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Fractional powers of the generating function for the partition function

by

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1. Introduction. A *partition* of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \dots, \lambda_r$ such that

$$\sum_{i=1}^{r} \lambda_i = n.$$

We denote the number of partitions of n by $p(n)$. By convention, we set $p(0) = 1$. It is well known (see [13, Section 19.3]) that the generating function of $p(n)$ is

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty},$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \qquad |q| < 1.$$

It was observed by S. Ramanujan [21] that $p(n)$ satisfies the congruences

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

For Ramanujan's discussion of (1.1)–(1.3), see [21, 22].

Let k be an integer and define $p_k(n)$ by

$$\sum_{n=0}^{\infty} p_k(n) q^n = (q;q)_\infty^k. \tag{1.4}$$

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Observe that $p(n) = p_{-1}(n)$. When k is a positive integer, $p_{-k}(n)$ enumerates the number of multipartitions with k components of n [1]. The arithmetic properties of $p_{-k}(n)$ have been extensively studied. For example, A. O. L. Atkin [2] gave a list of congruences modulo arbitrary powers of 2, 3, 5 and 7 satisfied by $p_{-k}(n)$. B. Gordon [12] established congruences modulo arbitrary powers of 11 for $p_{-k}(n)$ for $k \in \mathbb{Z}$. From their work, we know that there are many congruences of the form

$$p_{-k}(\ell n + r) \equiv 0 \pmod{\ell}, \tag{1.5}$$

where ℓ is a prime and $0 \le r \le \ell - 1$. I. Kiming and J. Olsson [17] proved that if $\ell \ge 5$ is a prime, $1 \le k \le \ell - 1$ and $k \notin \{\ell-3, \ell-1\}$, then a congruence of the form (1.5) exists only if k is an odd integer and $24r - k \equiv 0 \pmod{\ell}$. M. Boylan [5] has found all possible congruences of the form (1.5) when k is a positive odd integer not exceeding 47. Recently, by using the theory of modular forms, M. Locus and I. Wagner [19] obtained some congruences of the form (1.5) for positive integer k with some restrictions on ℓ and r.

Around 2003, S. T. Ng [20], following a suggestion of the first author, considered $p_k(n)$ defined in (1.4) when k is a negative rational number. He proved, using the theory of modular forms, that for any $n \ge 0$,

$$p_{-2/3}(19n + 9) \equiv 0 \pmod{19}. \tag{1.6}$$

It was also mentioned in [20, 26] that Y. F. Yang showed in an unpublished work that for any $n \ge 0$,

$$p_{-1/2}(17n + 11) \equiv 0 \pmod{17}. \tag{1.7}$$

In this article, we prove numerous congruences satisfied by $p_k(n)$ when k is a rational number. We first introduce some notations. For any real number x, we denote by $\lfloor x \rfloor$ the integer part of x. For any integer n and prime p, we use $\mathrm{ord}_p(n)$ to denote the integer m such that $p^m \mid n$ and $p^{m+1} \nmid n$. For any rational number x, we write it in reduced form $x = u/v$ with $u, v \in \mathbb{Z}$, $\gcd(u, v) = 1$ and $v \ge 1$, and we call $\mathrm{denom}(x) = v$ the *denominator* of x. In the following theorem, we determine the denominator of $p_k(n)$.

Theorem 1.1. *Let $k = a/b$, where $a, b \in \mathbb{Z}$, $b \ge 1$ and $\gcd(a, b) = 1$. We have*

$$\mathrm{denom}(p_k(n)) = b^n \prod_{p \mid b} p^{\alpha_p(n)} \tag{1.8}$$

where

$$\alpha_p(n) = \mathrm{ord}_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots . \tag{1.9}$$

This theorem implies that b and the denominator of $p_{a/b}(n)$ share the same prime divisors. For instance, from the series expansions

$$(q;q)_\infty^{-1/2} = 1 + \tfrac{1}{2}q + \tfrac{7}{8}q^2 + \tfrac{17}{16}q^3 + \tfrac{203}{128}q^4 + \tfrac{455}{256}q^5 + \tfrac{2723}{1024}q^6 + \tfrac{6001}{2048}q^7 + \tfrac{133107}{32768}q^8 + \tfrac{312011}{65536}q^9 + \tfrac{1613529}{262144}q^{10} + \cdots \tag{1.10}$$

and

$$(q;q)_\infty^{1/3} = 1 - \tfrac{1}{3}q - \tfrac{4}{9}q^2 - \tfrac{23}{81}q^3 - \tfrac{82}{243}q^4 - \tfrac{34}{729}q^5 - \tfrac{1711}{6561}q^6 + \tfrac{2254}{19683}q^7 - \tfrac{5117}{59049}q^8 + \tfrac{124025}{1594323}q^9 + \tfrac{183415}{4782969}q^{10} + \cdots, \tag{1.11}$$

we observe that the denominators of $p_{-1/2}(n)$ and $p_{1/3}(n)$ are powers of 2 and 3, respectively.

From Theorem 1.1, we know that it is meaningful to study congruences modulo m satisfied by $p_{a/b}(n)$ for any positive integer m such that $\gcd(m,b) = 1$. By using the known series expansion of $(q;q)_\infty^d$ where $d \in \{1,3,4,6,8,10,14,26\}$, we obtain the following result:

THEOREM 1.2. *Suppose* $a,b,d \in \mathbb{Z}$, $b \geq 1$ *and* $\gcd(a,b) = 1$. *Let* ℓ *be a prime divisor of* $a+db$ *and* $0 \leq r < \ell$. *Suppose* d, ℓ *and* r *satisfy any of the following conditions:*

1. $d = 1$ *and* $24r+1$ *is a quadratic non-residue modulo* ℓ*;*
2. $d = 3$ *and* $8r+1$ *is a quadratic non-residue modulo* ℓ *or* $8r+1 \equiv 0 \pmod{\ell}$*;*
3. $d \in \{4,8,14\}$, $\ell \equiv 5 \pmod 6$ *and* $24r+d \equiv 0 \pmod{\ell}$*;*
4. $d \in \{6,10\}$, $\ell \geq 5$ *and* $\ell \equiv 3 \pmod 4$ *and* $24r+d \equiv 0 \pmod{\ell}$*;*
5. $d = 26$, $\ell \equiv 11 \pmod{12}$ *and* $24r+d \equiv 0 \pmod{\ell}$.

