau)}{(q^\ell; q^\ell)_\infty^\ell} - \frac{1}{(q^{\ell^2}; q^{\ell^2})_\infty} \right) \\
 (4.9) \qquad &\quad - \left(\frac{12}{\ell} \right) - 1 + O(q).
 \end{aligned}$$

We now claim that

$$(4.10) \qquad \mathcal{B}_\ell(\tau) = 1 + 2\ell q^{(\ell-1)/2} + \dots,$$

so that

$$\frac{\mathcal{B}_\ell(\tau)}{(q^\ell; q^\ell)_\infty^\ell} - \frac{1}{(q^{\ell^2}; q^{\ell^2})_\infty} = 2\ell q^{(\ell-1)/2} + \dots.$$

Recall that $\mathcal{B}_\ell(\tau)$ is defined to be the theta series associated to the lattice whose Gram matrix is $\ell A_{\ell-1}^{-1}$, where A_n^{-1} is given by (2.2). In other words, we have

$$\mathcal{B}_\ell(\tau) = \sum_{m_1, \dots, m_{\ell-1} \in \mathbf{Z}} q^{Q'(m_1, \dots, m_{\ell-1})}, \quad q = e^{2\pi i \tau},$$

where

$$\begin{aligned}
 Q'(m_1, \dots, m_{\ell-1}) &= \frac{\ell-1}{2} (m_1^2 + \dots + m_{\ell-1}^2) - \frac{1}{2} \sum_{i \neq j} m_i m_j \\
 &= \frac{1}{2} (\ell(m_1^2 + \dots + m_{\ell-1}^2) - (m_1 + \dots + m_{\ell-1})^2).
 \end{aligned}$$

For each $(m_1, \dots, m_{\ell-1}) \in \mathbf{Z}^{\ell-1} \setminus \{0\}$, let r be the number of non-zero entries in the tuple. By the Cauchy-Schwarz inequality, we have

$$(m_1 + \dots + m_{\ell-1})^2 \leq r(m_1^2 + \dots + m_{\ell-1}^2).$$

Then

$$Q'(m_1, \dots, m_{\ell-1}) \geq \frac{1}{2}(\ell - r)(m_1^2 + \dots + m_{\ell-1}^2) \geq \frac{1}{2}(\ell - r)r \geq \frac{\ell - 1}{2}.$$

Therefore, the coefficient of q^j in $\mathcal{B}_\ell(\tau)$ vanishes for $j = 1, \dots, (\ell-1)/2 - 1$. Also, the contribution to the $q^{(\ell-1)/2}$ term comes from the cases where $r = 1$ or $r = \ell - 1$ and equality holds for each of the inequalities above. In other words, the contribution

to $q^{(\ell-1)/2}$ comes from the tuples where exactly one of m_j is ± 1 and all the others are 0 or $(m_1, \dots, m_{\ell-1}) = \pm(1, \dots, 1)$. We conclude that the coefficient of $q^{(\ell-1)/2}$ in $\mathcal{B}_\ell(\tau)$ is 2ℓ . This proves the claim (4.10).

For the cases $\ell = 5$ and $\ell = 7$, we have $(\ell^2 - 1)/24 < (\ell - 1)/2$ and $(\frac{12}{\ell}) = -1$. Therefore,

$$(4.11) \quad (f_\ell(\tau) - g_\ell(\tau)) \Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = O(q).$$

When $\ell = 11$, we have $(\ell^2 - 1)/24 = (\ell - 1)/2$ and $(\frac{12}{\ell}) = 1$. Again, (4.9) implies that (4.11) holds in this case. For other cases, we note that in general we have

$$\mathcal{B}_\ell(\tau) = 1 + 2\ell q^{(\ell-1)/2} + \ell(\ell - 1)q^{\ell-2} + \dots,$$

and hence

$$q^{-(\ell^2-1)/24}(q; q)_\infty \left(\frac{\mathcal{B}_\ell(\tau)}{(q^\ell; q^\ell)_\infty^\ell} - \frac{1}{(q^{\ell^2}; q^{\ell^2})_\infty} \right) = 2\ell q^{-(\ell^2-1)/24+(\ell-1)/2}(1-q-q^2+\dots)$$

for $\ell \geq 11$. When $\ell = 13$, we have $-(\ell^2 - 1)/24 + (\ell - 1)/2 = -1$ and $(\frac{12}{13}) = 1$. Then from (4.9), we deduce that

$$(f_\ell(\tau) - g_\ell(\tau)) \Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = 2q^{-1} - 4 + O(q).$$

For other primes $\ell \geq 17$, (4.9) yields

$$(f_\ell(\tau) - g_\ell(\tau)) \Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} = 2q^{-(\ell^2-1)/24+(\ell-1)/2}(1 - q + O(q^2))$$

instead. This completes the proof of the lemma. □

Theorem 4.1. *Let $\ell \geq 5$ be a prime. Let*

$$f_\ell(\tau) = (q^\ell; q^\ell)_\infty \text{C}\Phi_\ell(q)$$

and

$$g_\ell(\tau) = 1 + \ell(q^\ell; q^\ell)_\infty \sum_{n=1}^\infty p\left(\ell n - \frac{\ell^2 - 1}{24}\right) q^n.$$

- (a) *If $\ell = 5, 7, 11$, then $f_\ell(\tau) = g_\ell(\tau)$.*
- (b) *If $\ell = 13$, then*

$$f_{13}(\tau) = g_{13}(\tau) + 26 \frac{\eta^2(13\tau)}{\eta^2(\tau)}.$$

- (c) *If $\ell \geq 17$, then*

$$(4.12) \quad h_\ell(\tau) = f_\ell(\tau) - g_\ell(\tau) - 2\ell^{(\ell-11)/2} \left(\frac{\eta(\ell\tau)}{\eta(\tau)} \right)^{\ell-11}$$

is a modular function on $\Gamma_0(\ell)$ with a zero at ∞ and a pole of order $(\ell + 1)(\ell - 13)/24$ at 0, and

$$h_\ell(\tau)(\eta(\tau)\eta(\ell\tau))^{\ell-13}$$

is a holomorphic modular form of weight $\ell - 13$ with a zero of order $(\ell - 1)(\ell - 11)/24$ at ∞ .

(d) We have

$$h_\ell(\tau) \equiv 0 \pmod{\begin{cases} 170 & \text{when } \ell = 17, \\ 266 & \text{when } \ell = 19, \\ 506 & \text{when } \ell = 23. \end{cases}}$$

(e) For any prime $\ell > 11$,

$$h_\ell(\tau) \equiv \ell F_\ell(\tau) \pmod{\ell^2},$$

where $F_\ell(\tau)$ is a non-zero modular form of weight $\ell - 1$ on $SL(2, \mathbf{Z})$.

Proof. We first remark that the functions $f_\ell(\tau)$ and $g_\ell(\tau)$ are both holomorphic on the upper half-plane. Thus, to prove that $f_\ell(\tau) = g_\ell(\tau)$ for the cases $\ell = 5, 7, 11$, we only need to verify that $f_\ell(\tau) - g_\ell(\tau)$ does not have poles at cusps and $f_\ell(\tau) - g_\ell(\tau)$ vanishes at one particular point in these three cases. Indeed, by Lemma 4.1, $f_\ell(\tau) - g_\ell(\tau)$ vanishes at both cusps in the three cases since $p(\ell - \delta_\ell) = \ell$ for $\ell = 5, 7$, and 11 . This proves (a). We remark that in fact it suffices to know that $f_\ell(\tau) - g_\ell(\tau)$ has no pole at the cusp 0 since it would mean that $f_\ell(\tau) - g_\ell(\tau)$ is a constant. Since the expansion at ∞ begins with $\ell(\ell - p(\ell - \delta_\ell))q$, the only possibility that $f_\ell(\tau) - g_\ell(\tau)$ is a constant is when $p(\ell - \delta_\ell) = \ell$. In other words, without listing out the partitions of $5, 7$, and 11 , we know that $p(4) = 5, p(5) = 7$, and $p(6) = 11$.

We next consider the case $\ell = 13$. By Lemma 4.1, the Fourier expansion of $f_\ell(\tau) - g_\ell(\tau)$ at 0 is

$$2q^{-1} - 4 + \dots$$

Now we observe that $\eta(13\tau)^2/\eta(\tau)^2$ is also a modular function on $\Gamma_0(13)$ and satisfies

$$\frac{\eta^2(13\tau)}{\eta^2(\tau)} \Big| \begin{pmatrix} 0 & -1 \\ 13 & 0 \end{pmatrix} = \frac{1}{13} \frac{\eta^2(\tau)}{\eta^2(13\tau)} = \frac{1}{13}(q^{-1} - 2 + \dots).$$

Therefore, $f_\ell(\tau) - g_\ell(\tau) - 26\eta(13\tau)^2/\eta(\tau)^2$ is a modular function on $\Gamma_0(13)$ that has no poles and vanishes at the cusps. We conclude that $f_\ell(\tau) - g_\ell(\tau) - 26\eta(13\tau)^2/\eta(\tau)^2$ is identically 0 , and the proof of (b) is complete.

Similarly, for primes $\ell \geq 17$, using Lemma 4.1 and the transformation formula of $\eta(\tau)$, we find that

$$\begin{aligned} h_\ell(\tau) \Big| \begin{pmatrix} 0 & -1 \\ \ell & 0 \end{pmatrix} &= 2q^{-(\ell^2-1)/24+(\ell-1)/2}((1 - q + O(q^2)) - (1 - (\ell - 11)q + O(q^2))) \\ &= 2(\ell - 12)q^{-(\ell+1)(\ell-13)/24} + \dots \end{aligned}$$

Therefore, $h_\ell(\tau)$ has a pole of order $(\ell + 1)(\ell - 13)/24$ for $\ell \geq 17$. From Lemma 4.1, it is clear that $h_\ell(\tau)$ has a zero at ∞ . It follows that $h_\ell(\tau)(\eta(\tau)\eta(\ell\tau))^{\ell-13}$ is a holomorphic modular form of weight $\ell - 13$ on $\Gamma_0(\ell)$, and this completes the proof of (c).

The congruences in (d) can be verified using Sturm’s criterion [39].

Next, observe from (1.19) that

$$f_\ell(\tau) \equiv 1 \pmod{\ell^2}.$$

For $\ell > 13$,

$$h_\ell(\tau) \equiv f_\ell(\tau) - g_\ell(\tau) \equiv -\ell(q^\ell; q^\ell)_\infty \sum_{j=1}^\infty p\left(\ell j - \frac{\ell^2 - 1}{24}\right) q^j \pmod{\ell^2}.$$

It is known that (see [3, Corollary 5.15.1, p. 157] for a proof given by J.P. Serre)

$$(q^\ell; q^\ell)_\infty \sum_{j=1}^\infty p\left(\ell j - \frac{\ell^2 - 1}{24}\right) q^j = F_\ell(\tau) + \ell E_\ell(\tau),$$

where $F_\ell(\tau)$ is a cusp form on $SL(2, \mathbf{Z})$ of weight $\ell - 1$. This implies that

$$h_\ell(\tau) \equiv -\ell F_\ell(\tau) \pmod{\ell^2}.$$

The fact that $F_\ell(\tau)$ is non-zero follows from the result of S. Ahlgren and M. Boylan [1, Theorem 1]. □

We now give another representation for $C\Phi_{17}(q)$. Let

$$h_1(\tau) = \eta^8(17\tau) \sum_{k=0}^7 \mathcal{E}_{3^k}^{-1}(\tau) \mathcal{E}_{2 \cdot 3^k}^{-2}(\tau) \mathcal{E}_{5 \cdot 3^k}^{-1}(\tau) = q^4 + 3q^5 + 8q^6 + 5q^7 + \dots$$

and

$$h_2(\tau) = \eta^8(17\tau) \sum_{k=0}^7 \mathcal{E}_{7 \cdot 3^k}(\tau) \mathcal{E}_{3^k}^{-2}(\tau) \mathcal{E}_{3^{k+1}}^{-1}(\tau) \mathcal{E}_{5 \cdot 3^k}^{-1}(\tau) \mathcal{E}_{8 \cdot 3^k}^{-1}(\tau) = q^4 + q^5 + 8q^6 + \dots,$$

where $\mathcal{E}_a(\tau)$ is given by (3.27). Then

$$h_{17}(\tau) \eta^4(\tau) \eta^4(17\tau) = 595h_1(\tau) - 425h_2(\tau).$$

This gives the identity

$$(4.13) \quad C\Phi_{17}(q) = \frac{1}{(q^{17}; q^{17})_\infty} + 17 \sum_{j=1}^\infty p(17j - 12) q^j + 2 \cdot 17^3 q^4 \frac{(q^{17}; q^{17})_\infty^5}{(q; q)_\infty^6} + \frac{1}{q^3 (q; q)_\infty^4 (q^{17}; q^{17})_\infty^5} (595h_1(\tau) - 425h_2(\tau)).$$

Note the simplicity of (4.13) as compared to (3.28). Identities similar to (4.13) exist for $k = 19, 23$, and other primes. These identities involve the function $\mathcal{E}_a(\tau)$.

5. GENERATING FUNCTION OF $c\phi_k(n)$ FOR $k = 9$ AND 15

There are two cases to consider in this section, namely, $k = 9$ and 15 .

5.1. Case $k = 9$.

Let

$$\begin{aligned} E_{9,1} &= \frac{1}{240} + \sum_{k=1}^\infty \frac{k^3 q^k}{1 - q^k}, \\ E_{9,2} &= \frac{1}{240} + \sum_{k=1}^\infty \frac{k^3 q^{3k}}{1 - q^{3k}}, \\ E_{9,3} &= \frac{1}{240} + \sum_{k=1}^\infty \frac{k^3 q^{9k}}{1 - q^{9k}}, \end{aligned}$$

and

$$E_{9,4} = \sum_{n=1}^\infty \binom{n}{3} \sum_{d|n} d^3 q^n.$$

These are Eisenstein series of $M_4(\Gamma_0(9))$.

Theorem 5.1. *We have*

$$\begin{aligned}
 (5.1) \quad C\Phi_9(q) &= 324q \frac{(q^3; q^3)_\infty^8}{(q; q)_\infty^9} + 19683q^4 \frac{(q^9; q^9)_\infty^{12}}{(q; q)_\infty^9 (q^3; q^3)_\infty^4} - 240q \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty^4} \\
 &\quad - 1458q^2 \frac{(q^9; q^9)_\infty^6}{(q; q)_\infty^3 (q^3; q^3)_\infty^4} + \frac{(q; q)_\infty^3}{(q^3; q^3)_\infty^4} \\
 (5.2) \quad &= \frac{1}{(q; q)_\infty^9} (81E_{9,1} - 84E_{9,2} + 243E_{9,3} - 3E_{9,4} - 6q(q^3; q^3)_\infty^8).
 \end{aligned}$$

Proof. By Theorem 2.1, we find that $\mathfrak{A}_9(q) \in M_4(\Gamma_0(9))$. Next, from [31, Theorem 1.34], we find that $\dim M_4(\Gamma_0(9)) = 5$ and the basis is given by

$$\begin{aligned}
 B_{9,1} &= \eta^8(3\tau), \quad B_{9,2} = \frac{\eta^{12}(9\tau)}{\eta^4(3\tau)}, \quad B_{9,3} = \frac{\eta^9(\tau)\eta^3(9\tau)}{\eta^4(3\tau)}, \\
 B_{9,4} &= \frac{\eta^6(\tau)\eta^6(9\tau)}{\eta^4(3\tau)}, \quad B_{9,5} = \frac{\eta^{12}(\tau)}{\eta^4(3\tau)}.
 \end{aligned}$$

By comparing Fourier coefficients of $\mathfrak{A}_9(q)$ and $B_{9,j}, 1 \leq j \leq 5$, we deduce that

$$(5.3) \quad \mathfrak{A}_9(q) = 324B_{9,1} + 19683B_{9,2} - 240B_{9,3} - 1458B_{9,4} + B_{9,5}.$$

This proves (5.1).

We can replace the basis $\{B_{9,j} | 1 \leq j \leq 5\}$ by $\{B_{9,1}, E_{9,j} | 1 \leq j \leq 4\}$. Using these modular forms as a basis for $M_4(\Gamma_0(9))$, we deduce (5.2). \square

Theorem 5.2. *For any integer $n \geq 0$, we have*

$$(5.4) \quad c\phi_9(9n + 3) \equiv c\phi_9(9n + 6) \equiv 0 \pmod{9},$$

$$(5.5) \quad c\phi_9(3n + 1) \equiv 0 \pmod{81},$$

and

$$(5.6) \quad c\phi_9(3n + 2) \equiv 0 \pmod{729}.$$

Proof. From [40, Lemma 2.5], we find that

$$(5.7) \quad (q; q)_\infty^3 = S(q^3) - 3q(q^9; q^9)_\infty^3,$$

where

$$(5.8) \quad S(q) = (q; q)_\infty (\Theta_3(q)\Theta_3(q^3) + \Theta_2(q)\Theta_2(q^3)).$$

From (5.1), we deduce that

$$\begin{aligned}
 (5.9) \quad \sum_{n=0}^{\infty} c\phi_9(n)q^n &\equiv 2^2 \cdot 3^4 q \frac{(q^3; q^3)_\infty^8}{(q; q)_\infty^9} - 240q \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty^4} + \frac{(q; q)_\infty^3}{(q^3; q^3)_\infty^4} \pmod{729} \\
 &\equiv 2^2 \cdot 3^4 q (q^3; q^3)_\infty^5 - 240q \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty^4} + \frac{S(q^3)}{(q^3; q^3)_\infty^4} - 3q \frac{(q^9; q^9)_\infty^3}{(q^3; q^3)_\infty^4} \pmod{729}.
 \end{aligned}$$

Comparing the coefficients of q^{3n+2} on both sides, we deduce that

$$c\phi_9(3n + 2) \equiv 0 \pmod{729}.$$

Extracting the terms of the form q^{3n+1} on both sides of (5.9), dividing by q , and replacing q^3 by q , we deduce that

$$\begin{aligned}
 \sum_{n=0}^{\infty} c\phi_9(3n+1)q^n &\equiv 2^2 \cdot 3^4(q; q)_{\infty}^5 - 240 \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}^4} - 3 \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}^4} \pmod{729} \\
 &\equiv 2^2 \cdot 3^4(q; q)_{\infty}^5 - 243 \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}^4} \pmod{729} \\
 &\equiv 2^2 \cdot 3^4(q; q)_{\infty}^5 - 3^5(q; q)_{\infty}^5 \pmod{729} \\
 (5.10) \quad &\equiv 3^4(q; q)_{\infty}^5 \pmod{729},
 \end{aligned}$$

which implies (5.5).

Extracting the terms of the form q^{3n} on both sides of (5.9) and replacing q^3 by q , we find that

$$\begin{aligned}
 \sum_{n=0}^{\infty} c\phi_9(3n)q^n &\equiv \frac{S(q)}{(q; q)_{\infty}^4} \pmod{729} \\
 (5.11) \quad &\equiv \frac{1}{(q; q)_{\infty}^3} (\Theta_3(q)\Theta_3(q^3) + \Theta_2(q)\Theta_2(q^3)) \pmod{729}.
 \end{aligned}$$

From [40, Lemma 2.6], we deduce that

$$(5.12) \quad \frac{1}{(q; q)_{\infty}^3} = \frac{(q^9; q^9)_{\infty}^3}{(q^3; q^3)_{\infty}^{12}} (S^2(q^3) + 3qS(q^3)(q^9; q^9)_{\infty}^3 + 9q^2(q^9; q^9)_{\infty}^6).$$

From [8, Corollaries (i) and (ii), p. 49], we find that

$$(5.13) \quad \Theta_3(q) = \Theta_3(q^9) + 2qf(q^3, q^{15})$$

and

$$(5.14) \quad \Theta_2(q) = \Theta_2(q^9) + 2q^{1/4}f(q^6, q^{12}),$$

where

$$f(a, b) = (-a; ab)_{\infty}(-b; ab)_{\infty}(ab; ab)_{\infty}.$$

Substituting (5.12)–(5.14) into (5.11), we deduce that

$$\begin{aligned}
 \sum_{n=0}^{\infty} c\phi_9(3n)q^n &\equiv \frac{(q^9; q^9)_{\infty}^3}{(q^3; q^3)_{\infty}^{12}} \left(S^2(q^3) + 3qS(q^3)(q^9; q^9)_{\infty}^3 \right) \\
 (5.15) \quad &\times \left(\Theta_3(q^3) \left(\Theta_3(q^9) + 2qf(q^3, q^{15}) \right) + \Theta_2(q^3) \left(\Theta_2(q^9) + 2q^{1/4}f(q^6, q^{12}) \right) \right) \pmod{9}.
 \end{aligned}$$

Extracting the terms of the form q^{3n+1} on both sides of (5.15), dividing by q , and replacing q^3 by q , applying (5.8), we deduce that

$$\begin{aligned}
 & \sum_{n=0}^{\infty} c\phi_9(9n+3)q^n \\
 & \equiv S^2(q) \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}^{12}} \left(3 \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}} + 2 \left(\Theta_3(q)f(q, q^5) + q^{-1/4} \Theta_2(q)f(q^2, q^4) \right) \right) \pmod{9} \\
 & \equiv S^2(q) \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}^{12}} \\
 (5.16) \quad & \times \left(3 \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}} + 2 \frac{(q^2; q^2)_{\infty}^7 (q^3; q^3)_{\infty} (q^{12}; q^{12})_{\infty}}{(q; q)_{\infty}^3 (q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}} + 4 \frac{(q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}^2}{(q^2; q^2)_{\infty}^2 (q^{12}; q^{12})_{\infty}} \right) \pmod{9},
 \end{aligned}$$

where the last congruence follows by converting

$$\Theta_3(q)f(q, q^5) + q^{-1/4} \Theta_2(q)f(q^2, q^4)$$

to infinite products.