Then for $n \geq 0$,

$$p_{-a/b}(\ell n + r) \equiv 0 \pmod{\ell}. \tag{1.12}$$

Let $(a,b) = (1,1)$ in Theorem 1.2. Then by setting (d,ℓ,r) to be $(4,5,4)$, $(6,7,5)$ and $(10,11,6)$, we obtain Ramanujan's congruences (1.1), (1.2) and (1.3), respectively. Since the arithmetic properties of $p_k(n)$ when $k \in \mathbb{Z}$ have already been extensively studied, we will concentrate on the cases when $k \in \mathbb{Q} \setminus \mathbb{Z}$. In this direction, Theorem 1.2 gives many explicit congruences. For example, we have

$$p_{1/2}(11n+8) \equiv 0 \pmod{11}, \tag{1.13}$$

$$p_{1/3}(41n+37) \equiv 0 \pmod{41}, \tag{1.14}$$

$$p_{3/5}(59n+53) \equiv 0 \pmod{59}, \tag{1.15}$$

$$p_{-1/2}(29n+26) \equiv 0 \pmod{29}, \tag{1.16}$$

$$p_{-1/3}(31n+28) \equiv 0 \pmod{31}, \tag{1.17}$$

$$p_{-3/4}(43n+39) \equiv 0 \pmod{43}, \tag{1.18}$$

$$p_{-1/5}(71n+29) \equiv 0 \pmod{71}. \tag{1.19}$$

Besides congruences implied by Theorem 1.2, we also discover several congruences modulo powers of primes. Here is a sample:

$$p_{1/5}(7n+6) \equiv 0 \pmod{49}, \tag{1.20}$$

$$p_{-1/2}(49n+r) \equiv 0 \pmod{49}, \quad r \in \{20, 34, 41, 48\}, \tag{1.21}$$

$$p_{-2/3}(49n+r) \equiv 0 \pmod{49}, \quad r \in \{22, 29, 43\}. \tag{1.22}$$

The paper is organized as follows. In Section 2, we give proofs to Theorems 1.1 and 1.2. In Section 3, we present many congruences satisfied by $p_{a/b}(n)$ where $1 \le |a| < b \le 5$ modulo primes or prime powers. Our study of the functions $p_k(n)$, with negative rational numbers k, also leads to new proofs of Ramanujan's congruences (1.1) and (1.2).

The partition function $p(n)$ satisfies congruences modulo powers of $5, 7$ and 11. Our motivation in studying $p_k(n)$ is to find congruences modulo prime powers ℓ^s with primes $\ell > 11$. An example is, for any $n \ge 0$,

$$p_{-1/2}(289n+283) \equiv 0 \pmod{289}. \tag{1.23}$$

Many other such congruences are presented as conjectures in Section 3.

2. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Note that

$$\sum_{n=0}^{\infty} p_{a/b}(n) q^n = \prod_{m=1}^{\infty} (1-q^m)^{a/b}. \tag{2.1}$$

We deduce, using the generalized binomial theorem, that

$$(1-q^m)^{a/b} = \sum_{n=0}^{\infty} c_{a/b}(n)(-1)^n q^{mn}$$

where

$$\begin{aligned} c_{a/b}(n) &= \frac{1}{n!}\frac{a}{b}\left(\frac{a}{b}-1\right)\left(\frac{a}{b}-2\right)\cdots\left(\frac{a}{b}-n+1\right) \\ &= \frac{a(a-b)(a-2b)\cdots(a-(n-1)b)}{b^n n!}. \end{aligned} \tag{2.2}$$

We want to show that

$$a(a-b)(a-2b)\cdots(a-(n-1)b)b^{n-1} \equiv 0 \pmod{n!}. \tag{2.3}$$

Let $\nu = \operatorname{ord}_p(n!)$. Since

$$\nu = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \cdots < \frac{n}{2} + \frac{n}{4} + \cdots = n,$$

we conclude that $\nu \le n-1$. Therefore, if $p \mid b$, then $p^\nu \mid b^{n-1}$.

If $p \nmid b$, then for any integer $m \geq 0$, the set

$$\{a - p^t m, a - (p^t m + 1)b, \ldots, a - (p^t(m+1) - 1)b\}$$

forms a complete set of residues modulo p^t. Therefore, if $0 \leq r < p^t$, then r will appear $\lfloor n/p^t \rfloor$ times when the integers in the set

$$S = \{a, a - b, a - 2b, \ldots, a - (n-1)b\}$$

are written in terms of their least non-negative residues modulo p^t. So the set S contains at least $\lfloor n/p^t \rfloor$ integers divisible by p^t. This implies that

$$\operatorname{ord}_p\Big(\prod_{i=0}^{n-1}(a - ib)\Big) \geq \sum_{t\geq 1}\left\lfloor \frac{n}{p^t}\right\rfloor = \nu.$$

Therefore, for any prime p, the order of p dividing $n!$ cannot be greater than the order of p dividing the left hand side of (2.3). Hence (2.3) holds.

From (2.2) and (2.3), we find that $\operatorname{denom}(c_{a/b}(n))$ divides b^{2n-1}. Therefore, any prime factor of $\operatorname{denom}(c_{a/b}(n))$ divides b. Moreover, since $\gcd(a,b) = 1$, we find that $\prod_{i=0}^{n-1}(a - ib)$ is divisible by no prime $p \mid b$. By (2.2), we deduce that

$$\operatorname{ord}_p(\operatorname{denom}(c_{a/b}(n))) = n \operatorname{ord}_p(b) + \operatorname{ord}_p(n!). \tag{2.4}$$

From (2.1), we obtain

$$p_{a/b}(n) = \sum_{\substack{m_1 n_1 + \cdots + m_r n_r = n \\ 0 < m_1 < \cdots < m_r,\, r \geq 1}} c_{a/b}(n_1) \cdots c_{a/b}(n_r)(-1)^{n_1 + \cdots + n_r}. \tag{2.5}$$

For each prime $p \mid b$, we deduce from (2.4) that

$$\begin{aligned} &\operatorname{ord}_p(\operatorname{denom}(c_{a/b}(n_1) \cdots c_{a/b}(n_r))) \\ &\quad = (n_1 + \cdots + n_r)\operatorname{ord}_p(b) + \sum_{i=1}^{r}\left(\left\lfloor \frac{n_i}{p}\right\rfloor + \left\lfloor \frac{n_i}{p^2}\right\rfloor + \cdots\right) \\ &\quad \leq (n_1 + \cdots + n_r)\operatorname{ord}_p(b) + \left(\left\lfloor \frac{\sum_{i=1}^r n_i}{p}\right\rfloor + \left\lfloor \frac{\sum_{i=1}^r n_i}{p^2}\right\rfloor + \cdots\right) \\ &\quad \leq n \operatorname{ord}_p(b) + \left(\left\lfloor \frac{n}{p}\right\rfloor + \left\lfloor \frac{n}{p^2}\right\rfloor + \cdots\right) \end{aligned} \tag{2.6}$$

where for the second last inequality of (2.6), we have used the fact that

$$\sum_{i=1}^{m}\lfloor x_i \rfloor \leq \Big\lfloor \sum_{i=1}^{m} x_i \Big\rfloor, \qquad x_1, \ldots, x_m \in \mathbb{R}, \tag{2.7}$$

and for the last inequality of (2.6), we have used the inequality

$$\sum_{i=1}^{r} n_i \leq \sum_{i=1}^{r} m_i n_i = n. \tag{2.8}$$

We observe that equality holds in (2.6) only if equality holds in (2.8). Since $m_1 < \cdots < m_r$, this happens only if $r = 1$, $m_1 = 1$ and $n_1 = n$. In this case, we do have

$$\operatorname{ord}_p(\operatorname{denom}(c_{a/b}(n))) = n\operatorname{ord}_p(b) + \left(\left\lfloor \frac{n}{p}\right\rfloor + \left\lfloor \frac{n}{p^2}\right\rfloor + \cdots\right). \tag{2.9}$$