From [41, (3.75), (3.38)], we find that

$$(5.17) \quad \frac{(q^3; q^3)_{\infty}}{(q; q)_{\infty}^3} = \frac{(q^4; q^4)_{\infty}^6 (q^6; q^6)_{\infty}^3}{(q^2; q^2)_{\infty}^9 (q^{12}; q^{12})_{\infty}^2} + 3q \frac{(q^4; q^4)_{\infty}^2 (q^6; q^6)_{\infty} (q^{12}; q^{12})_{\infty}^2}{(q^2; q^2)_{\infty}^7}$$

and

$$(5.18) \quad \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}} = \frac{(q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}^2}{(q^2; q^2)_{\infty}^2 (q^{12}; q^{12})_{\infty}} + q \frac{(q^{12}; q^{12})_{\infty}^3}{(q^4; q^4)_{\infty}}.$$

By (5.17), we find that

$$\begin{aligned}
 (5.19) \quad & 2 \frac{(q^2; q^2)_{\infty}^7 (q^3; q^3)_{\infty} (q^{12}; q^{12})_{\infty}}{(q; q)_{\infty}^3 (q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}} \\
 & = 2 \frac{(q^2; q^2)_{\infty}^7 (q^{12}; q^{12})_{\infty}}{(q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}} \left(\frac{(q^4; q^4)_{\infty}^6 (q^6; q^6)_{\infty}^3}{(q^2; q^2)_{\infty}^9 (q^{12}; q^{12})_{\infty}^2} + 3q \frac{(q^4; q^4)_{\infty}^2 (q^6; q^6)_{\infty} (q^{12}; q^{12})_{\infty}^2}{(q^2; q^2)_{\infty}^7} \right) \\
 (5.20) \quad & = 2 \frac{(q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}^2}{(q^2; q^2)_{\infty}^2 (q^{12}; q^{12})_{\infty}} + 6q \frac{(q^{12}; q^{12})_{\infty}^3}{(q^4; q^4)_{\infty}},
 \end{aligned}$$

and this implies that

$$\begin{aligned}
 & 2 \frac{(q^2; q^2)_{\infty}^7 (q^3; q^3)_{\infty} (q^{12}; q^{12})_{\infty}}{(q; q)_{\infty}^3 (q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}} + 4 \frac{(q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}^2}{(q^2; q^2)_{\infty}^2 (q^{12}; q^{12})_{\infty}} \\
 & = 6 \frac{(q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}^2}{(q^2; q^2)_{\infty}^2 (q^{12}; q^{12})_{\infty}} + 6q \frac{(q^{12}; q^{12})_{\infty}^3}{(q^4; q^4)_{\infty}} \\
 (5.21) \quad & = 6 \frac{(q^3; q^3)_{\infty}^3}{(q; q)_{\infty}},
 \end{aligned}$$

where we have used (5.18) in the last equality. Substituting (5.21) into (5.16), we deduce that

$$c\phi_9(9n+3) \equiv 0 \pmod{9}.$$

Extracting the terms of the form q^{3n+2} on both sides of (5.15), dividing by q^2 , and replacing q^3 by q , we deduce that

$$\begin{aligned} & \sum_{n=0}^{\infty} c\phi_9(9n+6)q^n \\ & \equiv S(q) \frac{(q^3; q^3)_{\infty}^6}{(q; q)_{\infty}^{12}} \left(6 \frac{(q^2; q^2)_{\infty}^7 (q^3; q^3)_{\infty} (q^{12}; q^{12})_{\infty}}{(q; q)_{\infty}^3 (q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}} + 3 \frac{(q^4; q^4)_{\infty}^3 (q^6; q^6)_{\infty}^2}{(q^2; q^2)_{\infty}^2 (q^{12}; q^{12})_{\infty}} \right) \pmod{9} \\ & \equiv 0 \pmod{9}, \end{aligned}$$

where we have used (5.20) to deduce the last congruence. Hence

$$c\phi_9(9n+6) \equiv 0 \pmod{9}.$$

□

Congruences (5.4) and (5.5) can also be established using congruences discovered by Kolitsch. In [22], Kolitsch generalized Andrews’ congruence (3.29) and proved that

$$(5.22) \quad \sum_{d|(k,n)} \mu(d)c\phi_{\frac{k}{d}}\left(\frac{n}{d}\right) \equiv 0 \pmod{k^2},$$

where $\mu(n)$ is the Möbius function (see for example [4, Section 2.2]). We now prove a generalization of (5.4) and (5.5). For any non-negative integer k , we set $c\phi_k(x) = 0$ whenever $x \notin \mathbf{Z}$. We can then rewrite (5.22) as

$$(5.23) \quad \sum_{d|k} \mu(d)c\phi_{\frac{k}{d}}\left(\frac{n}{d}\right) \equiv 0 \pmod{k^2}.$$

Theorem 5.3. *Let p be a prime and let N be a positive integer which is not divisible by p . For any integers $\alpha \geq 1$ and $n \geq 0$, we have*

$$(5.24) \quad c\phi_{p^\alpha N}(n) \equiv c\phi_{p^{\alpha-1}N}(n/p) \pmod{p^{2\alpha}}$$

or, equivalently,

$$(5.25) \quad c\phi_{p^\alpha N}(pn+r) \equiv 0 \pmod{p^{2\alpha}}, \quad 1 \leq r \leq p-1,$$

and

$$(5.26) \quad c\phi_{p^\alpha N}(pn) \equiv c\phi_{p^{\alpha-1}N}(n) \pmod{p^{2\alpha}}.$$

Proof. Let $\Omega(N)$ be the number of prime divisors of N (counting multiplicities). We proceed by induction on $\Omega(N)$. If $\Omega(N) = 0$, then $N = 1$. Setting $k = p^\alpha$ in (5.23), we deduce that

$$(5.27) \quad c\phi_{p^\alpha}(n) \equiv c\phi_{p^{\alpha-1}}(n/p) \pmod{p^{2\alpha}}.$$

Thus, (5.24) is true if $\Omega(N) = 0$. Assume that (5.24) is true if $\Omega(N) < h$, where h is a positive integer. When $\Omega(N) = h$, we set $k = p^\alpha N$ in (5.23). Since p does not divide N , any positive divisor of $p^\alpha N$ has the form $p^j d$ where $0 \leq j \leq \alpha$ and $d|N$. In particular, if $j \geq 2$, then $\mu(p^j d') = 0$. Hence by (5.23), we obtain

$$(5.28) \quad \sum_{d|N} \left(\mu(d)c\phi_{\frac{p^\alpha N}{d}}\left(\frac{n}{d}\right) + \mu(pd)c\phi_{\frac{p^{\alpha-1}N}{d}}\left(\frac{n}{pd}\right) \right) \equiv 0 \pmod{p^{2\alpha}}.$$

According to $d = 1$ or $d > 1$, we separate the summands on the left hand side of (5.28) and deduce that

$$(5.29) \quad c\phi_{p^\alpha N}(n) - c\phi_{p^{\alpha-1}N}\left(\frac{n}{p}\right) + \sum_{d|N, d>1} \mu(d) \left(c\phi_{\frac{p^\alpha N}{d}}\left(\frac{n}{d}\right) - c\phi_{\frac{p^{\alpha-1}N}{d}}\left(\frac{n}{pd}\right) \right) \equiv 0 \pmod{p^{2\alpha}}.$$

Note that in the summand, since $d > 1$, we have $\Omega(\frac{N}{d}) < h$, and hence by assumption

$$(5.30) \quad c\phi_{\frac{p^\alpha N}{d}}\left(\frac{n}{d}\right) - c\phi_{\frac{p^{\alpha-1}N}{d}}\left(\frac{n}{pd}\right) \equiv 0 \pmod{p^{2\alpha}}.$$

From (5.30) and (5.29), we deduce that

$$c\phi_{p^\alpha N}(n) - c\phi_{p^{\alpha-1}N}\left(\frac{n}{p}\right) \equiv 0 \pmod{p^{2\alpha}}.$$

Hence (5.24) is true when $\Omega(N) = h$. This completes the proof of (5.24).

Replacing n in (5.24) by $pn + r$, where $0 \leq r \leq p - 1$, and observing that

$$c\phi_{p^{\alpha-1}N}\left(\frac{pn+r}{p}\right) = 0, \quad 1 \leq r \leq p - 1,$$

we deduce (5.25) and (5.26). □

Let $(p, \alpha, N) = (3, 2, 1)$ in Theorem 5.3. By (5.25), we deduce that

$$c\phi_9(3n + 1) \equiv c\phi_9(3n + 2) \equiv 0 \pmod{81},$$

and this gives another proof of (5.5). Similarly, by (5.26), we deduce that

$$(5.31) \quad c\phi_9(3n) \equiv c\phi_3(n) \pmod{81}.$$

By (3.29), we find that

$$c\phi_3(3n + 1) \equiv c\phi_3(3n + 2) \equiv 0 \pmod{9}.$$

Substituting these congruences into (5.31), we complete the proof of (5.4).

5.2. Case $k = 15$.

Let

$$\begin{aligned} f_{15}(\tau) &= \frac{\eta^2(\tau)\eta^2(15\tau)}{\eta(3\tau)\eta(5\tau)}, \\ h_{15}(\tau) &= \eta^4(\tau)\eta^4(5\tau) - 9\eta^4(3\tau)\eta^4(15\tau), \\ g_{15,1}(\tau) &= -\frac{1}{8}(E_2(\tau) + 3E_2(3\tau) - 5E_2(5\tau) - 15E_2(15\tau)), \\ g_{15,2}(\tau) &= -\frac{1}{12}(E_2(\tau) - 3E_2(3\tau) + 5E_2(5\tau) - 15E_2(15\tau)), \\ g_{15,3}(\tau) &= \eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau), \end{aligned}$$

and

$$g_{15,4}(\tau) = \frac{1}{8}(E_2(\tau) - 3E_2(3\tau) - 5E_2(5\tau) + 15E_2(15\tau)),$$

where

$$E_2(\tau) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k}.$$

Using dimension formula [31, Theorem 1.34], we find that

$$\dim M_7\left(\Gamma_0(15), \left(\frac{-15}{\cdot}\right)\right) = 14.$$

The modular forms

$$\begin{aligned} B_{15,1} &= f_{15}g_{15,1}^3, & B_{15,2} &= f_{15}g_{15,1}^2g_{15,2}, \\ B_{15,3} &= f_{15}g_{15,1}g_{15,2}^2, & B_{15,4} &= f_{15}g_{15,2}^3, \\ B_{15,5} &= f_{15}g_{15,1}^2g_{15,3}, & B_{15,6} &= f_{15}g_{15,1}^2g_{15,4}, \\ B_{15,7} &= f_{15}g_{15,1}g_{15,2}g_{15,3}, & B_{15,8} &= f_{15}g_{15,1}g_{15,2}g_{15,4}, \\ B_{15,9} &= f_{15}g_{15,2}^2g_{15,3}, & B_{15,10} &= f_{15}g_{15,2}^2g_{15,4}, \\ B_{15,11} &= f_{15}g_{15,1}h_{15}, & B_{15,12} &= f_{15}g_{15,2}h_{15}, \\ B_{15,13} &= \frac{\eta^{14}(3\tau)\eta^{14}(5\tau)}{\eta^7(\tau)\eta^7(15\tau)}, & \text{and } B_{15,14} &= \frac{\eta^{17}(\tau)\eta^2(5\tau)}{\eta^4(3\tau)\eta(15\tau)} \end{aligned}$$

form a basis for $M_7\left(\Gamma_0(15), \left(\frac{-15}{\cdot}\right)\right)$.