Hence, in the sum on the right side of (2.5), the order of p of the denominator of each term is at most $n\operatorname{ord}_p(b) + \operatorname{ord}_p(n!)$, and exactly one term achieves this maximal order. Therefore,

$$\operatorname{ord}_p(\operatorname{denom}(p_{a/b}(n))) = n\operatorname{ord}_p(b) + \operatorname{ord}_p(n!). \tag{2.10}$$

This proves the theorem since any prime divisor of $\operatorname{denom}(p_{a/b}(n))$ also divides b. ■

To prove Theorem 1.2, we need the following lemma.

Lemma 2.1. *Let $k = a/b$, where $a, b \in \mathbb{Z}$, $b \geq 1$ and $\gcd(a,b) = 1$. Let p be a prime such that $p \nmid b$. Then*

$$(1-x)^{p^j k} \equiv (1-x^p)^{p^{j-1}k} \pmod{p^j} \tag{2.11}$$

and for any positive integer t,

$$(q^t;q^t)_\infty^{p^j k} \equiv (q^{pt};q^{pt})_\infty^{p^{j-1}k} \pmod{p^j}. \tag{2.12}$$

Proof. It suffices to prove (2.11) since (2.12) follows from (2.11).

By the binomial theorem and the fact that for any $0 < j < p$,

$$\binom{p}{j} \equiv 0 \pmod{p},$$

we have

$$(1-x)^p = \sum_{j=0}^{p} \binom{p}{j}(-1)^j x^j \equiv 1 - x^p \pmod{p}. \tag{2.13}$$

By induction on j, we deduce that

$$(1-x)^{p^j} \equiv (1-x^p)^{p^{j-1}} \pmod{p^j}.$$

Let

$$(1-x)^{p^j} = (1-x^p)^{p^{j-1}} + p^j F(x),$$

where $F(x)$ is a power series in x with integer coefficients. From the proof of Theorem 1.1, we know that the denominator of $c_{a/b}(n)$ (in reduced form) divides b^{2n-1}, and hence is not divisible by p. Therefore,

$$\begin{aligned}(1-x)^{p^j a/b} &= \left((1-x^p)^{p^{j-1}} + p^j F(x)\right)^{a/b}\\ &= (1-x^p)^{p^{j-1}a/b} \sum_{n=0}^{\infty} c_{a/b}(n) p^{jn}\left(\frac{F(x)}{(1-x^p)^{p^{j-1}}}\right)^n\\ &\equiv (1-x^p)^{p^{j-1}a/b} \pmod{p^j}. \ ■\end{aligned}$$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Since $\ell \mid (a+db)$, we may let $a+db = \ell m$ for some integer m. Next, $\gcd(a,b) = 1$ and $\ell \mid (a+db)$ implies that $\gcd(\ell, b) = 1$. Since $\gcd(\ell, b) = 1$, by Lemma 2.1 we find that

$$\sum_{n=0}^{\infty} p_{-a/b}(n)q^n = \frac{(q;q)_\infty^d}{(q;q)_\infty^{(a+db)/b}} \equiv \frac{(q;q)_\infty^d}{(q^\ell;q^\ell)_\infty^{m/b}} \pmod{\ell}. \tag{2.14}$$

We now divide our proof according to the values of d.

Case $d = 1$: By Euler's pentagonal number theorem [3, Corollary 1.3.5], we find that

$$(q;q)_\infty = \sum_{i=-\infty}^{\infty} (-1)^i q^{i(3i+1)/2}. \tag{2.15}$$

Note that

$$N = i(3i+1)/2 \tag{2.16}$$

is equivalent to

$$24N + 1 = (6i+1)^2. \tag{2.17}$$

Therefore, if $24N + 1$ is a quadratic non-residue modulo ℓ, then there are no integers i satisfying (2.16). The congruence (1.12) follows by comparing the coefficients of $q^{\ell n + r}$ on both sides of (2.14).

Case $d = 3$: By Jacobi's identity [3, Theorem 1.3.9], we find that

$$(q;q)_\infty^3 = \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2}. \tag{2.18}$$

Note that

$$N = j(j+1)/2 \tag{2.19}$$

is equivalent to

$$8N + 1 = (2j+1)^2. \tag{2.20}$$

If $8N + 1$ is a quadratic non-residue modulo ℓ, then there are no integers j satisfying (2.19). Hence from (2.14) we conclude that $p_{-a/b}(\ell n + r) \equiv 0 \pmod{\ell}$.

If $8N + 1 \equiv 0 \pmod{\ell}$, then (2.20) implies that $2j + 1 \equiv 0 \pmod{\ell}$. Again by (2.18) and (2.14), we deduce (1.12).

Case $d = 4$: From (2.15) and (2.18), we find that

$$(q;q)_\infty^4 = \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} (-1)^{i+j} (2j+1) q^{i(3i+1)/2 + j(j+1)/2}. \tag{2.21}$$

Now, observe that

$$N = i(3i+1)/2 + j(j+1)/2$$

if and only if

$$24N + 4 = (6i+1)^2 + 3(2j+1)^2.$$

If $\ell \equiv 5 \pmod{6}$, then $\left(\frac{-3}{\ell}\right) = -1$. This implies that

$$24N + 4 \equiv 0 \pmod{\ell}$$

if and only if

$$6i + 1 \equiv 0 \pmod{\ell} \quad \text{and} \quad 2j + 1 \equiv 0 \pmod{\ell}.$$

Using (2.21) and comparing the coefficients of $q^{\ell n + r}$ on both sides of (2.14), we obtain (1.12).

Case $d = 6$: We deduce from (2.18) that

$$(q;q)_\infty^6 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}(2i+1)(2j+1)q^{i(i+1)/2+j(j+1)/2}. \tag{2.22}$$

Observe that

$$N = i(i+1)/2 + j(j+1)/2$$

is equivalent to

$$8N + 2 = (2i+1)^2 + (2j+1)^2.$$

If $\ell \equiv 3 \pmod{4}$, then $\left(\frac{-1}{\ell}\right) = -1$. This implies that

$$8N + 2 \equiv 0 \pmod{\ell}$$

if and only if

$$2i + 1 \equiv 0 \pmod{\ell} \quad \text{and} \quad 2j + 1 \equiv 0 \pmod{\ell}.$$

Congruence (1.12) follows by comparing the coefficients of $q^{\ell n + r}$ on both sides of (2.14).

Case $d = 8$: We need the following identity (see [18]):

$$\begin{aligned}(q;q)_\infty^8 = \frac{4}{3}&\left(\sum_{m=-\infty}^{\infty}(3m+1)^3 q^{3m^2+2m}\right)\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)\\ &-\frac{1}{3}\left(\sum_{m=-\infty}^{\infty}(6m+1)^3 q^{3m^2+m}\right)\left(\sum_{n=0}^{\infty} q^{n^2+n}\right).\end{aligned} \tag{2.23}$$

Note that

$$N = 3m^2 + 2m + n^2$$

is equivalent to

$$3N + 1 = (3m+1)^2 + 3n^2.$$

Suppose $3m+1$ and n are non-zero modulo ℓ. Then $3N+1 \equiv 0 \pmod{\ell}$ implies

$$u^2 \equiv -3 \pmod{\ell}$$

for some integer u. But since $\ell \equiv 5 \pmod 6$, we have $\left(\frac{-3}{\ell}\right) = -1$, and hence such an integer u cannot exist. Therefore, if $3m+1$ and n are non-zero modulo ℓ, then $3N+1$ is non-zero modulo ℓ. In other words,

$$3N+1 \equiv 0 \pmod{\ell}$$

if and only if

$$3m+1 \equiv 0 \pmod{\ell} \quad \text{and} \quad n \equiv 0 \pmod{\ell}.$$

Similarly,

$$N = 3m^2 + m + n^2 + n$$

is equivalent to

$$4(3N+1) = (6m+1)^2 + 3(2n+1)^2.$$

This identity implies, as in the previous case, that

$$3N+1 \equiv 0 \pmod{\ell}$$

if and only if

$$6m+1 \equiv 0 \pmod{\ell} \quad \text{and} \quad 2n+1 \equiv 0 \pmod{\ell}.$$

Therefore, from (2.14) and (2.23) we see that $p_{-a/b}(\ell n + r) \equiv 0 \pmod{\ell}$.

Case $d = 10$: From [9, Corollary 4.2], we find that

$$(q;q)_\infty^{10} = \frac{4}{3}\left(\sum_{m=-\infty}^{\infty} (3m+1)^3 q^{3m^2+2m}\right) \times \left(\sum_{n=-\infty}^{\infty} (6n+1) q^{3n^2+n}\right) - \left(\sum_{m=-\infty}^{\infty} (3m+1) q^{3m^2+2m}\right) \times \left(\sum_{n=-\infty}^{\infty} (6n+1)^3 q^{3n^2+n}\right). \tag{2.24}$$

Observe that

$$N = 3m^2 + 2m + 3n^2 + n$$

is equivalent to

$$12N + 5 = (6m+2)^2 + (6n+1)^2.$$

If $\ell \equiv 3 \pmod 4$, then $\left(\frac{-1}{\ell}\right) = -1$. We know that

$$12N + 5 \equiv 0 \pmod{\ell}$$

if and only if

$$3m+1 \equiv 0 \pmod{\ell} \quad \text{and} \quad 6n+1 \equiv 0 \pmod{\ell}.$$

From (2.24), congruence (1.12) follows by comparing the coefficients of $q^{\ell n + r}$ on both sides of (2.14).

Case $d=14$: Recall from [7, Theorem 5.3] that

$$(q;q)_\infty^{14} = -\frac{1}{15}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^m(3m+1)(4n+1)(6m+4n+3) \tag{2.25}$$

$$\times(6m-4n+1)(6m+12n+5)(6m-12n-1)q^{(4(3m+1)^2+3(4n+1)^2-7)/12}.$$

We observe that

$$N=(4(3m+1)^2+3(4n+1)^2-7)/12$$

is equivalent to

$$12N+7=4(3m+1)^2+3(4n+1)^2.$$

If $\ell\equiv 5 \pmod 6$, then $\left(\frac{-3}{\ell}\right)=-1$. We deduce that

$$12N+7\equiv 0 \pmod{\ell}$$

if and only if

$$3m+1\equiv 0 \pmod{\ell}\quad\text{and}\quad 4n+1\equiv 0 \pmod{\ell}.$$

The congruence $p_{-a/b}(\ell n+r)\equiv 0 \pmod{\ell}$ now follows from the comparison of the coefficients of $q^{\ell n+r}$ on both sides of (2.14).

Case $d=26$: Let

$$f(m,n)=\sum_{j=0}^{12}\binom{12}{2j}(-1)^j m^j n^{6-j}. \tag{2.26}$$

From [6, Theorem 3], we find

$$(q;q)_\infty^{26} = \frac{q^{-13/12}}{16308864} \tag{2.27}$$

$$\times\left(\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}f\left(\frac{(6i+1)^2}{2},\frac{(6j+1)^2}{2}\right)q^{((6i+1)^2+(6j+1)^2)/24}\right.$$

$$\left.+\sum_{i=-\infty}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}f(12i^2,(6j+1)^2)q^{i^2+(6j+1)^2/12}\right).$$

Observe that

$$N=\tfrac{1}{24}\left((6i+1)^2+(6j+1)^2-26\right)$$

is equivalent to

$$24N+26=(6i+1)^2+(6j+1)^2.$$

If $\ell\equiv 11 \pmod{12}$, then $\left(\frac{-1}{\ell}\right)=-1$. Hence

$$24N+26\equiv 0 \pmod{\ell}$$

if and only if

$$6i+1\equiv 0 \pmod{\ell}\quad\text{and}\quad 6j+1\equiv 0 \pmod{\ell},$$

in which case

$$f\left(\frac{(6i+1)^2}{2},\frac{(6j+1)^2}{2}\right)\equiv 0 \pmod{\ell^{12}}.$$

Similarly,

$$N = \tfrac{1}{12}(12i^2 + (6j+1)^2 - 13)$$

is equivalent to

$$12N + 13 = 12i^2 + (6j+1)^2.$$

If $\ell \equiv 11 \pmod{12}$, then $\left(\frac{-12}{\ell}\right) = -1$. Hence

$$12N + 13 \equiv 0 \pmod{\ell}$$

if and only if

$$i \equiv 0 \pmod{\ell} \quad \text{and} \quad 6j+1 \equiv 0 \pmod{\ell},$$

in which case

$$f(12i^2, (6j+1)^2) \equiv 0 \pmod{\ell^{12}}.$$

Note that $16308864 = 2^7 \cdot 3^4 \cdot 11^2 \cdot 13$. Using (2.27) and comparing the coefficients of $q^{\ell n + r}$ on both sides of (2.14), we obtain (1.12). ■

REMARK 1. For $d \in \{2, 4, 6, 8, 10, 14, 26\}$, there are some other double series expressions for $(q;q)_\infty^d$. For example, formulas similar to (2.24) for $(q;q)_\infty^{10}$ can be found in works of B. C. Berndt et al. [4], S. H. Chan [8], M. D. Hirschhorn [14, 16] and L. Winquist [25]. Formulas for $(q;q)_\infty^{10}$ were also discussed in [10].