Using the fact that $\mathfrak{A}_{15}(q) \in M_7\left(\Gamma_0(15), \left(\frac{-15}{\cdot}\right)\right)$, we deduce the following.

Theorem 5.4. For $|q| < 1$,

$$\begin{aligned} C\Phi_{15}(q) &= \frac{1}{(q; q)_{\infty}^{15}} \left(\frac{18125225}{1156} B_{15,1} - \frac{845079}{34} B_{15,2} - \frac{87564447}{1156} B_{15,3} \right. \\ &\quad + \frac{2491641}{34} B_{15,4} + \frac{147166525}{1156} B_{15,5} + \frac{341957}{68} B_{15,6} \\ &\quad - \frac{483081}{17} B_{15,7} - \frac{28623}{4} B_{15,8} - \frac{9784683}{68} B_{15,9} \\ &\quad - \frac{1168839}{34} B_{15,10} + \frac{7263781}{68} B_{15,11} - \frac{97629}{4} B_{15,12} \\ &\quad \left. + 3375 B_{15,13} - 3374 B_{15,14} \right). \end{aligned}$$

6. GENERATING FUNCTION OF $c\phi_k(n)$ FOR EVEN INTEGER $2 < k < 16$

In this section, we derive alternative expressions for $C\Phi_k(n)$ when $k > 2$ is even.

6.1. Case $k = 4$.

Theorem 6.1. We have

$$(6.1) \quad C\Phi_4(q) = \frac{1}{(q; q)_{\infty}^4} \left(\Theta_3^3(q^2) + 3\Theta_3(q^2)\Theta_2^2(q^2) \right)$$

$$(6.2) \quad = \frac{\Theta_3^4(q)}{(q; q)_{\infty}^4 \Theta_3(q^2)} + \frac{\Theta_3^2(-q)\Theta_2^2(q^2)}{(q; q)_{\infty}^4 \Theta_3(q^2)}.$$

Proof. Let $k = 4$ in Theorem 2.1. We deduce that $\mathfrak{A}_4(q)\Theta_3(q) \in M_2\left(\Gamma_0(8), \left(\frac{2}{\cdot}\right)\right)$. From [31, Theorem 1.34], we deduce that

$$\dim M_2\left(\Gamma_0(8), \left(\frac{2}{\cdot}\right)\right) = 3.$$

It can be verified that

$$\Theta_3(q)\Theta_3^3(q^2), \quad \Theta_3(q)^3\Theta_3(q^2), \quad \text{and} \quad \Theta_3(q^2)\Theta_2^2(q^2)\Theta_3(q)$$

form a basis of $M_2(\Gamma_0(8), (\frac{2}{\cdot}))$. Comparing the Fourier coefficients of $\mathfrak{A}_4(q)\Theta_3(q)$ and the given basis of $M_2(\Gamma_0(8), (\frac{2}{\cdot}))$, we deduce that

$$\mathfrak{A}_4(q)\Theta_3(q) = \left(\Theta_3^3(q^2) + 3\Theta_3(q^2)\Theta_2^2(q^2)\right)\Theta_3(q),$$

which proves (6.1).

Theorem 2.1 also implies that $\Theta_3(q^2)\mathfrak{A}_4(q) \in M_2(\Gamma_0(16))$. From [31, Theorem 1.34], we find that $\dim M_2(\Gamma_0(16)) = 5$. Identity (6.2) then follows from the fact that

$$\Theta_3^4(q), \quad \Theta_3^4(q^2), \quad \Theta_3^4(q^4), \quad \Theta_3^2(-q)\Theta_3^2(-q^2), \quad \text{and} \quad \Theta_3^2(-q)\Theta_2^2(q^2)$$

form a basis of $M_2(\Gamma_0(16))$. □

Remark 6.1. The representation (6.2) was first deduced by W. Zhang and C. Wang [43] from (6.1), where they used it to give an elementary proof of the congruence

$$c\phi_4(7n + 5) \equiv 0 \pmod{7}.$$

6.2. Case $k = 6$.

Theorem 6.2. *We have*

$$\begin{aligned} C\Phi_6(q) &= \frac{4}{9} \frac{(q; q)_\infty^5 (q^4; q^4)_\infty^2}{(q^2; q^2)_\infty^5 (q^3; q^3)_\infty^3} - \frac{1}{3} \frac{(q^2; q^2)_\infty^4 (q^4; q^4)_\infty^2}{(q; q)_\infty^4 (q^6; q^6)_\infty^3} \\ &\quad + \frac{8}{9} \frac{(q^4; q^4)_\infty^{11}}{(q; q)_\infty^4 (q^2; q^2)_\infty^5 (q^{12}; q^{12})_\infty^3} + 36q \frac{(q^4; q^4)_\infty^2 (q^3; q^3)_\infty^9}{(q; q)_\infty^7 (q^2; q^2)_\infty^5} \\ (6.3) \quad &\quad + 27q^2 \frac{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty^9}{(q; q)_\infty^4 (q^2; q^2)_\infty^8} + 72q^4 \frac{(q^{12}; q^{12})_\infty^9}{(q; q)_\infty^4 (q^2; q^2)_\infty^5 (q^4; q^4)_\infty}. \end{aligned}$$

Proof. Let $k = 6$ in Theorem 2.1. We deduce that $\Theta_3(q)\mathfrak{A}_6(q) \in M_3(\Gamma_0(12), (\frac{-12}{\cdot}))$. From [31, Theorem 1.34], we deduce that

$$\dim M_3\left(\Gamma_0(12), \left(\frac{-12}{\cdot}\right)\right) = 7.$$

Let

$$\begin{aligned} B_{6,1} &= \frac{\eta^9(\tau)}{\eta^3(3\tau)}, \quad B_{6,2} = \frac{\eta^9(2\tau)}{\eta^3(6\tau)}, \quad B_{6,3} = \frac{\eta^9(4\tau)}{\eta^3(12\tau)}, \quad B_{6,4} = \frac{\eta^9(3\tau)}{\eta^3(\tau)}, \\ B_{6,5} &= \frac{\eta^9(6\tau)}{\eta^3(2\tau)}, \quad B_{6,6} = \frac{\eta^9(12\tau)}{\eta^3(4\tau)}, \quad \text{and} \quad B_{6,7} = \eta^3(2\tau)\eta^3(6\tau). \end{aligned}$$

The set $\{B_{6,j} | 1 \leq j \leq 7\}$ forms a basis of $M_3(\Gamma_0(12), (\frac{-12}{\cdot}))$, and by comparing the Fourier coefficients of $\Theta_3(q)\mathfrak{A}_6(q)$ and modular forms in $\{B_{6,j} | 1 \leq j \leq 7\}$, we deduce that

$$(6.4) \quad \Theta_3(q)\mathfrak{A}_6(z) = \frac{4}{9}B_{6,1} - \frac{1}{3}B_{6,2} + \frac{8}{9}B_{6,3} + 36B_{6,4} + 27B_{6,5} + 72B_{6,6}.$$

This proves (6.3). □

Congruences for $c\phi_6(n)$ have drawn much attention in recent years. For example, Baruah and Sarmah [7] established 3-dissections of $C\Phi_6(q)$ and proved that

$$(6.5) \quad c\phi_6(3n + 1) \equiv 0 \pmod{9}$$

and

$$(6.6) \quad c\phi_6(3n + 2) \equiv 0 \pmod{9}.$$

We remark here that the congruences above follow directly from (5.25) with $(p, \alpha, N) = (3, 1, 2)$. Moreover, setting $(p, \alpha, N) = (2, 1, 3)$ in (5.25), we deduce that

$$(6.7) \quad c\phi_6(2n + 1) \equiv 0 \pmod{4}.$$

Congruence (6.7) appeared in [7] as Corollary 3.1.

For more congruences satisfied by $c\phi_6(n)$, see a recent paper of C. Gu, L. Wang, and E.X.W. Xia [18] and their list of references.

6.3. Case $k = 8$.

Theorem 6.3. *We have*

$$(6.8) \quad C\Phi_8(q) = \frac{1}{(q; q)_\infty^8} \left(\Theta_3^7(q^4) + 28\Theta_3^6(q^4)\Theta_2(q^4) + 105\Theta_3^5(q^4)\Theta_2^2(q^4) \right. \\ \left. + 112\Theta_3^4(q^4)\Theta_2^3(q^4) + 147\Theta_3^3(q^4)\Theta_2^4(q^4) + 84\Theta_3^2(q^4)\Theta_2^5(q^4) + 35\Theta_3(q^4)\Theta_2^6(q^4) \right).$$

Proof. Let $k = 8$ in Theorem 2.1. We deduce that $\Theta_3(q)\mathfrak{A}_8(q) \in M_4(\Gamma_0(16))$. From [31, Theorem 1.34], we find that

$$\dim M_4(\Gamma_0(16)) = 9,$$

and one can verify that

$$\begin{aligned} B_{8,1} &= \Theta_3(q)\Theta_3^7(q^4), & B_{8,2} &= \Theta_3(q)\Theta_3^6(q^4)\Theta_2(q^4), \\ B_{8,3} &= \Theta_3(q)\Theta_3^5(q^4)\Theta_2^2(q^4), & B_{8,4} &= \Theta_3(q)\Theta_3^4(q^4)\Theta_2^3(q^4), \\ B_{8,5} &= \Theta_3(q)\Theta_3^3(q^4)\Theta_2^4(q^4), & B_{8,6} &= \Theta_3(q)\Theta_3^2(q^4)\Theta_2^5(q^4), \\ B_{8,7} &= \Theta_3(q)\Theta_2^7(q^4), & B_{8,8} &= \Theta_2^8(q^4), \\ \text{and } B_{8,9} &= \Theta_3(q)\Theta_3(q^4)\Theta_2^6(q^4) \end{aligned}$$

form a basis for $M_4(\Gamma_0(16))$. By comparing the Fourier coefficients of the basis and those of $\Theta_3(q)\mathfrak{A}_8(q)$, we find that

$$(6.9) \quad \Theta_3(q)\mathfrak{A}_8(q) = B_{8,1} + 28B_{8,2} + 105B_{8,3} + 112B_{8,4} + 147B_{8,5} + 84B_{8,6} + 35B_{8,9}.$$

This completes the proof of (6.8). □

By (5.27), we find that

$$(6.10) \quad c\phi_8(n) \equiv c\phi_4(n/2) \pmod{64}.$$

In [6], Baruah and Sarmah proved that

$$(6.11) \quad c\phi_4(2n + 1) \equiv 0 \pmod{4^2},$$

$$(6.12) \quad c\phi_4(4n + 2) \equiv 0 \pmod{4},$$

and

$$(6.13) \quad c\phi_4(4n + 3) \equiv 0 \pmod{4^4}.$$

Combining (6.11)–(6.13) with (6.10), we obtain the following congruences for $c\phi_8(n)$.

Theorem 6.4. For any integer $n \geq 0$,

$$(6.14) \quad c\phi_8(2n + 1) \equiv 0 \pmod{64},$$

$$(6.15) \quad c\phi_8(4n + 2) \equiv 0 \pmod{16},$$

$$(6.16) \quad c\phi_8(8n + 4) \equiv 0 \pmod{4},$$

and

$$(6.17) \quad c\phi_8(8n + 6) \equiv 0 \pmod{64}.$$

6.4. Case $k = 10$.

By Theorem 2.1, we have $\Theta_3(q)\mathfrak{A}_{10}(q) \in M_5(\Gamma_0(20), (\frac{-20}{\cdot}))$. From [31, Theorem 1.34], we deduce that

$$\dim M_5(\Gamma_0(20), (\frac{-20}{\cdot})) = 14.$$

Let

$$\begin{aligned} B_{10,1} &= \Theta_3^9(q)\Theta_3(q^5), & B_{10,2} &= \Theta_3(q)\Theta_2^3(q^{1/2})\Theta_2^3(q)\Theta_2^3(q^{5/2}), \\ B_{10,3} &= \Theta_3(q)\Theta_3^3(q^5)\Theta_2^2(q^{1/2})\Theta_2^2(q)\Theta_2^2(q^{5/2}), & B_{10,4} &= \Theta_3(q)\Theta_3(q^5)\Theta_2^2(q^{1/2})\Theta_2^6(q^{5/2}), \\ B_{10,5} &= \Theta_3(q)\Theta_3(q^5)\Theta_2^8(q), & B_{10,6} &= \Theta_3^7(q)\Theta_3^3(q^5), \\ B_{10,7} &= \Theta_3^6(q^5)\Theta_2^3(q^{1/2})\Theta_2(q^{5/2}), & B_{10,8} &= \Theta_3(q)\Theta_3^5(q^5)\Theta_2^4(q), \\ B_{10,9} &= \Theta_3^3(q^5)\Theta_2(q^{1/2})\Theta_2^5(q)\Theta_2(q^{5/2}), & B_{10,10} &= \Theta_3^6(q)\Theta_2^3(q)\Theta_2(q^5), \\ B_{10,11} &= \Theta_3(q)\Theta_3(q^5)\Theta_2^8(q^{1/2}), & B_{10,12} &= \Theta_3(q)\Theta_3(q^5)\Theta_2^6(q)\Theta_2^2(q^5), \\ B_{10,13} &= \Theta_3^5(q)\Theta_3^5(q^5), \text{ and } & B_{10,14} &= \Theta_3^3(q)\Theta_3^3(q^5)\Theta_2^3(q^{1/2})\Theta_2(q^{5/2}). \end{aligned}$$

The set $\{B_{10,j} | 1 \leq j \leq 14\}$ forms a basis of $M_5(\Gamma_0(20), (\frac{-20}{\cdot}))$, and we deduce the following.

Theorem 6.5. We have

$$\begin{aligned} C\Phi_{10}(q) &= \frac{1}{\Theta_3(q)(q; q)_{\infty}^{10}} \left(\frac{13}{8}B_{10,1} + \frac{435}{32}B_{10,2} + \frac{9275}{128}B_{10,3} + \frac{175}{32}B_{10,4} \right. \\ &\quad - \frac{31}{8}B_{10,5} - \frac{15}{4}B_{10,6} + \frac{225}{4}B_{10,7} - \frac{775}{32}B_{10,8} + \frac{221}{32}B_{10,10} - \frac{857}{512}B_{10,11} \\ &\quad \left. + \frac{155}{8}B_{10,12} + \frac{25}{8}B_{10,13} \right). \end{aligned} \tag{6.18}$$

Let

$$\begin{aligned} f_{10} &= \frac{\eta(\tau)\eta(2\tau)\eta(10\tau)\eta(20\tau)}{\eta(4\tau)\eta(5\tau)}, \\ f_{10,1} &= \Theta_3(q)\Theta_3(q^5)\frac{\eta^{10}(\tau)}{\eta^2(5\tau)}, \end{aligned}$$

and

$$f_{10,2} = \Theta_3^6(q^5)\frac{\eta^5(20\tau)}{\eta(4\tau)}.$$

Let

$$\begin{aligned}
 g_{10,1} &= \frac{1}{6} (E_2(\tau) - 4E_2(2\tau) + 4E_2(4\tau) + 5E_2(5\tau) - 20E_2(10\tau) + 20E_2(20\tau)), \\
 g_{10,2} &= \Theta_3^2(q)\Theta_3^2(q^5), \\
 g_{10,3} &= \frac{1}{4} (-E_2(2\tau) + 5E_2(10\tau)), \\
 g_{10,4} &= -\frac{1}{24} (E_2(\tau) + E_2(2\tau) + 4E_2(4\tau) - 5E_2(5\tau) - 5E_2(10\tau) - 20E_2(20\tau)), \\
 g_{10,5} &= \eta^2(2\tau)\eta^2(10\tau),
 \end{aligned}$$

and

$$g_{10,6} = \frac{5}{4}\Theta_3^4(q^5) - \frac{1}{4}\Theta_3^4(q).$$

Let

$$\begin{aligned}
 B_{10,1}^* &= f_{10}g_{10,1}^2, & B_{10,2}^* &= f_{10}g_{10,1}g_{10,2}, \\
 B_{10,3}^* &= f_{10}g_{10,2}^2, & B_{10,4}^* &= f_{10}g_{10,1}g_{10,3}, \\
 B_{10,5}^* &= f_{10}g_{10,1}g_{10,4}, & B_{10,6}^* &= f_{10}g_{10,1}g_{10,5}, \\
 B_{10,7}^* &= f_{10}g_{10,1}g_{10,6}, & B_{10,8}^* &= f_{10}g_{10,2}g_{10,3}, \\
 B_{10,9}^* &= f_{10}g_{10,2}g_{10,4}, & B_{10,10}^* &= f_{10}g_{10,2}g_{10,5}, \\
 B_{10,11}^* &= f_{10}g_{10,2}g_{10,6}, & B_{10,12}^* &= f_{10}g_{10,3}^2, \\
 B_{10,13}^* &= f_{10,1}, & \text{and } B_{10,14}^* &= f_{10,2}.
 \end{aligned}$$

We can replace the basis $\{B_{10,j} \mid 1 \leq j \leq 14\}$ by the basis $\{B_{10,j}^* \mid 1 \leq j \leq 14\}$ and deduce that

$$\begin{aligned}
 C\Phi_{10}(q) &= \frac{1}{\Theta_3(q)(q; q)_{\infty}^{10}} \left(\frac{5075}{2}B_{10,1}^* + \frac{4525}{4}B_{10,2}^* + \frac{29375}{4}B_{10,3}^* + \frac{4525}{2}B_{10,4}^* \right. \\
 &\quad - 4525B_{10,5}^* - 6525B_{10,6}^* + \frac{6275}{4}B_{10,7}^* - 4950B_{10,8}^* + 2300B_{10,9}^* \\
 (6.19) \quad &\quad \left. - 22375B_{10,10}^* + \frac{10325}{4}B_{10,11}^* - 10150B_{10,12}^* + B_{10,13}^* + 200000B_{10,14}^* \right).
 \end{aligned}$$

Identity (6.19) leads immediately to

$$(6.20) \quad C\Phi_{10}(q) \equiv \frac{\Theta_3(q^5)}{(q^5; q^5)_{\infty}^2} \pmod{5^2}.$$

Remark 6.2. Congruence (6.20) is the motivation behind the discovery of Theorem 5.3. Theorem 5.3, when interpreted in terms of generating functions, yields the congruence

$$(6.21) \quad C\Phi_{p\ell}(q) \equiv C\Phi_{\ell}(q^p) \pmod{p^2}$$

for any distinct primes p and ℓ . Congruence (6.20) is a special case of (6.21) once we identify the right hand side of (6.20) with $C\Phi_2(q^5)$ (see (3.1)).

Theorem 6.6. *For any integer $n \geq 0$, we have*

$$(6.22) \quad c\phi_{10}(2n + 1) \equiv 0 \pmod{4},$$

$$(6.23) \quad c\phi_{10}(5n + r) \equiv 0 \pmod{5^2}, \quad 1 \leq r \leq 5,$$

and

$$(6.24) \quad c\phi_{10}(25n + 15) \equiv 0 \pmod{5}.$$

Proof. Congruences (6.22) and (6.23) follow from Theorem 5.3 by setting $(p, \alpha, N) = (2, 1, 5)$ and $(5, 1, 2)$, respectively. Congruence (6.23) also follows from (6.20). Furthermore, from (6.20), we deduce that

$$(6.25) \quad \sum_{n=0}^{\infty} c\phi_{10}(5n)q^n \equiv \frac{\Theta_3(q)}{(q; q)_{\infty}^2} \equiv \frac{\Theta_3(q)(q; q)_{\infty}^3}{(q; q)_{\infty}^5} \pmod{5}$$

$$\equiv \frac{1}{(q^5; q^5)_{\infty}} \left(\sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} (-1)^j (2j + 1) q^{i^2 + j(j+1)/2} \right) \pmod{5}.$$

Note that

$$n = i^2 + \frac{j(j + 1)}{2} \quad \text{if and only if } 8n + 1 = 8i^2 + (2j + 1)^2.$$

Since $\left(\frac{-8}{5}\right) = -1$, we find that $8n + 1 \equiv 0 \pmod{5}$ or, equivalently, that

$$n \equiv 3 \pmod{5} \quad \text{if and only if } i \equiv 2j + 1 \equiv 0 \pmod{5}.$$

Hence, by (6.25), we deduce that

$$c\phi_{10}(5(5n + 3)) \equiv 0 \pmod{5}.$$

□

Remark 6.3. One can prove (6.24) by first observing that (6.20) implies that

$$c\phi_{10}(5n) \equiv c\phi_2(n) \pmod{5^2}.$$

Using (3.17), we deduce (6.24).