3. Explicit congruences for $p_k(n)$ where $k \in \mathbb{Q}\backslash\mathbb{Z}$. In this section, we give explicit congruences for $p_{-a/b}(n)$ with $1 \le |a| < b \le 5$. Most of them are special cases of Theorem 1.2 but there are some congruences which require more technical arguments.

First, we present some explicit congruences satisfied by $p_{-a/b}(n)$ where $1 \le -a < b \le 5$.

THEOREM 3.1. *For any integer $n \ge 0$,*

$$p_{1/2}(5n + r) \equiv 0 \pmod{5}, \qquad r \in \{2, 3, 4\}, \tag{3.1}$$

$$p_{1/2}(11n + 8) \equiv 0 \pmod{11}, \tag{3.2}$$

$$p_{1/2}(19n + 17) \equiv 0 \pmod{19}, \tag{3.3}$$

$$p_{1/3}(11n + 9) \equiv 0 \pmod{11}, \tag{3.4}$$

$$p_{1/3}(17n + 4) \equiv 0 \pmod{17}, \tag{3.5}$$

$$p_{1/3}(23n + 15) \equiv 0 \pmod{23}, \tag{3.6}$$

$$p_{1/3}(41n + 37) \equiv 0 \pmod{41}, \tag{3.7}$$

$$p_{2/3}(5n + 4) \equiv 0 \pmod{5}, \tag{3.8}$$

$$p_{2/3}(7n + r) \equiv 0 \pmod{7}, \qquad r \in \{2, 4, 5, 6\}, \tag{3.9}$$

$$p_{2/3}(11n + 7) \equiv 0 \pmod{11}, \tag{3.10}$$

$$p_{1/4}(5n+4) \equiv 0 \pmod{5}, \tag{3.11}$$

$$p_{1/4}(11n+r) \equiv 0 \pmod{11}, \quad r \in \{2,4,5,7,9,10\}, \tag{3.12}$$

$$p_{1/4}(23n+17) \equiv 0 \pmod{23}, \tag{3.13}$$

$$p_{3/4}(7n+5) \equiv 0 \pmod{7}, \tag{3.14}$$

$$p_{3/4}(29n+19) \equiv 0 \pmod{29}, \tag{3.15}$$

$$p_{3/4}(53n+48) \equiv 0 \pmod{53}, \tag{3.16}$$

$$p_{1/5}(7n+r) \equiv 0 \pmod{7}, \quad r \in \{2,4,5,6\}, \tag{3.17}$$

$$p_{1/5}(7n+6) \equiv 0 \pmod{49}, \tag{3.18}$$

$$p_{1/5}(23n+9) \equiv 0 \pmod{23}, \tag{3.19}$$

$$p_{2/5}(7n+5) \equiv 0 \pmod{7}, \tag{3.20}$$

$$p_{2/5}(13n+r) \equiv 0 \pmod{13}, \quad r \in \{4,5,7,8,9,11,12\}, \tag{3.21}$$

$$p_{2/5}(17n+15) \equiv 0 \pmod{17}, \tag{3.22}$$

$$p_{3/5}(17n+14) \equiv 0 \pmod{17}, \tag{3.23}$$

$$p_{3/5}(47n+27) \equiv 0 \pmod{47}, \tag{3.24}$$

$$p_{3/5}(59n+53) \equiv 0 \pmod{59}, \tag{3.25}$$

$$p_{4/5}(11n+r) \equiv 0 \pmod{11}, \quad r \in \{2,4,5,7,8,9\}, \tag{3.26}$$

$$p_{4/5}(23n+13) \equiv 0 \pmod{23}. \tag{3.27}$$

Proof. Except for congruence (3.18), all congruences follow from Theorem 1.2 with respective parameters given in Table 1.

Table 1

Eq.	(3.1)	(3.2)	(3.3)	(3.4)	(3.5)	(3.6)	(3.7)	(3.8)	(3.9)
a	-1	-1	-1	-1	-1	-1	-1	-2	-2
b	2	2	2	3	3	3	3	3	3
d	3	6	10	4	6	8	14	4	3
ℓ	5	11	19	11	17	23	41	5	7

Eq.	(3.10)	(3.11)	(3.12)	(3.13)	(3.14)	(3.15)	(3.16)	(3.17)	(3.19)
a	-2	-1	-1	-1	-3	-3	-3	-1	-1
b	3	4	4	4	4	4	4	5	5
d	8	4	3	6	6	8	14	3	14
ℓ	11	5	11	23	7	29	53	7	23

Eq.	(3.20)	(3.21)	(3.22)	(3.23)	(3.24)	(3.25)	(3.26)	(3.27)
a	-2	-2	-2	-3	-3	-3	-4	-4
b	5	5	5	5	5	5	5	5
d	6	3	14	4	10	26	3	10
ℓ	7	13	17	17	47	59	11	23

We now prove (3.18). By Lemma 2.1, we find that

$$\sum_{n=0}^{\infty} p_{1/5}(n)q^n = \frac{(q;q)_\infty^{10}}{(q;q)_\infty^{49/5}} \equiv \frac{(q;q)_\infty^{10}}{(q^7;q^7)_\infty^{7/5}} \pmod{49}. \tag{3.28}$$

Observe that

$$N = 3m^2 + 2m + 3n^2 + n$$

is equivalent to

$$12N + 5 = (6m+2)^2 + (6n+1)^2.$$

Since $\left(\frac{-1}{7}\right) = -1$, we know that

$$12N + 5 \equiv 0 \pmod 7$$

if and only if

$$3m + 1 \equiv 0 \pmod 7 \quad \text{and} \quad 6n + 1 \equiv 0 \pmod 7.$$

Using (2.24) and comparing the coefficients of q^{7n+6} on both sides of (3.28), we obtain (3.18). ∎

Numerical evidence suggests that the following congruences hold.