6.5. Case $k = 12$.

By Theorem 2.1, we have $\Theta_3(q)\mathfrak{A}_{12}(q) \in M_6(\Gamma_0(24), (\frac{24}{\cdot}))$. By [31, Theorem 1.34], we deduce that

$$\dim M_6(\Gamma_0(24), (\frac{24}{\cdot})) = 22.$$

Let

$$\begin{aligned} B_{12,1} &= \Theta_3^3(q)\Theta_3^9(q^6), & B_{12,2} &= \Theta_3^{11}(q)\Theta_3(q^6), \\ B_{12,3} &= \Theta_3(q^2)\Theta_3(q^3)\Theta_2^{10}(q^2), & B_{12,4} &= \Theta_3^2(q)\Theta_3(q^2)\Theta_3(q^3)\Theta_2^4(q)\Theta_2^4(q^3), \\ B_{12,5} &= \Theta_3^2(q)\Theta_3(q^2)\Theta_3^5(q^3)\Theta_2^2(q)\Theta_2^2(q^3), & B_{12,6} &= \Theta_3^2(q)\Theta_3(q^2)\Theta_3^9(q^3), \\ B_{12,7} &= \Theta_3(q)\Theta_3^3(q^6)\Theta_2^4(q^2)\Theta_2^4(q^6), & B_{12,8} &= \Theta_3(q)\Theta_3^5(q^2)\Theta_2^3(q^2)\Theta_2^3(q^6), \\ B_{12,9} &= \Theta_3(q)\Theta_3(q^2)\Theta_2^5(q^2)\Theta_2^5(q^6), & B_{12,10} &= \Theta_3(q)\Theta_3^5(q^2)\Theta_2^3(q)\Theta_2^3(q^3), \\ B_{12,11} &= \Theta_3(q)\Theta_3(q^2)\Theta_2^5(q)\Theta_2^5(q^3), & B_{12,12} &= \Theta_3(q)\Theta_3^7(q^6)\Theta_2^2(q)\Theta_2^2(q^3), \\ B_{12,13} &= \Theta_3(q)\Theta_3^3(q^6)\Theta_2^4(q)\Theta_2^4(q^3), & B_{12,14} &= \Theta_3(q)\Theta_3^5(q^2)\Theta_2(q)\Theta_2^2(q^2)\Theta_2(q^3)\Theta_2^2(q^6), \\ B_{12,15} &= \Theta_3(q)\Theta_3(q^2)\Theta_2(q)\Theta_2^8(q^2)\Theta_2(q^3), & B_{12,16} &= \Theta_3(q)\Theta_3^3(q^2)\Theta_2(q)\Theta_2^6(q^2)\Theta_2(q^3), \\ B_{12,17} &= \Theta_3(q)\Theta_3^5(q^2)\Theta_2(q)\Theta_2^4(q^2)\Theta_2(q^3), & B_{12,18} &= \Theta_3(q)\Theta_3^9(q^2)\Theta_2(q)\Theta_2(q^3), \\ B_{12,19} &= \Theta_3(q)\Theta_3^3(q^2)\Theta_3(q^6)\Theta_2^8(q^2), & B_{12,20} &= \Theta_3(q)\Theta_3^6(q^2)\Theta_3(q^6)\Theta_2^4(q^2), \\ B_{12,21} &= \Theta_3(q)\Theta_3^8(q^2)\Theta_3(q^6)\Theta_2^2(q^2), \quad \text{and} & B_{12,22} &= \Theta_3(q)\Theta_3^{10}(q^2)\Theta_3(q^6). \end{aligned}$$

The set $\{B_{12,j} | 1 \leq j \leq 22\}$ forms a basis of $M_6(\Gamma_0(24), (\frac{24}{\cdot}))$. Using the above basis, we deduce the following identity.

Theorem 6.7. *We have*

$$\begin{aligned}
 C\Phi_{12}(q) = & \frac{1}{\Theta_3(q)(q; q)_\infty^{12}} \left(-\frac{36207}{160}B_{12,1} + \frac{923091}{4000}B_{12,4} + \frac{35829}{100}B_{12,5} + \frac{891}{4}B_{12,6} \right. \\
 & - \frac{1485}{8}B_{12,7} - \frac{143247}{1000}B_{12,8} - \frac{891}{4}B_{12,9} - \frac{8109}{160}B_{12,10} \\
 & - \frac{582717}{4000}B_{12,11} + \frac{227691}{200}B_{12,12} + \frac{714249}{8000}B_{12,13} + \frac{8109}{80}B_{12,14} \\
 & + \frac{33}{8}B_{12,15} + \frac{1179561}{4000}B_{12,16} - \frac{16503}{400}B_{12,17} - \frac{99}{8}B_{12,18} + \frac{10559}{200}B_{12,19} \\
 (6.26) \quad & \left. - \frac{128807}{100}B_{12,20} + \frac{25647}{160}B_{12,21} + \frac{727}{160}B_{12,22} \right).
 \end{aligned}$$

Next, we give some congruences satisfied by $c\phi_{12}(n)$.

Theorem 6.8. *We have*

$$(6.27) \quad c\phi_{12}(2n + 1) \equiv 0 \pmod{16},$$

$$(6.28) \quad c\phi_{12}(3n + 1) \equiv 0 \pmod{9},$$

and

$$(6.29) \quad c\phi_{12}(3n + 2) \equiv 0 \pmod{9}.$$

Proof. This follows directly from Theorem 5.3 by setting $(p, \alpha, N) = (2, 2, 3)$ and $(3, 1, 4)$. □

6.6. Case $k = 14$.

By Theorem 2.1, we know that $\Theta_3(q)\mathfrak{A}_{14}(q) \in M_7(\Gamma_0(28), (\frac{-28}{\cdot}))$. By [31, Theorem 1.34], we deduce that

$$\dim M_7(\Gamma_0(28), (\frac{-28}{\cdot})) = 27.$$

Let

$$\begin{aligned}
 B_{14,1} &= \Theta_3^{13}(q)\Theta_3(q^7), & B_{14,2} &= \Theta_3^7(q)\Theta_3^3(q^7)\Theta_2^2(q^{1/2})\Theta_2^2(q^{7/2}), \\
 B_{14,3} &= \Theta_3^5(q)\Theta_3(q^7)\Theta_2^4(q^{1/2})\Theta_2^4(q^{7/2}), & B_{14,4} &= \Theta_3(q)\Theta_3^5(q^7)\Theta_2^4(q^{1/2})\Theta_2^4(q^{7/2}), \\
 B_{14,5} &= \Theta_3^5(q)\Theta_3^3(q^7)\Theta_2^2(q)\Theta_2^4(q^{7/2}), & B_{14,6} &= \Theta_3^3(q)\Theta_3^7(q^7)\Theta_2^2(q^{1/2})\Theta_2^2(q^{7/2}), \\
 B_{14,7} &= \Theta_3^{11}(q)\Theta_3^3(q^7), & B_{14,8} &= \Theta_3^{12}(q)\Theta_2(q^{1/2})\Theta_2(q^{7/2}), \\
 B_{14,9} &= \Theta_3^8(q)\Theta_2^3(q^{1/2})\Theta_2^3(q^{7/2}), & B_{14,10} &= \Theta_3^4(q)\Theta_2^5(q^{1/2})\Theta_2^5(q^{7/2}), \\
 B_{14,11} &= \Theta_3(q)\Theta_3(q^5)\Theta_2^8(q^{1/2}), & B_{14,12} &= \Theta_3^{12}(q)\Theta_2(q)\Theta_2(q^7), \\
 B_{14,13} &= \Theta_3^8(q)\Theta_2^3(q)\Theta_2^3(q^7), & B_{14,14} &= \Theta_3^4(q)\Theta_2^5(q)\Theta_2^5(q^7), \\
 B_{14,15} &= \Theta_2^7(q)\Theta_2^7(q^7), & B_{14,16} &= \Theta_3^8(q)\Theta_3(q^7)\Theta_2^2(q^{1/2})\Theta_2^3(q), \\
 B_{14,17} &= \Theta_3^6(q)\Theta_3^3(q^7)\Theta_2^2(q^{1/2})\Theta_2^3(q), & B_{14,18} &= \Theta_3^3(q)\Theta_2(q)\Theta_2^4(q^2)\Theta_2(q^7), \\
 B_{14,19} &= \Theta_2^3(q)\Theta_3^7(q^7)\Theta_2^2(q^{1/2})\Theta_2^3(q), & B_{14,20} &= \Theta_3^9(q^7)\Theta_2^2(q^{1/2})\Theta_2^3(q), \\
 B_{14,21} &= \Theta_3^{10}(q)\Theta_3(q^7)\Theta_2(q)\Theta_2^2(q^{7/2}), & B_{14,22} &= \Theta_3^4(q)\Theta_3(q^7)\Theta_2^3(q)\Theta_2^6(q^{7/2}), \\
 B_{14,23} &= \Theta_2^3(q)\Theta_3^3(q^7)\Theta_2^3(q)\Theta_2^6(q^{7/2}), & B_{14,24} &= \Theta_3^5(q^7)\Theta_2^3(q)\Theta_2^6(q^{7/2}), \\
 B_{14,25} &= \Theta_3^{11}(q)\Theta_2(q^{1/2})\Theta_2(q)\Theta_2^3(q^{7/2}), & B_{14,26} &= \Theta_2^3(q)\Theta_3^4(q^7)\Theta_2(q^{1/2})\Theta_2^2(q)\Theta_2^5(q^{7/2}),
 \end{aligned}$$

and

$$B_{14,27} = \Theta_3^4(q)\Theta_3^2(q^7)\Theta_2(q^{1/2})\Theta_2^2(q)\Theta_2^5(q^{7/2}).$$

The set $\{B_{14,j} | 1 \leq j \leq 27\}$ forms a basis of $M_7(\Gamma_0(28), (\frac{-28}{\cdot}))$. This basis allows us to derive the following identity.

Theorem 6.9. *We have*

$$\begin{aligned}
 C\Phi_{14}(q) &= \frac{1}{\Theta_3(q)(q; q)_{\infty}^{14}} \left(-\frac{3}{4}B_{14,1} - \frac{332339}{1024}B_{14,2} + \frac{255927}{4096}B_{14,3} - \frac{197519}{4096}B_{14,4} \right. \\
 &\quad + \frac{17325}{64}B_{14,5} + \frac{1407329}{2048}B_{14,6} + \frac{7}{4}B_{14,7} + \frac{3}{4}B_{14,8} - \frac{13765}{256}B_{14,9} \\
 &\quad - \frac{52045}{1024}B_{14,10} + \frac{3861}{512}B_{14,12} + \frac{429}{16}B_{14,13} + \frac{6623}{64}B_{14,16} - \frac{79799}{512}B_{14,17} \\
 &\quad + \frac{29407}{512}B_{14,19} - \frac{3989}{64}B_{14,21} + \frac{19803}{128}B_{14,22} - \frac{16807}{256}B_{14,23} \\
 (6.30) \quad &\quad \left. + \frac{50421}{256}B_{14,26} - \frac{6895}{256}B_{14,27} \right).
 \end{aligned}$$

By setting $(p, \alpha, N) = (7, 1, 2)$ in (5.24), we get

$$(6.31) \quad c\phi_{14}(n) \equiv c\phi_2(n/7) \pmod{49}.$$

By (5.25), we deduce that

$$(6.32) \quad c\phi_{14}(7n + r) \equiv 0 \pmod{49}, \quad 1 \leq r \leq 6.$$

Moreover, setting $(p, \alpha, N) = (2, 1, 7)$ in (5.25), we deduce that

$$(6.33) \quad c\phi_{14}(2n + 1) \equiv 0 \pmod{4}.$$

6.7. **Case $k = 16$.**

By Theorem 2.1, we know that $\Theta_3(q)\mathfrak{A}_{16}(q) \in M_8(\Gamma_0(32), (\frac{-2}{\cdot}))$. By [31, Theorem 1.34], we deduce that

$$\dim M_8(\Gamma_0(32), (\frac{-2}{\cdot})) = 32.$$

Let

$$\begin{aligned} B_{16,j} &= \Theta_3(q)\Theta_3^{j-1}(q^2)\Theta_2^{16-j}(q^8), \quad 1 \leq j \leq 15, \\ B_{16,j} &= \Theta_3^3(q)\Theta_3^{j-16}(q^2)\Theta_2^{29-j}(q^8), \quad 16 \leq j \leq 29, \\ B_{16,30} &= \Theta_3^5(q)\Theta_3^2(q^2)\Theta_3^9(q^8), \\ B_{16,31} &= \Theta_3^9(q)\Theta_3(q^4)\Theta_3^6(q^8), \end{aligned}$$

and

$$B_{16,32} = \Theta_3^3(q)\Theta_3(q^4)\Theta_3(q^8)\Theta_2^{11}(q^8).$$

The set $\{B_{16,j} | 1 \leq j \leq 32\}$ forms a basis of $M_8(\Gamma_0(32), (\frac{-2}{\cdot}))$. Hence, we deduce the following identity.

Theorem 6.10. *We have*

$$\begin{aligned} C\Phi_{16}(q) &= \frac{1}{\Theta_3(q)(q; q)_{\infty}^{15}} \left(-16384B_{16,1} + 122880B_{16,2} - 431024B_{16,3} \right. \\ &\quad + 10384B_{16,4} + 3956568B_{16,5} - 12663584B_{16,6} \\ &\quad + 21477101B_{16,7} - 23125005B_{16,8} + 15986724B_{16,9} \\ &\quad - 6153988B_{16,10} + 108966B_{16,11} + 1259002B_{16,12} - 678464B_{16,13} \\ &\quad + 162042B_{16,14} - 15218B_{16,15} + 61440B_{16,18} - 337920B_{16,19} \\ &\quad + 844918B_{16,20} - 870438B_{16,21} - 327528B_{16,22} + 122540544B_{16,23} \\ &\quad - 2366700B_{16,24} + 1511404B_{16,25} - 484664B_{16,26} + 34128B_{16,27} \\ (6.34) \quad &\quad \left. + 20722B_{16,28} - 58B_{16,29} + 59B_{16,30} \right). \end{aligned}$$

By Theorem 5.3, we obtain

$$(6.35) \quad c\phi_{16}(2n + 1) \equiv 0 \pmod{256}$$

and

$$(6.36) \quad c\phi_{16}(2n) \equiv c\phi_8(n) \pmod{256}.$$

7. MÖBIUS INVERSION AND KOLITSCH'S CONGRUENCE (5.22)

In this section, we will use a different notation for k -colored generalized Frobenius symbol λ . The color of a part will be placed on the left hand side of the part. In other words, our symbol λ is now written as

$$(7.1) \quad \lambda = \begin{pmatrix} c_1(z_1) & c_2(z_2) & \cdots & c_d(z_d) \\ c'_1(z'_1) & c'_2(z'_2) & \cdots & c'_d(z'_d) \end{pmatrix},$$

where c_j and c'_j denote colors from the set $\{1, 2, \dots, k\}$ and z_j, z'_j denote the parts. For example, the 2-colored generalized Frobenius symbol

$$\begin{pmatrix} 2_2 & 2_1 \\ 1_2 & 0_1 \end{pmatrix}$$

is now written as

$$\begin{pmatrix} 2(2) & 1(2) \\ 2(1) & 1(0) \end{pmatrix}.$$

Let σ_k be the k -cycle $(1 \ 2 \ \dots \ k)$. Let the symbol

$$\begin{pmatrix} \dots\dots \\ \dots\dots \end{pmatrix}^{\text{sort}}$$

denote sorting the resulting rows to be strictly decreasing according to (1.1). We say that λ has order ℓ with respect to σ_k if ℓ is the smallest positive integer for which the equality of the following symbols holds:

$$\begin{pmatrix} c_1(z_1) & c_2(z_2) & \dots & c_d(z_d) \\ c'_1(z'_1) & c'_2(z'_2) & \dots & c'_d(z'_d) \end{pmatrix} = \begin{pmatrix} \sigma_k^\ell(c_1)(z_1) & \sigma_k^\ell(c_2)(z_2) & \dots & \sigma_k^\ell(c_d)(z_d) \\ \sigma_k^\ell(c'_1)(z'_1) & \sigma_k^\ell(c'_2)(z'_2) & \dots & \sigma_k^\ell(c'_d)(z'_d) \end{pmatrix}^{\text{sort}}.$$

For example, with respect to the 4-cycle $\sigma_4 = (1 \ 2 \ 3 \ 4)$, the 4-colored generalized Frobenius symbol

$$\begin{pmatrix} 3(1) & 1(1) \\ 4(2) & 2(2) \end{pmatrix}$$

has order 2, while

$$\begin{pmatrix} 2(1) & 1(1) \\ 4(2) & 2(2) \end{pmatrix}$$

has order 4.

Let $\Psi_{k,\ell}(n)$ be the number of k -colored generalized Frobenius symbols of n that have order ℓ . When $\ell = k$, we follow Kolitsch and denote $\Psi_{k,k}(n)$ by $\overline{c\phi}_k(n)$. For example, we have $\overline{c\phi}_2(2) = 8$ since there are eight 2-colored generalized Frobenius symbols of 2 that have order 2:

$$\begin{pmatrix} 1(1) \\ 1(0) \end{pmatrix}, \begin{pmatrix} 2(1) \\ 1(0) \end{pmatrix}, \begin{pmatrix} 1(1) \\ 2(0) \end{pmatrix}, \begin{pmatrix} 2(1) \\ 2(0) \end{pmatrix}, \\ \begin{pmatrix} 1(0) \\ 1(1) \end{pmatrix}, \begin{pmatrix} 1(0) \\ 2(1) \end{pmatrix}, \begin{pmatrix} 2(0) \\ 1(1) \end{pmatrix}, \begin{pmatrix} 2(0) \\ 2(1) \end{pmatrix}.$$

The function $\overline{c\phi}_k(n)$ is implicitly mentioned by Kolitsch in [22], and the following identity was later given by him in [23, p. 220].

Theorem 7.1. *Let k and n be positive integers. Then*

$$(7.2) \quad \overline{c\phi}_k(n) = \sum_{\ell|(k,n)} \mu(\ell) c\phi_{\frac{k}{\ell}} \left(\frac{n}{\ell} \right).$$

With (7.2), (5.22) can be written as

$$(7.3) \quad \overline{c\phi}_k(n) \equiv 0 \pmod{k^2}.$$

Congruence (7.3) provides an elegant analogue of Andrews' original congruence (3.29), which states that

$$c\phi_p(n) \equiv 0 \pmod{p^2}$$

for primes p not dividing n . Using the definition of $\overline{c\phi}_k(n)$, we can rewrite (1.13), (1.14), and (1.15) [26, Theorem 3] as

$$\overline{c\phi}_5(n) = 5p(5n - 1), \quad \overline{c\phi}_7(n) = 7p(7n - 2) \quad \text{and} \quad \overline{c\phi}_{11}(n) = 11p(11n - 5),$$

where n is any positive integer.

In this section, we prove the following.

Theorem 7.2. *Let k and n be positive integers. Then*

$$(7.4) \quad c\phi_k(n) = \sum_{\ell|k} \overline{c\phi}_\ell \left(\frac{n}{(k/\ell)} \right) = \sum_{\ell|k} \overline{c\phi}_{k/\ell} \left(\frac{n}{\ell} \right).$$

We then establish (7.2) using Theorem 7.2. We will also take this opportunity to present Kolitsch’s proof of (7.3) (see Theorem 7.3). Our presentation of Kolitsch’s proof contains more details than that given in [25]. We feel that it is important for us (and perhaps the reader) to fully understand Koltisch’s proof as it is an important congruence and it is essential in our proof of Theorem 5.3.

We now begin our proof of Theorem 7.2.

Proof of Theorem 7.2. Every k -colored generalized Frobenius symbol has an order ℓ with respect to σ_k . We first show that the order of a k -colored generalized Frobenius symbol λ must divide k . Suppose not. Let $m = ds$ be the order of λ with $d = (m, k)$ and $s > 1$. Observe that σ_k^d splits into a product of d disjoint cycles $C_j, 1 \leq j \leq d$, of length k/d . Since $(s, k/d) = 1$, $(\sigma_k^d)^s$ is again a product of d disjoint cycles $C'_j, 1 \leq j \leq d$, and the integers in C'_j are the same as those in C_j . Hence, if σ_k^m leaves λ invariant, it would have been left invariant under σ_k^d , but this contradicts the minimality of m . Therefore, the order of λ must be a divisor of k , and we deduce that

$$c\phi_k(n) = \sum_{\ell|k} \Psi_{k,\ell}(n).$$

To prove (7.4), it suffices to show that

$$(7.5) \quad \Psi_{k,\ell}(n) = \overline{c\phi}_\ell \left(\frac{n}{(k/\ell)} \right).$$

For $\ell|k$, we know that σ_k^ℓ splits into ℓ disjoint cycles $C_j, j = 1, 2, \dots, \ell$ of length k/ℓ . Now, if λ is a k -colored generalized Frobenius symbol of order ℓ , then it means that if an entry $c_\nu(z)$, with c_ν appearing in C_j , appears in λ , then $c_\mu(z)$ must appear in λ for every color c_μ that appears in the cycle C_j . We now replace all the colors in this cycle where c_ν belongs by the color represented by the smallest integer, which can be chosen to be less than ℓ . In this way, we will obtain an ℓ -colored generalized Frobenius symbol where each entry $c_j(z)$ appears k/ℓ times. In other words, from

$$n = d + \sum_{i=1}^d c_i(z_i) + \sum_{i=1}^d c'_i(z'_i),$$

we obtain an ℓ -colored generalized Frobenius symbol giving rise to the partition

$$n = d + \frac{k}{\ell} \left(\sum_{i=1}^{d/(k/\ell)} c_{j_i}(z_i) + \sum_{i=1}^{d/(k/\ell)} c'_{j_i}(z'_i) \right),$$

which implies that

$$\frac{n}{(k/\ell)} = \frac{d}{(k/\ell)} + \sum_{i=1}^{d/(k/\ell)} c_{j_i}(z_i) + \sum_{i=1}^{d/(k/\ell)} c'_{j_i}(z'_i).$$

We have thus constructed from λ the ℓ -colored generalized Frobenius symbol of $n/(k/\ell)$, which we denote as λ^* . We claim that λ^* has order ℓ with respect to

$$\gamma = (1 \ 2 \ \cdots \ \ell).$$

If λ^* is of order m less than ℓ , then this means that

$$\gamma^m = \prod_{j=1}^m C'_j,$$

where each C'_j is an ℓ/m cycle, leaves λ^* invariant. Since $m < \ell$, at least two of the integers u and v between 1 and ℓ are in some cycle C'_j . When we reverse the above process of obtaining an ℓ -colored generalized Frobenius symbol of $n/(k/\ell)$ from a k -colored generalized Frobenius symbol of n of order ℓ , we would obtain a symbol λ which is fixed by a cycle that includes both u and v . But u and v are in disjoint cycles in the decomposition of σ_k^ℓ , and this contradicts the fact that λ has order ℓ . Hence, λ^* cannot have order strictly less than ℓ and its order must be ℓ .

Conversely, given an ℓ -colored generalized Frobenius symbol of $n/(k/\ell)$ of order ℓ with respect to γ , we reverse the process to obtain a k -colored generalized Frobenius symbol of n of order ℓ . Hence, we have (7.5), and the proof of Theorem 7.2 is complete. \square

Theorem 7.1 now follows from Theorem 7.2 by using the following lemma with $F(n, k) = c\phi_k(n)$ and $G(n, k) = \overline{c\phi_k}(n)$.