Conjecture 3.1. *For any integer $n \geq 0$,*

$$p_{1/2}(125n + r) \equiv 0 \pmod{25}, \quad r \in \{38, 63, 88, 113\}, \tag{3.29}$$

$$p_{2/3}(25n + r) \equiv 0 \pmod{25}, \quad r \in \{19, 24\}, \tag{3.30}$$

$$p_{2/3}(121n + 84) \equiv 0 \pmod{121}, \tag{3.31}$$

$$p_{1/4}(25n + r) \equiv 0 \pmod{25}, \quad r \in \{14, 24\}, \tag{3.32}$$

$$p_{1/4}(25n + 19) \equiv 0 \pmod{125}, \tag{3.33}$$

$$p_{1/4}(121n + 92) \equiv 0 \pmod{121}, \tag{3.34}$$

$$p_{1/5}(49n + r) \equiv 0 \pmod{343}, \quad r \in \{27, 34, 48\}, \tag{3.35}$$

$$p_{2/5}(49n + 40) \equiv 0 \pmod{49}. \tag{3.36}$$

Next, we present some explicit congruences satisfied by $p_{-a/b}(n)$ where $1 \leq a < b \leq 5$.

Theorem 3.2. *For any integer $n \geq 0$,*

$$p_{-1/2}(7n + r) \equiv 0 \pmod 7, \quad r \in \{2, 4, 5, 6\}, \tag{3.37}$$

$$p_{-1/2}(49n + r) \equiv 0 \pmod{49}, \quad r \in \{20, 34, 41, 48\}, \tag{3.38}$$

$$p_{-1/2}(17n + 11) \equiv 0 \pmod{17}, \tag{3.39}$$

$$p_{-1/2}(29n + 26) \equiv 0 \pmod{29}, \tag{3.40}$$

$$p_{-1/3}(5n + r) \equiv 0 \pmod 5, \quad r \in \{2, 3, 4\}, \tag{3.41}$$

$$p_{-1/3}(5n + 3) \equiv 0 \pmod{25}, \tag{3.42}$$

$$p_{-1/3}(19n + 14) \equiv 0 \pmod{19}, \tag{3.43}$$

$$p_{-1/3}(31n+28) \equiv 0 \pmod{31}, \tag{3.44}$$

$$p_{-2/3}(5n+r) \equiv 0 \pmod{5}, \qquad r \in \{3,4\}, \tag{3.45}$$

$$p_{-2/3}(11n+r) \equiv 0 \pmod{11}, \qquad r \in \{2,4,5,7,8,9\}, \tag{3.46}$$

$$p_{-1/4}(5n+r) \equiv 0 \pmod{5}, \qquad r \in \{3,4\}, \tag{3.47}$$

$$p_{-1/4}(13n+r) \equiv 0 \pmod{13}, \qquad r \in \{4,5,7,8,9,11,12\}, \tag{3.48}$$

$$p_{-3/4}(5n+r) \equiv 0 \pmod{5}, \qquad r \in \{2,3,4\}, \tag{3.49}$$

$$p_{-3/4}(43n+39) \equiv 0 \pmod{43}, \tag{3.50}$$

$$p_{-3/4}(59n+24) \equiv 0 \pmod{59}, \tag{3.51}$$

$$p_{-3/4}(107n+97) \equiv 0 \pmod{107}, \tag{3.52}$$

$$p_{-1/5}(31n+23) \equiv 0 \pmod{31}, \tag{3.53}$$

$$p_{-1/5}(71n+29) \equiv 0 \pmod{71}, \tag{3.54}$$

$$p_{-1/5}(131n+119) \equiv 0 \pmod{131}, \tag{3.55}$$

$$p_{-2/5}(7n+r) \equiv 0 \pmod{7}, \qquad r \in \{3,4,6\}, \tag{3.56}$$

$$p_{-2/5}(11n+9) \equiv 0 \pmod{11}, \tag{3.57}$$

$$p_{-2/5}(17n+r) \equiv 0 \pmod{17}, \qquad r \in \{2,5,7,8,9,12,13,14,16\}, \tag{3.58}$$

$$p_{-3/5}(11n+8) \equiv 0 \pmod{11}, \tag{3.59}$$

$$p_{-4/5}(11n+7) \equiv 0 \pmod{11}, \tag{3.60}$$

$$p_{-4/5}(19n+r) \equiv 0 \pmod{19}, \tag{3.61}$$

$$r \in \{4,5,7,8,11,12,13,14,16,18\}.$$

Proof. Except for (3.38) and (3.42), all congruences follow directly from Theorem 1.2 with respective parameters given in Table 2.

Now we prove (3.38). By Lemma 2.1,

$$\begin{aligned} \sum_{n=0}^{\infty} p_{-1/2}(n)q^{n+1} &= \frac{q(q;q)_\infty^{24}}{(q;q)_\infty^{49/2}} \\ &\equiv \frac{1}{(q^7;q^7)_\infty^{7/2}} \sum_{n\geq 0} \tau(n)q^n \pmod{49}, \end{aligned} \tag{3.62}$$

where $\tau(n)$ is Ramanujan's tau function.

For any prime p, it is known [23, Chapter VII] that

$$\tau(pn) = \tau(p)\tau(n) - p^{11}\tau(n/p).$$

Hence,

$$\tau(7n) = -16744\tau(n) - 7^{11}\tau(n/7) \equiv 14\tau(n) \pmod{49}. \tag{3.63}$$

Extracting the terms of the form q^{7n} on both sides of (3.62), replacing q^7 by q and using(3.63), we deduce that

Table 2

Eq.	(3.37)	(3.39)	(3.40)	(3.41)	(3.43)	(3.44)	(3.45)	(3.46)
a	1	1	1	1	1	1	2	2
b	2	2	2	3	3	3	3	3
d	3	8	14	3	6	10	1	3
ℓ	7	17	29	5	19	31	5	11

Eq.	(3.47)	(3.48)	(3.49)	(3.50)	(3.51)	(3.52)	(3.53)	(3.54)
a	1	1	3	3	3	3	1	1
b	4	4	4	4	4	4	5	5
d	1	3	3	10	14	26	6	14
ℓ	5	13	5	43	59	107	31	71

Eq.	(3.55)	(3.56)	(3.57)	(3.58)	(3.59)	(3.60)	(3.61)
a	1	2	2	2	3	4	4
b	5	5	5	5	5	5	5
d	26	1	4	3	6	8	3
ℓ	131	7	11	17	11	11	19

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-1/2}(7n+6)q^{n+1} &\equiv \frac{1}{(q;q)_\infty^{7/2}}\sum_{n=0}^{\infty}\tau(7n)q^n\\ &\equiv \frac{14}{(q;q)_\infty^{7/2}}\sum_{n=0}^{\infty}\tau(n)q^n \pmod{49},\end{aligned}$$

which implies, by Lemma 2.1, that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{7}p_{-1/2}(7n+6)q^{n+1} &\equiv \frac{2}{(q;q)_\infty^{7/2}}\sum_{n=0}^{\infty}\tau(n)q^n\\ &\equiv \frac{2}{(q^7;q^7)_\infty^{1/2}}\sum_{n=0}^{\infty}\tau(n)q^n \pmod{7}.\end{aligned}$$

Hence,

$$(q^7;q^7)_\infty^{1/2}\sum_{n=0}^{\infty} p_{-1/2}(7n+6)q^{n+1} \equiv 14\sum_{n=0}^{\infty}\tau(n)q^n \pmod{49}. \tag{3.64}$$

We recall from [23, p. 97, Eq. (56)] that

$$\tau(n)\equiv n\sigma_3(n) \pmod{7}, \tag{3.65}$$

where $\sigma_3(n)=\sum_{d|n} d^3$. We claim that if the residue of n modulo 7 is 3, 5 or 6, then $\sigma_3(n)\equiv 0 \pmod{7}$. Indeed, in these cases, n cannot be a square and $n^3\equiv -1 \pmod{7}$. Therefore, when $n\equiv 3,5$ or $6 \pmod{7}$, we find that

$$\sigma_3(n) = \sum_{\substack{d|n \\ d<\sqrt{n}}} \left(d^3 + \left(\frac{n}{d}\right)^3\right) = \sum_{\substack{d|n \\ d<\sqrt{n}}} \frac{d^6-1}{d^3} \equiv 0 \pmod{7},$$

where the last congruence follows from Fermat's little theorem.

By (3.65) we deduce that

$$\tau(7n+s) \equiv 0 \pmod{7}, \quad s \in \{0,3,5,6\}.$$

Using this result and (3.64), we deduce that

$$p_{-1/2}(7(7n+s-1)+6) \equiv 0 \pmod{49}, \quad s \in \{0,3,5,6\}.$$

The congruences in (3.38) are proved.

Next, we prove (3.42). By Lemma 2.1, we find that

$$\sum_{n=0}^{\infty} p_{-1/3}(n)q^n = \frac{(q;q)_\infty^8}{(q;q)_\infty^{25/3}} \equiv \frac{(q;q)_\infty^8}{(q^5;q^5)_\infty^{5/3}} \pmod{25}. \tag{3.66}$$

Now we use the the expansion (2.23) of $(q;q)_\infty^8$. Observe that

$$N = 3m^2 + 2m + n^2$$

is equivalent to

$$3N+1 = (3m+1)^2 + 3n^2.$$

Since $\left(\frac{-3}{5}\right) = -1$, we find that

$$3N+1 \equiv 0 \pmod{5} \quad (\text{or equivalently } N \equiv 3 \pmod{5})$$

if and only if

$$3m+1 \equiv 0 \pmod{5} \quad \text{and} \quad n \equiv 0 \pmod{5}.$$

Similarly,

$$N = 3m^2 + m + n^2 + n$$

is equivalent to

$$4(3N+1) = (6m+1)^2 + 3(2n+1)^2.$$

We know that

$$3N+1 \equiv 0 \pmod{5} \quad (\text{or equivalently } N \equiv 3 \pmod{5}))$$

if and only if

$$6m+1 \equiv 0 \pmod{5} \quad \text{and} \quad 2n+1 \equiv 0 \pmod{5}.$$

Therefore, (2.23) and (3.66) imply

$$p_{-1/3}(5n+3) \equiv 0 \pmod{25}. \blacksquare$$

As an interesting application of the congruences in this section, using (3.41) and (3.45) we can give a new proof of (1.1).

Corollary 3.1. *For any integer $n \ge 0$,*

$$p(5n+4) \equiv 0 \pmod{5}.$$

Proof. Since

$$\sum_{n=0}^{\infty} p(n)q^n = \Big(\sum_{n=0}^{\infty} p_{-1/3}(n)q^n\Big)\Big(\sum_{n=0}^{\infty} p_{-2/3}(n)q^n\Big),$$

we have

$$p(n) = \sum_{k=0}^{n} p_{-1/3}(k)p_{-2/3}(n-k). \tag{3.67}$$

Note that for any integers k and n, either the least non-negative residue of k modulo 5 belongs to $\{2,3,4\}$ or the least non-negative residue of $5n+4-k$ modulo 5 belongs to $\{3,4\}$. Hence by (3.41) and (3.45), we always have

$$p_{-1/3}(k)p_{-2/3}(5n+4-k) \equiv 0 \pmod{5}.$$

This proves the corollary. ■

Similarly, by using (3.37) we give a new proof of (1.2).

Corollary 3.2. *For any integer $n \ge 0$,*

$$p(7n+5) \equiv 0 \pmod{7}.$$

Proof. Since

$$\sum_{n=0}^{\infty} p(n)q^n = \Big(\sum_{n=0}^{\infty} p_{-1/2}(n)q^n\Big)^2,$$

we have

$$p(n) = \sum_{k=0}^{n} p_{-1/2}(k)p_{-1/2}(n-k). \tag{3.68}$$

Note that for any integers k and n, at least one of k or $7n+5-k$ must be congruent to 2, 4, 5 or 6. By (3.37) and (3.68), we conclude that $p(7n+5)$ is always divisible by 7. ■

Numerical evidence suggests that the following conjecture holds.

Conjecture 3.2. *For any integer $n \ge 0$, we have*

$$p_{-1/2}(343n+293) \equiv 0 \pmod{343}, \tag{3.69}$$

$$p_{-1/2}(2401n+r) \equiv 0 \pmod{2401}, \quad r \in \{979, 1665, 2008, 2351\}, \tag{3.70}$$

$$p_{-1/2}(289n+283) \equiv 0 \pmod{289}, \tag{3.71}$$

$$p_{-1/3}(25n+r) \equiv 0 \pmod{125}, \quad r \in \{18, 23\}, \tag{3.72}$$

$$p_{-1/3}(361n+356) \equiv 0 \pmod{361}, \tag{3.73}$$

$$p_{-2/3}(49n+r) \equiv 0 \pmod{7}, \quad r \in \{22, 29, 43\}, \tag{3.74}$$

$$p_{-3/4}(25n+r) \equiv 0 \pmod{25}, \qquad r \in \{13, 23\}, \tag{3.75}$$

$$p_{-3/4}(25n+18) \equiv 0 \pmod{125}, \tag{3.76}$$

$$p_{-3/4}(125n+r) \equiv 0 \pmod{3125}, \quad r \in \{93, 118\}. \tag{3.77}$$

4. Modular approach to some congruences. It is possible to prove some congruences in Conjectures 3.1 and 3.2 using the theory of modular forms. We illustrate the method by proving (3.71). Let

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

We denote by $M_k(\mathrm{SL}_2(\mathbb{Z}))$ (resp. $S_k(\mathrm{SL}_2(\mathbb{Z}))$) the space of modular forms (resp. cusp forms) of weight k on $\mathrm{SL}_2(\mathbb{Z})$. For any positive integer m, we define the Hecke operator $T(m)$ and the U-operator $U(m)$ which send a function

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n$$

to

$$f(z)|_{T(m)} := \sum_{n=0}^{\infty} \biggl(\sum_{d|(m,n)} d^{k-1} a\Bigl(\frac{nm}{d^2}\Bigr) \biggr) q^n$$

and

$$f(z)|_{U(m)} := \sum_{n=0}^{\infty} a(mn) q^n,$$

respectively. It is known that if $f(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, then $f(z)|_{T(m)} \in M_k(\mathrm{SL}_2(\mathbb{Z}))$.