Lemma 7.1. *Let $F(n, k)$ and $G(n, k)$ be two-variable arithmetical functions. Then*

$$(7.6) \quad F(n, k) = \sum_{\ell|(n,k)} G(n/\ell, k/\ell)$$

if and only if

$$(7.7) \quad G(n, k) = \sum_{\ell|(n,k)} \mu(\ell)F(n/\ell, k/\ell).$$

Proof. To prove (7.7), we set $n = dn'$ and $k = dk'$ where $d = (n, k)$. From (7.6), we have

$$F(n'd, k'd) = \sum_{\ell|d} G(n'd/\ell, k'd/\ell).$$

Using the Möbius inversion formula, we deduce that

$$G(n'd, k'd) = \sum_{\ell|d} \mu(\ell)F(n'd/\ell, k'd/\ell)$$

or

$$G(n, k) = \sum_{\ell|d} \mu(\ell)F(n/\ell, k/\ell).$$

The converse follows in a similar way from the Möbius inversion formula. \square

Remark 7.1. We observe that using the above inversion, we can find an expression of Möbius function in terms of Ramanujan’s sum $c_q(n)$. We will write Ramanujan’s sum as $c(q, n)$. It is known that [4, Section 8.3]

$$c(q, n) = \sum_{\ell|(q,n)} \mu(q/\ell)\ell.$$

Now, we observe that

$$\frac{c(q, n)}{(q, n)} = \sum_{\ell|(q, n)} \mu(q/\ell) \frac{\ell}{(q, n)}.$$

Using the inversion formula with

$$F(q, n) = \frac{c(q, n)}{(q, n)} \quad \text{and} \quad G(q, n) = \frac{\mu(q)}{(q, n)},$$

we deduce that

$$\frac{\mu(q)}{(q, n)} = \sum_{\ell|(q, n)} \frac{c(q/\ell, n/\ell)}{(q, n)} \ell \mu(\ell)$$

or

$$\mu(q) = \sum_{\ell|(q, n)} c(q/\ell, n/\ell) \ell \mu(\ell).$$

Theorem 7.3. *Let k and n be positive integers. Then*

$$\overline{c\phi}_k(n) \equiv 0 \pmod{k^2}.$$

We will next prove Theorem 7.3.

Proof of Theorem 7.3. Given a k -colored generalized Frobenius symbol λ represented by (7.1), we say that the color difference of λ is m when m is the sum of the numerical values of the colors on the first row minus the sum of the numerical values of the colors on the second row of λ . In other words,

$$m = c_1 + c_2 + \cdots + c_d - (c'_1 + c'_2 + \cdots + c'_d).$$

Let $\overline{c\phi}_k(m, n)$ denote the number of k -colored generalized Frobenius symbols λ of n with color difference m and order k . Let $c\phi_k(m, n)$ denote the number of k -colored generalized Frobenius symbols λ of n with color difference m . These functions satisfy the following analogue of (7.4):

$$(7.8) \quad c\phi_k(m, n) = \sum_{\ell|k} \overline{c\phi}_\ell \left(\frac{m}{k/\ell}, \frac{n}{k/\ell} \right).$$

The proof of (7.8) is the same as (7.4) by checking that there is a one-to-one correspondence between a k -colored generalized Frobenius symbol of n with color difference m and order k and an ℓ -colored generalized Frobenius symbol of $n/(k/\ell)$ with color difference $m/(k/\ell)$ and order ℓ . The only additional step we need to observe is that under our previous construction, when we replace the k -colored generalized Frobenius symbol λ of n with a k -colored generalized Frobenius symbol λ^\dagger with only colors j with $1 \leq j \leq \ell$ (by identifying colors belonging to the cycle containing j), the color difference of λ^\dagger becomes $m/(k/\ell)$. This is because if a color j appears in λ , then the rest of the colors belonging to the cycle containing j are of the form $j + w\ell$, $1 \leq w < k/\ell$.

Using an inversion formula similar to Lemma 7.1 with two-variable arithmetical functions replaced by three-variable arithmetical functions, we deduce from (7.8) that

$$(7.9) \quad \overline{c\phi}_k(m, n) = \sum_{\ell|k} \mu(\ell) c\phi_{k/\ell} \left(\frac{m}{\ell}, \frac{n}{\ell} \right).$$

Now, the function

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c\phi_k(m, n)t^m q^n$$

is the constant term, i.e., coefficient of z^0 of the function

$$\prod_{j=1}^k (zt^j q; q)_{\infty} (z^{-1}t^{-j}; q)_{\infty},$$

which we shall write as

$$(7.10) \quad \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c\phi_k(m, n)t^m q^n = \text{CT} \left(\prod_{j=1}^k (zt^j q; q)_{\infty} (z^{-1}t^{-j}; q)_{\infty} \right).$$

See [2, Theorems 5.1 and 5.2, pp. 4–6] for examples of expressing generating functions of various partition functions as constant terms of infinite products involving z .

From (7.9) and (7.10), we deduce that

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \overline{c\phi}_k(m, n)t^m q^n &= \sum_{\ell|k} \mu(\ell) \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c\phi_{k/\ell} \left(\frac{m}{\ell}, \frac{n}{\ell} \right) t^m q^n \\ &= \sum_{\ell|k} \mu(\ell) \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} c\phi_{k/\ell}(m, n) t^{\ell m} q^{\ell n} \\ &= \sum_{\ell|k} \mu(\ell) \text{CT} \left(\prod_{j=1}^{k/\ell} (zt^{\ell j} q^{\ell}; q^{\ell})_{\infty} (z^{-1}t^{-\ell j}; q^{\ell})_{\infty} \right) \\ (7.11) \quad &= \sum_{\ell|k} \mu(\ell) \text{CT} \left(\prod_{j=1}^{k/\ell} (z^{\ell}t^{\ell j} q^{\ell}; q^{\ell})_{\infty} (z^{-\ell}t^{-\ell j}; q^{\ell})_{\infty} \right), \end{aligned}$$

where the last equality follows from the fact that (7.10) holds with z replaced by z^a for any positive integer a .

Next, we rewrite the left hand side of (7.10) as

$$\begin{aligned} \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \overline{c\phi}_k(m, n)t^m q^n &= \sum_{j=0}^{k-1} \sum_{s=-\infty}^{\infty} \sum_{n=0}^{\infty} \overline{c\phi}_k(sk + j, n)t^{sk+j} q^n \\ (7.12) \quad &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{k-1} \sum_{s=-\infty}^{\infty} \overline{c\phi}_k(sk + j, n)t^{sk+j} \right) q^n. \end{aligned}$$

Let

$$c_k(j, n) = \sum_{\substack{m=-\infty \\ m \equiv j \pmod{k}}}^{\infty} \overline{c\phi}_k(m, n).$$

Let $t = 1$ in (7.12). Note that

$$\overline{c\phi}_k(n) = \sum_{m=-\infty}^{\infty} \overline{c\phi}_k(m, n).$$

We find that

$$(7.13) \quad \sum_{j=0}^{k-1} c_k(j, n) = \overline{c\phi}_k(n).$$

Next, if $t = \zeta \neq 1$ is a primitive r -th root of unity with $r|k$, then from (7.12), we deduce that

$$(7.14) \quad \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \overline{c\phi}_k(m, n) \zeta^m q^n = \sum_{n=0}^{\infty} \sum_{j=0}^{k-1} c_k(j, n) \zeta^j q^n.$$

To complete the proof of (7.3), we need the following lemma.

Lemma 7.2. *Let ζ_k be a primitive k -th root of unity. Then ζ_k^s is a root of*

$$P_n(t) := \sum_{j=0}^{k-1} c_k(j, n) t^j$$

for all $1 \leq s \leq k - 1$.

Assume that Lemma 7.2 is true. It would imply that $P_n(t)$ is divisible by $Q(t) = 1 + t + \dots + t^{k-1}$, and since the degrees of $P_n(t)$ and $Q(t)$ are the same, we must conclude that $c_k(j, n) = c_k(0, n)$ are all equal for $1 \leq j \leq k - 1$. From (7.13), we conclude that

$$\overline{c\phi}_k(n) = kc_k(0, n).$$

Let S_0 be the set of k -colored generalized Frobenius symbols of n of order k with color difference divisible by k . Note that $|S_0| = c_k(0, n)$. If $\pi \in S_0$, then π under the action of σ_k^j , $1 \leq j \leq k - 1$, is also in S_0 since the residue of the color difference is invariant modulo k under the action of σ and the order of π is k . This implies that S_0 can be grouped into disjoint sets containing k elements in each set, which implies that k divides $c_k(0, n)$. Therefore,

$$\overline{c\phi}_k(n) \equiv 0 \pmod{k^2},$$

and this completes the proof of (7.3). □

It remains to prove Lemma 7.2.

Proof of Lemma 7.2. Given any integer j between 1 and $k - 1$, there exists an integer $r|k$ such that ζ_k^j is a primitive r -th root of unity. Therefore, to prove Lemma 7.2, it suffices to prove that $P_n(\zeta) = 0$ for any primitive r -th root of unity with $r|k$. From (7.11) and (7.14), we deduce that

$$(7.15) \quad \sum_{n=0}^{\infty} P_n(\zeta) q^n = \sum_{\ell|k} \mu(\ell) \text{CT} \left(\prod_{j=1}^{k/\ell} (z^\ell \zeta^{\ell j} q^\ell; q^\ell)_\infty (z^{-\ell} \zeta^{-\ell j}; q^\ell)_\infty \right).$$

The presence of the factor $\mu(\ell)$ in (7.15) shows that we only need to consider divisors of the squarefree part of k . Fix a prime p which divides r and separate the sum in (7.15) into a sum over divisors of the form d where $(p, d) = 1$ and a sum over divisors of the form pd . We only need to show that the term corresponding to d cancels with the term corresponding to pd .

Observe that since d is squarefree and $(d, p) = 1$, we can write $d = ww'$ where $w|r$ and $(w', r) = 1$. Note that the term corresponding to $d = ww'$ is

$$\begin{aligned} & \text{CT} \left(\mu(ww') \prod_{j=1}^{k/(ww')} (z^{ww'} \zeta^{ww'j} q^{ww'}; q^{ww'})_{\infty} (z^{-ww'} \zeta^{-ww'j}; q^{ww'})_{\infty} \right) \\ &= \mu(ww') \text{CT} \left((z^{rw'} q^{rw'}; q^{rw'})_{\infty}^{k/(rw')} (z^{-rw'}; q^{rw'})_{\infty}^{k/(rw')} \right) \end{aligned}$$

since ζ^w is an r/w -th primitive root of unity and

$$\prod_{j=0}^{\nu} (1 - z\zeta^j) = (1 - z^{\nu}).$$

Similarly, the term corresponding to $pd = pww'$ is

$$\begin{aligned} & \text{CT} \left(\mu(pww') \prod_{j=1}^{k/(pww')} (z^{pww'} \zeta^{pww'j} q^{pww'}; q^{pww'})_{\infty} (z^{-pww'} \zeta^{-pww'j}; q^{pww'})_{\infty} \right) \\ &= \mu(pww') \text{CT} \left((z^{rw'} q^{rw'}; q^{rw'})_{\infty}^{k/(rw')} (z^{-rw'}; q^{rw'})_{\infty}^{k/(rw')} \right). \end{aligned}$$

Clearly these two terms cancel as $\mu(pww') = -\mu(ww')$. This completes the proof of the lemma. □

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