Modular proof of (3.71). Let $q = e^{2\pi i \tau}$ with $\mathrm{Im}\,\tau > 0$. Recall that the discriminant modular form is

$$\Delta(\tau) := q(q;q)_\infty^{24}. \tag{4.1}$$

It is clear that $\Delta^6(\tau) \in S_{72}(\mathrm{SL}_2(\mathbb{Z}))$ is a cusp form. By Lemma 2.1, we deduce that

$$\begin{aligned} \Delta^6(\tau) &= q^6 \frac{(q;q)_\infty^{289/2}}{(q;q)_\infty^{1/2}} = (q;q)_\infty^{289/2} \sum_{n=0}^{\infty} p_{-1/2}(n) q^{n+6} \\ &\equiv (q^{17};q^{17})_\infty^{17/2} \sum_{n=0}^{\infty} p_{-1/2}(n) q^{n+6} \pmod{17^2}. \end{aligned} \tag{4.2}$$

Applying the Hecke operator T_{17} to both sides, and observing that applying T_{17} is the same as applying U_{17} modulo 17^2, we obtain

$$\varDelta^6(\tau)|_{T_{17}} \equiv (q;q)_\infty^{17/2} \sum_{n=1}^{\infty} p_{-1/2}(17n-6)q^n \pmod{17^2}. \tag{4.3}$$

From (3.39), we know that $p_{-1/2}(17n-6) \equiv 0 \pmod{17}$. By Lemma 2.1, we deduce from (4.3) that

$$\begin{aligned} \frac{1}{17}\varDelta^6(\tau)|_{T_{17}} &\equiv (q;q)_\infty^{17/2} \sum_{n=0}^{\infty} \frac{p_{-1/2}(17n-6)}{17} q^n \\ &\equiv (q^{17};q^{17})_\infty^{1/2} \sum_{n=0}^{\infty} \frac{p_{-1/2}(17n-6)}{17} q^n \pmod{17}, \end{aligned}$$

or

$$\varDelta^6(\tau)|_{T_{17}} \equiv (q^{17};q^{17})_\infty^{1/2} \sum_{n=0}^{\infty} p_{-1/2}(17n-6)q^n \pmod{17^2}. \tag{4.4}$$

Since $\varDelta^6(\tau)|_{T_{17}} \in S_{72}(\mathrm{SL}_2(\mathbb{Z}))$, we apply the Hecke operator T_{17} to both sides of (4.4) and deduce that

$$(\varDelta^6(\tau)|_{T_{17}})|_{T_{17}} \equiv (q;q)_\infty^{1/2} \sum_{n=0}^{\infty} p_{-1/2}(17^2 n-6)q^n \pmod{17^2}. \tag{4.5}$$

Now we recall the following Eisenstein series on $\mathrm{SL}_2(\mathbb{Z})$:

$$E_6 := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}. \tag{4.6}$$

Let

$$\begin{aligned} &B_1 := \varDelta^6(\tau), && B_2 := \varDelta^5(\tau)E_6^2, && B_3 := \varDelta^4(\tau)E_6^4, \\ &B_4 := \varDelta^3(\tau)E_6^6, && B_5 := \varDelta^2(\tau)E_6^8, && B_6 := \varDelta(\tau)E_6^{10}. \end{aligned}$$

It is not difficult to see that $\{B_1, B_2, B_3, B_4, B_5, B_6\}$ forms a basis of $S_{72}(\mathrm{SL}_2(\mathbb{Z}))$. By comparing the Fourier coefficients we find that

$$(\varDelta^6(\tau)|_{T_{17}})|_{T_{17}} = \sum_{i=0}^{6} a_i B_i, \tag{4.7}$$

where

$$\begin{aligned} a_1 = {} &280326642444401148696179366339442612394330680689384957359229 2\\ &186093616946565526483482308, \\ a_2 = {} &111323160254502459554314659620478214275489261082924623899091 9\\ &796002850856740428953088, \end{aligned}$$

$$\begin{aligned}
a_3 &= 4732834266810238479570385785097996159241875744074623451960104\\
&\quad 3621686163904563174 4,\\
a_4 &= -1554074151888843490223291792047379113908229307924982050366\\
&\quad 8479703217777938560,\\
a_5 &= -1604489154697352414421369082789170888441111798460125970130\\
&\quad 1088310976672,\\
a_6 &= 2160262250994438781921107036915961456818368902323 83902466304.
\end{aligned}$$

It is easy to verify that

$$\operatorname{ord}_{17}(a_1) = 3, \quad \operatorname{ord}_{17}(a_i) = 2, \quad 2 \le i \le 6.$$

From (4.5) and (4.7) we can complete the proof of (3.71). ∎

While we believe that this method is applicable to most of the congruences in Conjectures 3.1 and 3.2, we are not sure if one can establish these congruences without the use of modular forms.

5. Concluding remarks. Ramanujan's original proofs of (1.1) and (1.2) (see [21]) involve the fourth and sixth powers of $(q;q)_\infty$. In 1969, Winquist [25] discovered an identity for $(q;q)_\infty^{10}$ and gave a proof of (1.3) which is in the spirit of Ramanujan's proofs for (1.1) and (1.2). Recently, Hirschhorn [15] gave a simple proof of (1.3) that relies only on (2.15) and (2.18). One common feature of the identities used by Ramanujan and Winquist is that for $d = 4, 6$ and 10, $(q;q)_\infty^d$ can be expressed in the form

$$\sum_{m,n=-\infty}^{\infty} A(m,n) q^{Q(m,n)}, \tag{5.1}$$

where $A(m,n)$ is a polynomial in m and n and $Q(m,n)$ is a degree 2 polynomial in m and n. In 1985, J.-P. Serre [24] showed that if d is even, then $(q;q)_\infty^d$ can be expressed in the form given by (5.1) if and only if $d = 2, 4, 6, 8, 10, 14$ and 26. The proof of a series representation for $(q;q)_\infty^{26}$ was given for the first time in [24] although the identity in a different form was first discovered by A. O. L. Atkin (see [11]). For alternative representations of $(q;q)_\infty^{26}$, see the works [6, 7] by Chan, S. Cooper and P. C. Toh. In this work, we return to Ramanujan's original idea and derive congruences satisfied by $p_k(n)$ for a certain rational number k from the series representations for $(q;q)_\infty^d$. In particular, it seems that this is the first time that expansions of $(q;q)_\infty^{14}$ and $(q;q)_\infty^{26}$ are associated to congruences analogous to Ramanujan's partition congruences (1.1)–(1.3).

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